Parameter-free characterization of nuclear band spectra

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We show that the ground-state band spectra of very many rare-earth and actinide nuclei appear to obey simple recurrence relations. Our initial empirical observation is then refined to suggest a new method for predicting higher lying members of a band from lower lying known members.

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In a recent paper [1] we investigated the theory of similarities of nuclear spectra using a binary cluster model that has provided simple explanations of many features of nuclear collective excitations. The aim was to throw light on the phenomenon of so-called identical bands in neighboring nuclei. We found that the degree of similarity could be attributed to the fractional change, between nuclei, of the reduced masses μ_1 and μ_2 of the relevant cluster-core decompositions; that is, $\delta\mu/\mu$ gave a good measure of the fractional change in spectra, where $\delta\mu = (\mu_2 - \mu_1)$ and $\mu = (\mu_1 + \mu_2)/2$. We concluded that the identical bands were merely sporadic special cases in which $\delta\mu/\mu \approx 0$ and were not manifestations of some underlying symmetry.

This work led us to ask whether it might be possible to find some other relations between nuclear quasi-band spectra. To our surprise, we discovered that there exist internal structures in nuclear bands that appear to be common to nearly all deformed nuclei. These depend only on the energies and angular momenta of the levels in any specific band and have the same parameter-free form for most nuclei. So, in a sense, we can claim that almost all band spectra are identical.

These parameter-free relations may be written in various forms, but most conveniently as recurrence relations between pairs of band energies and their associated angular momenta. Writing $E(J)$ and $E(K)$ for the excitation energies of two levels of a band with spins $J > K$ we find, empirically, that

$$\frac{E(J) - E(K)}{(J - K)} = \frac{E(J + L) - E(K - L)}{(J + L) - (K - L)}, \tag{1}$$

where L changes in steps of 2 for $K^\pi = 0^+$ and $K^\pi = 0^-$ bands or in steps of 1 for other bands such as 2^+ gamma bands. As the spin values $(J + L)$, $(K - L)$ change with L we see that their sum remains equal to $(J + K)$. In this paper we restrict our attention to ground-state 0^+ bands in even-even nuclei, but it appears that other bands may satisfy the same equalities. To illustrate the results we apply Eq. (1) to a large number of medium and heavy mass nuclei with $144 \leqslant A \leqslant 248$ subject to the restrictions that we consider only the levels $J^\pi = 0^+, 2^+, \ldots, 12^+$ and that nucleon numbers N_p and N_n are ≥ 6 from closed shells. The reason for the first constraint is that many ground-state bands suffer band-crossing, and hence mixing with states of other bands, and then cease to be yrast beyond about $J \approx 14$. Secondly, as a closed shell is approached, ground-state bands in nuclei with N_p or $N_n < 6$ deviate markedly from the systematic behavior in other nuclei. In addition we omit a few isotopes in this range because of uncertainties in the data. This leaves us with a database of the 76 isotopes listed in Table I with properties taken from Ref. [2]. Having fixed our database, we next selected several pairs of energy differences, the same for the ground band in each nucleus, to show clearly how well the recurrences are obeyed. We concentrate here on the $[(J + L), (K - L) : (J, K)]$ pairs listed in Table II, noting that other choices give similar results.

Then, on rewriting Eq. (1) in the form

$$\frac{E(J + L) - E(K - L)}{E(J) - E(K)} = \frac{J - K + 2L}{(J - K)}, \tag{2}$$

we find that these pairings should yield the numbers 5/3, 2, 3, and 5, respectively. We do not include pairings involving $E(0) = 0$, because we shall later compare with an improved form that would involve division by zero in that case.

For each of the spin pairings specified, we plot, in Fig. 1, the left-hand side of Eq. (2) for all isotopes in the database, against their numerical position in Table I. We also show the horizontal straight lines implied by the right-hand side of Eq. (2). It is immediately obvious that the energy difference ratios fall slightly short of the spin ratios in many cases. The deviations are never more than a few percent, but in each nucleus they are highly correlated.

