Nuclear symmetry energy and stability of matter in neutron stars

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It is shown that the nuclear symmetry energy is the key quantity in the stability consideration in neutron star matter. The symmetry energy controls the position of crust-core transition and also may lead to new effects in the inner core of neutron star.

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I. INTRODUCTION

In neutron stars, the prevailing part of the interior is fulfilled with matter in the state called the beta equilibrium [1]. It concerns the liquid core as well as the inner parts of the star crust in the region of crust-core transition. The transitional region performs a nuclear matter being a subject of instability against clusterization to a two-phase system: neutron-rich nuclei immersed in dripped neutrons (and sometimes protons) [2]. As nuclei are arranged in a lattice they form solid state crust covering the star core being a homogeneous liquid.

In this work we intend to show that this kind of instability may be analyzed in terms of simple inequalities that show direct connections to the shape of nuclear symmetry energy. The symmetry energy behavior with density is still not well determined, especially at densities much above saturation point n_0, which are found in the neutron star core. There is no experimental evidence (from terrestrial laboratories) about the shape of E_s at these densities. To obtain any description we are forced to extrapolate a theory that is well tested merely around n_0. Most of models predict high values of E_s or at least steady increase with density [3]. However, recent observations of neutron star cooling suggest that fast cooling through direct URCA cycle does not work in stars [4], which would mean that E_s takes rather low values at higher densities. These "too hot" neutron stars may sustain direct URCA only after inclusion the proton superfluidity [5] but in this way the cooling observation cease to be conclusive for E_s behavior. In this work we find that a very low E_s leads to a new effect that changes the internal structure of neutron stars and may have observational consequences not connected with cooling only.

II. STABILITY CONDITIONS

The beta reactions that take place in a neutron star conserve charge Q and baryon number B. Having neglected the temperature, relevant only for a young hot star, the total energy U becomes a function of volume and conserved numbers $U(V, B, Q)$. To consider stability of single phase one needs to introduce intense (local) quantity $u = U/B$. The energy per particle u then becomes a function of other local quantities, taken per baryon number $v = V/B$ and $q = Q/B$. The first principle of thermodynamics takes the following form:

$$du = -Pdv - \mu dq, \tag{1}$$

where P is the pressure and μ the chemical potential of an electric charge. From beta equilibrium one may reads that

$$\mu = \mu_e = \mu_\mu = \mu_n - \mu_p. \tag{2}$$

The minus sign before μ in Eq. (1) comes from the definition of $Q = N_p - N_e - N_\mu$, which is negative for leptons.[1] The stability of any single phase, also called the *intrinsic* stability, is ensured by the convexity of $u(v, q)$ [6]. Thermodynamical identities allow us to express this requirement in terms of following inequalities [7]:

$$-\left(\frac{\partial P}{\partial v}\right)_q > 0 \quad -\left(\frac{\partial \mu}{\partial q}\right)_P > 0. \tag{3}$$

Usually, only the positive compressibility is examined; in particular, it is required for locally neutral matter that

$$-\left(\frac{\partial P}{\partial v}\right)_{q=0} > 0. \tag{4}$$

However, the second inequality in Eq. (3) is of the same importance. It concerns the stability of charge fluctuations and, as shown later, it is connected to the positive value of the screening length in matter. Not all nuclear models ensure the charge fluctuations to be stable. As was shown in the case of kaon condensation for wide class of models the system is not stable at any density [7]. In this work we show that a system like *npl* matter also represents a region of density where the instability occurs.

One may find another pair of inequalities that are equivalent to those in Eq. (3) and, as it shown later, are more convenient in further calculations:

$$-\left(\frac{\partial P}{\partial v}\right)_\mu > 0 \quad -\left(\frac{\partial \mu}{\partial q}\right)_v > 0. \tag{5}$$

The intrinsic stability is determined by the details of nucleon-nucleon interactions. To show that on the most general level, let's split the total energy per baryon into the nucleonic and leptonic part, $u = u^N + u^L$. The nucleonic contribution may be always expressed as a function of baryon number density, $n = B/V$, and the proton fraction, $x = N_p/B$. For leptons ε^L, the energy per volume, is completely determined by their