We point out at this stage that our proposed recursion formula, and a refinement to be discussed in the following, could be useful in the experimental investigation of band spectra. As a simple example we might suppose that the energy spacings in a 0^+ or 0^- band are known up to some $E(J)$. Then using the relation of Eq. (2) in the form

$$E(J + L) - E(K - L) = \frac{(J - K + 2L)}{(J - K)}[E(J) - E(K)], \tag{3}$$

with $L = 2$, makes it possible to estimate the next level energy $E(J + 2)$ and hence to indicate where to look for it. Also, the strong correlations of the deviations from the simple estimates in a particular nucleus can, in principle, be used to improve the original estimate. Any gross difference between the final estimate and the experimental energy would then be evidence of structural alteration in the proposed band (e.g., the band levels cease to be yrast levels as a result of

TABLE I. Database of 76 rare-earth and actinide nuclei.

Table position	Nucleus			Table position	Nucleus		
		Z	A			Z	A
01	Ba	56	144	39	Hf	72	168
02	Ba	56	146	40	Hf	72	170
03	Ce	58	148	41	Hf	72	172
04	Nd	60	150	42	Hf	72	174
05	Nd	60	152	43	Hf	72	176
06	Sm	62	150	44	Hf	72	178
07	Sm	62	152	45	Hf	72	180
08	Sm	62	154	46	W	74	168
09	Gd	64	152	47	W	74	170
10	Gd	64	154	48	W	74	172
11	Gd	64	156	49	W	74	174
12	Gd	64	158	50	W	74	180
13	Gd	64	160	51	W	74	182
14	Dy	66	154	52	W	74	186
15	Dy	66	156	53	Os	76	180
16	Dy	66	158	54	Os	76	184
17	Dy	66	160	55	Os	76	186
18	Dy	66	162	56	Os	76	188
19	Dy	66	164	57	Os	76	192
20	Er	68	156	58	Ra	88	222
21	Er	68	158	59	Ra	88	224
22	Er	68	160	60	Ra	88	226
23	Er	68	162	61	Th	90	222
24	Er	68	164	62	Th	90	226
25	Er	68	166	63	Th	90	228
26	Er	68	168	64	Th	90	230
27	Er	68	170	65	Th	90	232
28	Yb	70	158	66	Th	90	234
29	Yb	70	162	67	U	92	230
30	Yb	70	164	68	U	92	232
31	Yb	70	166	69	U	92	234
32	Yb	70	168	70	U	92	236
33	Yb	70	170	71	U	92	238
34	Yb	70	172	72	Pu	94	238
35	Yb	70	174	73	Pu	94	240
36	Yb	70	176	74	Pu	94	242
37	Hf	72	164	75	Pu	94	244
38	Hf	72	166	76	Cm	96	248

band-crossing). However, the better recursion formula to be derived in the following gives improved results with deviations still correlated. By these methods a partially known band can be extended by using its own bootstraps.

TABLE II. Selected J, K, and L values.

J	K	L	$J+L$	$K-L$	$\frac{J-K+2L}{J-K}$
10	4	2	12	2	5/3
8	4	2	10	2	2
8	6	2	10	4	3
8	6	4	12	2	5

We also find that applying Eq. (3) to the previously selected energy difference pairings, in all 76 nuclei of the database, leads to the fan plot presented in Fig. 2. The data points all lie close to straight lines, with slopes slightly less than the values 5/3, 2, 3, and 5 expected from Eq. (3).

To refine the difference formula of Eq. (1) we begin by showing how it can be transformed into an expression containing adjustable parameters specific to each individual nucleus. This involves solving the linear recursion relation under the assumption that the necessary parameters remain constant throughout the band. It is easy to verify algebraically that a solution for the energies $E(J)$ can be written in either of the equivalent forms

$$E(J) = Q + \alpha J + \beta J^2, \tag{4}$$

$$E(J) = Q + aJ + bJ(J+1), \tag{5}$$

for all J. For the ground-state 0^+ bands considered here, the constant Q is just the absolute energy of the bandhead. However, since Q drops out in the energy differences we can ignore it and write the effective solution as

$$E(J) = \alpha J + \beta J^2, \tag{6}$$

where the $E(J)$ are now taken to be excitation energies relative to $E(0) = 0$. We thus find that the parameter-free difference relations of Eq. (1) imply that the $E(J)$ can be written as suitably parametrized linear combinations of harmonic and rotational spacings, as originally proposed by Ejiri *et al.* [3]. Conversely, since various simple models (and extensions thereof) can be shown [4,5] to give rise to spectra with particular values of the parameters α and β in Eq. (6), it then follows that all such models will generate spectra obeying the parameter-free difference relations of Eq. (1).

We show further that our linear recursion relation has no general independent solution of the form γJ^n with integer $n > 2$. If we put $L = K$ in Eq. (1) and exclude the trivial identity obtained when $K = 0$ we have

$$\frac{[E(J) - E(K)]}{(J-K)} = \frac{[E(J+K) - E(0)]|}{(J+K)}, \tag{7}$$

which on substitution becomes

$$\frac{\gamma[J^n - K^n]}{(J-K)} = \frac{\gamma(J+K)^n}{(J+K)}. \tag{8}$$

This equation is satisfied only for $n \leqslant 2$.

It is now simple to generalize our results. Defining

$$e(J) = E(J)/J \tag{9}$$

we see that Eq. (6) implies

$$e(J) = \alpha + \beta J \tag{10}$$

and hence that $e(J)$ also satisfies the recursion relations of Eq. (1). But we already know, from our earlier discussions, that the general solution of the difference equations

$$\frac{e(J+L) - e(K-L)}{e(J) - e(K)} = \frac{(J+L) - (K-L)}{(J-K)} \tag{11}$$

must be

$$e(J) = \alpha + \beta J + \gamma J^2 \tag{12}$$

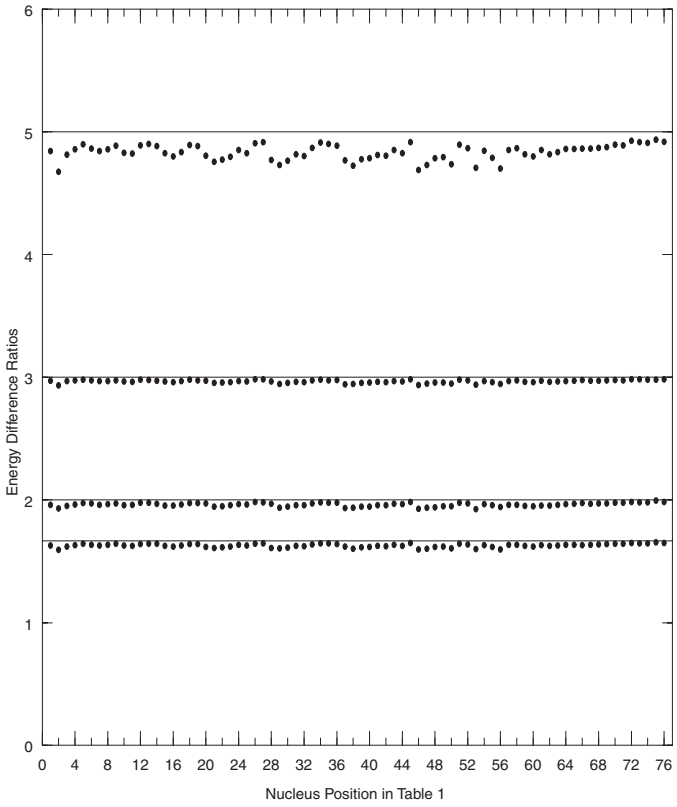

FIG. 1. Observed values of excitation energy ratios $[E(J+L)-E(K-L)]/[E(J)-E(K)]$ for all nuclei listed in Table I. The bold horizontal lines indicate the simple angular momentum ratios $(J-K+2L)/(J-K)$ of 5/3, 2, 3, and 5. See discussion of Eq. (2) in text and Table II for details.

so that the energies of band levels may be well described by the cubic expression

$$E(J) = \alpha J + \beta J^2 + \gamma J^3 \tag{13}$$

in which the coefficients for particular nuclei may be determined from observed data. Note that Eq. (13) is more general than Eq. (6), and hence it should be more accurate. The result of Eq. (13) has also been suggested by Zamfir and Casten [5]. who however did not introduce the parameter-free