[1] This convention of charge sign is opposite to that used in Ref. [7] and, of course, is more natural.

chemical potential μ. Such decomposition is also true for the total pressure, so one may write

$$u = u^N(n, x) + \varepsilon^L(\mu)/n \quad (6)$$
$$P = P^N(n, x) + P^L(\mu), \quad (7)$$

where $P^N = n^2 u_n^N$; henceforth, we indicate the partial derivatives of u^N or P^N by subscripts n or x. One may show that $\mu_n - \mu_p = (\partial u^N/\partial x)_n$ and then the beta equilibrium means that

$$\mu = -u_x^N. \quad (8)$$

Differentiation of that equation leads to the expression for proton fraction derivative hold under beta equilibrium

$$(\partial x/\partial n)_\mu = -u_{nx}^N / u_{xx}^N. \quad (9)$$

Above relation, together with $P^N = n^2 u_n^N$, allows us to express the stability conditions only in terms of nucleonic contribution u_N to the total energy u. Let's take first the compressibility appearing in Eq. (5)

$$-\left(\frac{\partial P}{\partial v}\right)_\mu = n^2\left[P_n^N + P_x^N\left(\frac{\partial x}{\partial n}\right)_\mu\right] \quad (10)$$
$$= n^4\left[2\frac{u_n^N}{n} + u_{nn}^N - \frac{\left(u_{nx}^N\right)^2}{u_{xx}^N}\right]. \quad (11)$$

Keeping in mind that charge per baryon is $q = x - n_L/n$ and using Eq. (8), we obtain

$$-\left(\frac{\partial q}{\partial \mu}\right)_n = -\frac{1}{u_x^N x} + \frac{n_L'(\mu)}{n}, \quad (12)$$

which is the inverse of derivative appearing in the second inequality in Eq. (5).

The obtained expression for derivatives appearing in the stability conditions require some comments now. The first two terms in Eq. (11) refer to the pressure and compressibility of pure nucleonic matter and they are positive for very fundamental reasons, whereas the third term (which comes from leptons presence) contributes negatively. It is the leptons that make the matter unstable. In the expression for $(\partial q/\partial\mu)_n$ one may recognize the derivative n_L' as the screening lengths for leptons [8], so the second stability condition in Eq. (5) is connected to the stable screening for leptons. However, the first term in Eq. (12) lacks this kind of interpretation; as a result, for the quantities like $\partial q/\partial\mu$ we adopt the name "electrical capacitance" of matter. It measures the energetic cost of change in electric charge held in matter.

For further discussion we introduce the compressibility and electric capacitance as

$$K_i = -v^2\left(\frac{\partial P}{\partial v}\right)_i = \left(\frac{\partial P}{\partial n}\right)_i, \quad i = q, \mu \quad (13)$$
$$\chi_j = -\left(\frac{\partial q}{\partial \mu}\right)_j, \quad j = P, v \quad (14)$$

then stability condition may be written as

$$K_\mu = 2nu_n^N + n^2 u_{nn}^N - \frac{\left(u_{nx}^N n\right)^2}{u_{xx}^N} > 0 \quad (15)$$
$$\chi_v = \frac{1}{u_{xx}^N} + \frac{\mu(k_e + k_\mu)}{n\pi^2} > 0. \quad (16)$$

Here we express $n_L'(\mu)$ in terms of Fermi momenta of leptons k_e, k_μ.

In the same manner as presented above one may express the first pair of conditions (3).

$$K_q = n^2 u_{nn}^N + 2nu_n^N + \frac{n_L^2 v^2 u_{xx}^N - u_{nx}^N\left(2n_L + n_L' n u_{nx}^N\right)}{1 + u_{xx}^N n_L' v} > 0 \quad (17)$$
$$\chi_P = \frac{P_n^N + u_{xx}^N n_L^2 v^2 - 2u_{nx} n_L}{P_n^N u_{xx}^N - \left(u_{nx}^N n\right)^2} + \frac{\mu(k_e + k_\mu)}{n\pi^2} > 0. \quad (18)$$

By use of the standard theorem for implicit functions one may get useful relations between compressibilities and capacitance

$$\chi_P - \chi_v = K_\mu \alpha_P^{-2} n^{-2} \quad (19)$$
$$K_q - K_\mu = \chi_v \alpha_q^2 n^2, \quad (20)$$

where $\alpha_j = (\partial\mu/\partial n)j$. For further analysis we choose the stability conditions expressed by the pair of equations (15) and (16) rather than (17) and (18). They lead to simpler expressions and moreover are more sensitive to the onset of instability. The right-hand side of equations (19) and (20) is always positive, so for any stable system the relation holds

$$K_\mu < K_q \quad \text{and} \quad \chi_v < \chi_P, \quad (21)$$

and if we approach to the instability point the K_μ vanishes before K_q or χ_v vanishes before χ_P.

Finally, we may express the stability conditions (15) and (16) in terms of symmetry energy. The isospin symmetry allows us to decompose the nucleonic contribution u^N in a series of even powers of $(1 - 2x)$ [9]:

$$u^N(n, x) = V(n) + E_s(n)(1 - 2x)^2 + \mathcal{O}(1 - 2x)^4, \quad (22)$$

where $V(n)$ is the isoscalar potential and $E_s(n)$ the symmetry energy corresponding to the interactions isovector channel. In further analysis we neglect the quartic term. Up to now we have no experimental constraint on its value. Some theoretical investigations, like in Ref. [9], show that the terms higher than $(1 - 2x)^2$ are negligible. However, it was recently shown that quartic terms are of great importance in the case of direct URCA cycle [10]. It would be interesting to check its relevance in the stability considerations, but we give up it in this work to keep simple picture and avoid additional complexity.

Applying Eq. (22) to Eqs. (15) and (16) we obtain

$$K_\mu = n^2(E_s''(1-2x)^2 + V'') + 2n(E_s'(1-2x)^2 + V') - \frac{2(1-2x)^2 E_s'^2 n^2}{E_s} > 0 \quad (23)$$
$$\chi_v = \frac{1}{8E_s(n)} + \frac{\mu(k_e + k_\mu)}{n\pi^2} > 0. \quad (24)$$

Above equations shows explicitly the importance of symmetry energy E_s in the stability considerations. The concrete shape of the function $E_s(n)$ is not well known. We know only its value at saturation point, $n_0 = 0.16$ fm^{-3}, where it takes about 30 MeV. Recent analysis on the isospin diffusion in heavy-ion collisions constrained significantly the slope of $E_s'(n_0)$ and the stiffness $E_s''(n_0)$ at saturation point [11,12]; however, these results do not determine the high-density behavior definitely. There is no experimental evidence about values of E_s at very high density that is available in the central parts of a neutron star. In such extrapolations we must rely on the model calculations. For all of them the symmetry energy at saturation point have positive slope but at higher densities, they lead to different conclusions. For most the E_s is a monotonically increasing function of n but some models lead to the E_s that saturates at higher densities or even bends down at some point and goes to zero [13,14]. This kind of behavior is especially interesting as the last term in Eq. (23) may then take arbitrary large negative values and lead to instability. From the other side, going to very low density, we encounter uncertainties as well. All models predict E_s decreasing to 0. However, recent experiments [15] show that symmetry energy saturates at the level about 10 MeV for very low densities.

III. NUCLEAR MODELS

To present the role played by the symmetry energy we apply a set of nuclear models. At low densities the isoscalar part is kept the same, whereas the symmetry energy takes different forms. The isoscalar potential $V(n)$ was taken from Ref. [16] and leads to the compressibility of symmetric matter equal to 240 MeV at saturation point. For E_s we used shapes applied by Chen *et al*. in Ref. [12]; they were numbered by a parameter $x = 1, 0, -1, -2$. Here we named them a, b, c, and d to avoid confusion with proton fraction x. The shapes of symmetry energy dependence at lower densities are presented in Fig. 1.