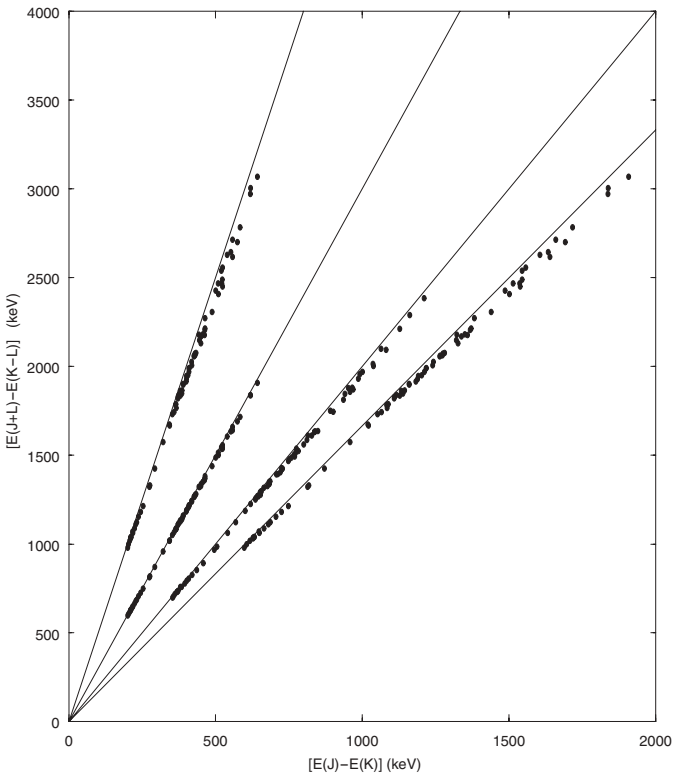

FIG. 2. Plots of excitation energies $E(J+L)-E(K-L)$ against $E(J)-E(K)$, all in keV, for all nuclei listed in Table I. Each nucleus generates a point for each of the four J, K, and L pairings listed in Table II. The expected gradients $(J-K+2L)/(J-K)$ of 5/3, 2, 3, and 5 are indicated by the solid lines through the origin. See discussion of Eq. (3) in text and Table II for details.

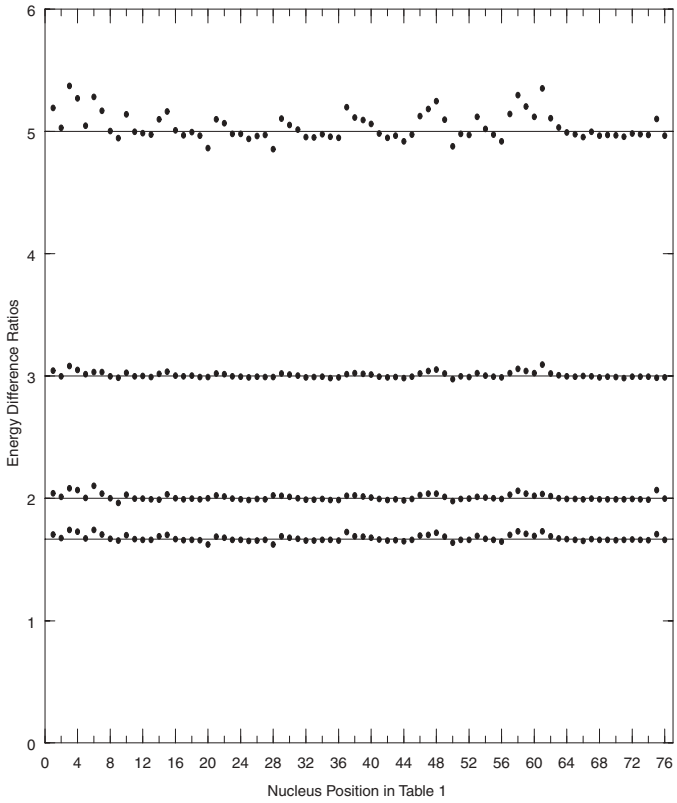

FIG. 3. Observed values of modified excitation energy ratios $[e(J+L)-e(K-L)]/[e(J)-e(K)]$, where $e(J)=E(J)/J$, for all nuclei listed in Table I. The bold horizontal lines indicate the simple angular momentum ratios $(J-K+2L)/(J-K)$ of 5/3, 2, 3, and 5. See discussion of Eq. (11) in text and Table II for details.

difference relations of Eqs. (1) and (11). These would have been helpful in subsequent analysis [6], where the differences $[E(J)-E(K)]$ were plotted against $[E(J')-E(K')]$ for a number of combinations of J, K, J', and K', none of which however obeying the critical condition $(J+K)=(J'+K')$. We remark in passing that the parameter-free Eq. (11) should be simpler to use in practice than Eq. (13) for extending a band from a knowledge of lower levels. Furthermore Eq. (11) implies most of the effects of a quartic term added to Eq. (13).

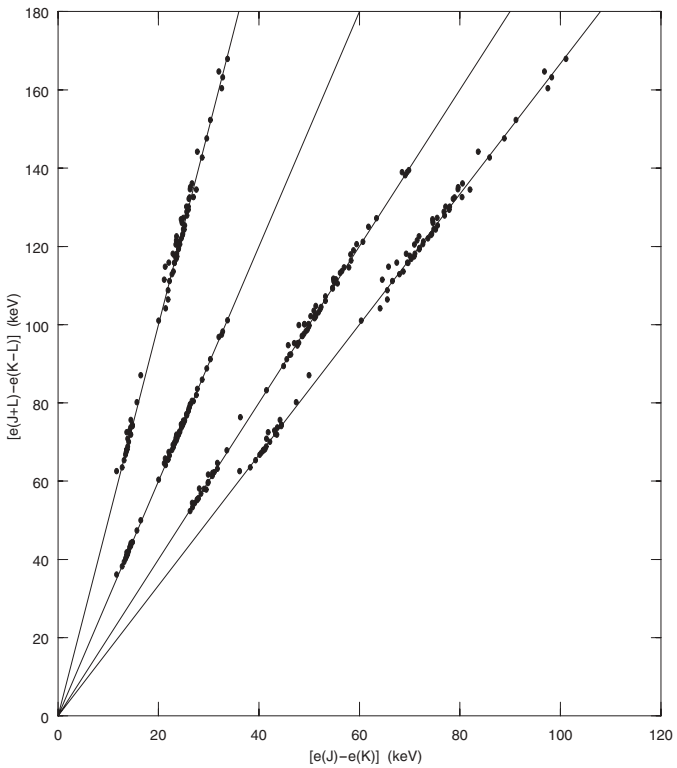

FIG. 4. Plots of modified excitation energies $e(J+L)-e(K-L)$ against $e(J)-e(K)$, where $e(J)=E(J)/J$, all in keV/$\hbar$, for all nuclei listed in Table I. Each nucleus generates a point for each of the four J, K, and L pairings listed in Table II. The expected gradients $(J-K+2L)/(J-K)$ of 5/3, 2, 3, and 5 are indicated by the solid lines through the origin. See discussion of Eq. (11) in text and Table II for details.

In Fig. 3 we present the results of applying the modified relations of Eq. (11) to the same choices of energy pairings as in Fig. 1. The data points are now seen to straddle the horizontal lines. Strong correlations of percentage residuals in individual nuclei remain. The fan plot implied by Eq. (11) is shown in Fig. 4. The results again indicate tight straight lines having slopes in even better agreement than before with the predictions of the corresponding angular momentum difference ratios.

In conclusion, we have shown that simple, parameter-free recursion relations apply accurately to the excitation energies of states belonging to the ground-state bands of an extensive set of even-even nuclei. One immediate use of these relations is in estimating the excitation energies of the higher lying members of a band from the known energies of lower lying members. In this context they are considerably more accurate than methods commonly used at present (e.g., the widespread fitting of A and B coefficients in Nuclear Data Sheets [2]). So far these relations are purely phenomenological, but they are sufficiently intriguing to prompt further research aimed at achieving a theoretical understanding.

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