At higher densities, much above n_0, we introduce two kinds of isoscalar potential $V(n)$: one from Ref. [16] (the same as in low density regime) and the second from Ref. [17]. The isoscalar potential mainly influences the stiffness of the equation of state. In this way one may test how the instability point is affected by the stiffness of the equation of state. The latter potential is stiffer and lead to stars with higher maximal mass and is in better agreement with recent observations of

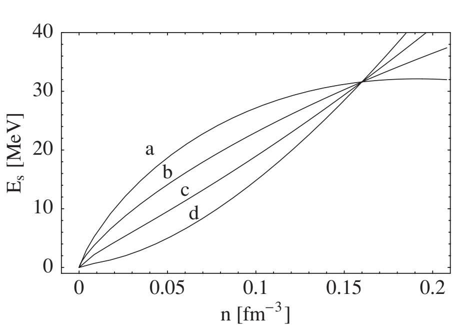

FIG. 1. The different shapes of the symmetry energy at densities below saturation point.

TABLE I. The parameterization of symmetry energy (n_1, n_2) and results for soft and stiff equations of state. All densities are in fm^{-3}.

	A	B	C
n_1	1.0	1.5	1.8
n_2	2.3	2.5	10.0
	Soft		
n_c	0.74	1.20	1.43
$n_{\rm cen}$	1.92	1.32	1.21
$M_{\rm max}/M_\odot$	1.64	1.73	1.84
	Stiff		
n_c	0.85	1.40	>1.6
$n_{\rm cen}$	1.35	1.22	1.17
$M_{\rm max}/M_\odot$	2.02	2.08	2.13

pulsar with mass $2.1 \pm 0.2 M_\odot$ [18]. For E_s we applied a "bent-down" function. This type of symmetry energy with low values at high density was typical in the past variational calculations based on realistic potentials [13]. This kind of behavior is not obtained in more recent calculations like in Ref. [17]. Nevertheless, there are also other modern approaches based on chiral dynamics [14] and Skyrme effective forces [19] or relativistic mean field [20] where very low values of E_s were obtained. Although the "bent-down" scenario seems to be less likely it cannot be abandoned completely. Here, for numerical simplicity (to avoid uncertainty in derivatives of interpolated function), we introduce the simple polynomial that imitates results of works mentioned above

$$E_s(n) = E_s^{(0)} \frac{n(n-n_1)(n-n_2)}{n_0(n_0-n_1)(n_0-n_2)}, \tag{25}$$

where $E_s^{(0)} = 30$ MeV and other parameters are given in Table I. The shapes of these functions, named A, B, and C, are shown in Fig. 2.

IV. RESULTS

The transition between the liquid core and the solid crust of the star is strictly connected to the breaking of the conditions (23) and (24). When at least one of them is broken the matter

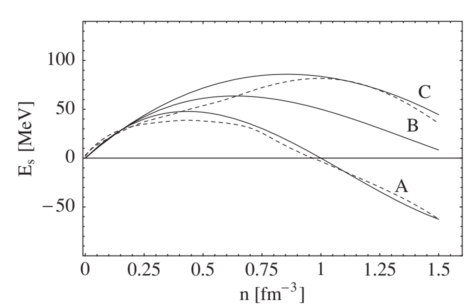

FIG. 2. Three different shapes of the symmetry energy at densities above saturation point (solid lines). For comparison the results of realistic potentials (dotted lines), the higher (UV14UVII) and the lower (UV14TNI).

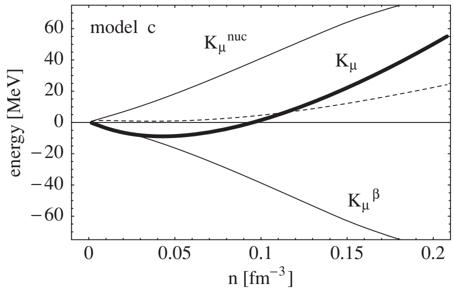

FIG. 3. The compressibility K_μ (thick) and its contributions (thin lines). The dotted line corresponds to the energy per baryon for neutral matter $u(n, 0)$.

can no longer be homogeneous; it splits into two phases. Figure 3 shows the compressibility under constant μ and its two contributions: "nuclear," K_μ^{nuc}, the two first terms in Eq. (23), and "beta," K_μ^{β}, the last term in Eq. (23), which comes from the presence of leptons. The "beta" contribution is always negative, hence there is always a competition between the positive "nuclear" compressibility and the beta reactions that tend to destabilize the matter. At some critical point, n_c, the compressibility, changes its sign and below n_c the matter cannot exist as a single, neutral phase. The actual splitting into two phases does not occur exactly at n_c but at slightly above n_c, because the system must find a state where the two charged phases may coexist. The correction is expected to be tiny, so the point for the vanishing of K_μ may be treated as a good estimation for the boundary of the liquid core in a neutron star.

Table I shows the critical density at which K_μ vanishes. It depends strongly on E_s but does not behaves monotonically with the E_s. For symmetry energy taking both high and low values (a, d case) the n_c moves to higher density close to saturation point. The lowest values of n_c are achieved with intermediate E_s (models b and c). There is no simple relation between values of E_s and n_c because the first and second derivatives of E_s are essential as well.

The stability of matter at densities much greater than n_0 do not need to be obvious. For the symmetry energy increasing in the whole range of density the matter is stable indeed. However, the chosen nuclear models with very low values of E_s lead again to the same kind of instability as occurs in the crust-core transition region. For all presented models (A, B, and C, soft and stiff), there is a critical density n_c where K_μ vanishes. The behavior of the compressibility K_μ for model B with soft isoscalar potential is shown in Fig. 4. It is worth noting that energy per baryon for neutral matter $u(n) \equiv u(n, 0)$ reveals no pathology—it is monotonically increasing and its convexity, $u_{nn}(n) = K_q/n^2$, remains positive at all densities. If one looks only at the energy per baryon behavior, one

TABLE II. The critical density for different models.

Model	a	b	c	d
n_c, fm^{-3}	0.119	0.092	0.095	0.160

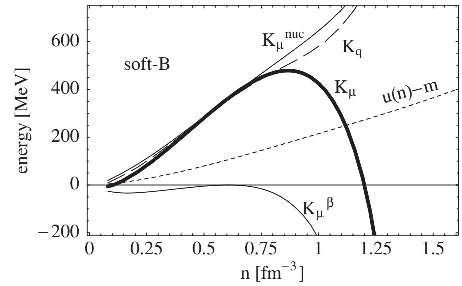

FIG. 4. The compressibility K_μ (thick) and its contributions (thin lines) for soft equation of state and B symmetry energy. The dashed line corresponds to K_q and dotted line energy per baryon for neutral matter, $u(n) \equiv u(n, 0)$.

may overlook that the matter becomes unstable at some point. Table II shows the value of n_c depends mainly on the symmetry energy model. The lower E_s is at higher density the lower n_c is. The stiffness of equation-of-state changes n_c by moving it to higher densities for all A, B, and C models.

Of course, the instability point has physical meaning only if it is attainable in a neutron star. Table II shows basic neutron star properties: the central density, n_{cen}, of a star with maximal mass, M_{max}, found by solving of the Tolman-Oppenheimer-Volkoff (TOV) equation. The results in Table II are in apparent contradiction to the common conviction that the equation of state is softer (maximal mass is lower) for larger E_s. That is true for the case when E_s has a positive slope $[E_s'(n) > 0]$. The pressure depends only on the slope of V and E_s (let's neglect the tiny leptonic contribution)

$$P = n^2(V' + (1 - 2x)^2 E_s').$$

For increasing E_s matter becomes more symmetric, $x \to 1/2$, and the contribution to the pressure from E_s becomes smaller and the equation of state is softer. For A, B, and C models matter gets less symmetric, $x \to 0$, but at some density $E_s'(n)$ changes its sign, and now low values of proton fraction increase the negative contribution from symmetry energy, making the equation of state softer again. Another comment concerns the concrete values of maximal masses in Table II. The equation of state applied in TOV equations assumes a one-phase system also for densities above n_c. It shows that n_c is attainable only for a given equation of state. Above n_c the equation of state should be corrected by a proper construction for a two-phase system and then n_{cen} and M_{max} may change, but, we suspect, that not much. As one may notice, in the models A and B for soft and in model A for the stiff equation of state the phase instability occurs for a sufficiently massive star. For such star, the central part of its core must contain separated phases. It is an open question about properties of matter in this state. The instability itself only signals formation of inhomogeneities with different charge and density. One may suspect formation of mixed phase with liquid properties or solidification of the central part of stellar core. To answer the question, what actually happens above the critical density requires a more detailed analysis, including the phase coexistence and fine-size effects like surface and Coulomb energy.

V. SUMMARY AND DISCUSSION

In this article we presented the simple connection between the symmetry energy E_s and the phase stability of dense matter filling the neutron star interior. It was shown that the relevant quantity in such considerations is K_μ, the compressibility under constant chemical potential, rather than K_q, the compressibility under constant charge. The instability of matter under low density below n_0 leads to phase separations and corresponds to the transition from the liquid core to the solid crust. The pulsar glitching phenomenon allows us to estimate the size of neutron star crust [21], so in this way one may get constraint on E_s behavior at low densities coming from pulsar observations.

The stability considerations were also carried out at a very high density. It was shown that for nuclear models with small values of E_s the instability indeed occurs. The value of critical density depends mainly on E_s but, moreover, the stiffness represented by the isoscalar potential V influences the onset of instability. The instability may lead to solidification of the internal parts of the core; this seems to be especially interesting in connection to rotational and magnetic properties of pulsars. Such research seems to be worth doing. Knowledge about possible observational consequences of the solid inner core will enable us to verify the nuclear models leading to the "bent-down" scenario for the symmetry energy.

[1] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, New York, 1983).

[2] J. M. Lattimer and D. G. Ravenhall, Astrophys. J. **223**, 314 (1978); C. J. Pethick, D. G. Ravenhall, and C. P. Lorenz, Nucl. Phys. **A584**, 675 (1995).

[3] A. W. Steiner, M. Prakash, J. M. Lattimer, and P. J. Ellis, Phys. Rept. **411**, 325 (2005).

[4] D. Page, J. M. Lattimer, M. Prakash, and A. W. Steiner, Astrophys. J. Suppl. **155**, 623 (2004).

[5] D. G. Yakovlev and C. J. Pethick, Ann. Rev. Astron. Astrophys. **42**, 169 (2004).

[6] H. B. Callen, *Thermodynamics* (John Wiley & Sons Inc., New York, 1960).

[7] S. Kubis, Phys. Rev. C **70**, 065804 (2004).

[8] H. Heiselberg, C. J. Pethick, and E. F. Staubo, Phys. Rev. Lett. **70**, 1355 (1993).

[9] C.-H. Lee, T. T. S. Kuo, G. Q. Li, and G. E. Brown, Phys. Rev. C **57**, 3488 (1998).

[10] A. W. Steiner, Phys. Rev. C **74**, 045808 (2006).

[11] L. W. Chen, C. M. Ko, and B. A. Li, Phys. Rev. C **72**, 064309 (2005).

[12] L. W. Chen, C. M. Ko, and B. A. Li, Phys. Rev. Lett. **94**, 032701 (2005).

[13] R. B. Wiringa, V. Fiks, and A. Fabrocini, Phys. Rev. C **38**, 1010 (1988).

[14] N. Kaiser, S. Fritsch, and W. Weise, Nucl. Phys. **A697**, 255 (2002).

[15] S. Kowalski *et al*., Phys. Rev. C **75**, 014601 (2007).

[16] M. Prakash, T. L. Ainsworth, and J. M. Lattimer, Phys. Rev. Lett. **61**, 2518 (1988).

[17] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Phys. Rev. C **58**, 1804 (1998).

[18] D. J. Nice, E. M. Splaver, I. H. Stairs, O. Loehmer, A. Jessner, M. Kramer, and J. M. Cordes, Astrophys. J. **634**, 1242 (2005).

[19] B. K. Agrawal, S. K. Dhiman, and R. Kumar, Phys. Rev. C **73**, 034319 (2006).

[20] S. Typel, Phys. Rev. C **71**, 064301 (2005).

[21] M. A. Alpar, H. F. Chau, K. S. Cheng, and D. Pines, Astrophys. J. **409**, 345 (1993).