Fully nonlinear excitations of non-Abelian plasmas

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We investigate fully nonlinear, non-Abelian excitations of quark-antiquark plasma using relativistic fluid theory in cold plasma approximation. There are mainly three important nonlinearities, coming from various sources such as non-Abelian interactions of Yang-Mills (YM) fields, Wong's color dynamics, and plasma nonlinearity, in our model. By neglecting nonlinearities due to plasma and color dynamics we obtain the earlier results of J. P. Blaizot and E. Iancu [Phys. Rev. Lett. **72**, 3317 (1994)]. Similarly, by neglecting YM field nonlinearity and plasma nonlinearity, the model reduces to the model of S. S. Gupta, P. K. Kaw, and J. C. Parikh [Phys. Lett. **B498**, 223 (2005)]. Thus we have the most general non-Abelian mode of quark-gluon plasma. Further, our model resembles the model of laser propagation through relativistic plasma [P. K. Paw, A. Sen, and E. J. Valeo, Physica **9D**, 96 (1983)] in the absence of all non-Abelian interactions.

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I. INTRODUCTION

Quark-gluon plasma (QGP) is a quasi-color-neutral gas of quarks and gluons that exhibits collective behavior. It is expected to be formed in relativistic heavy ion collision (RHICs) experiments, deep inside the neutron star and might have formed in early universe. A study of collective excitations of QGP is important to diagnose various parameters and signatures of QGP. It is also proposed that the chaotic collective modes of QGP give an estimate of thermalization of QGP in RHICs [1]. From the extensive study of electrodynamics plasma, we know that there exist various linear and nonlinear excitations in it, governed by electrodynamic interactions, which is an Abelian gauge theory. Here, in QGP, we also expect similar linear and nonlinear modes, but modified by the non-Abelian interaction, which itself is nonlinear. Therefore, in QGP, there are two types of nonlinear effects, one coming from usual plasma nonlinearity and another from non-Abelian effects. Nonlinear solutions of non-Abelian or Yang-Mills (YM) theory are studied extensively by Matinyan, Savvidy, and Ter-Arutyunyan-Savvidy [2] without plasma, but with Higgs order phase. Later, these studies were extended to QGP by Blaizot and Iancu [3] and various periodic, quasi-periodic, chaotic nonlinear modes, and transition from order to chaos by plasma collective effects were studied. A study of stabilization of QCD vacuum instability by plasma collective modes was examined earlier in Ref. [4]. There is another group of work along these lines by Gupta, Kaw, and Parikh [5] where nonlinear or non-Abelian modes, coming from the Wong's color dynamics [6], were studied, but without the nonlinearity of YM fields and plasma nonlinearity. In Ref. [3], the role of Wong's color dynamics is not explicit because it is based on quantum kinetic theory [7], whereas in classical theories of [5,8,9] and in our model color dynamics is explicit. Here, we present fully nonlinear, non-Abelian excitations, including all nonlinearities: plasma nonlinearity, YM field nonlinearity, and color dynamics nonlinearity.

II. RELATIVISTIC FLUID THEORY OF QGP

The relativistic fluid set of equations, in cold plasma limit, is given by [8]

$$m\frac{du^\mu}{d\tau} = g I_a G_a^{\mu\nu} u_\nu, \tag{1}$$

the equation of motion, where m is the mass, τ the proper time, g the coupling constant, a the color index, u^ν the 4-velocity, which is also the fluid velocity in cold plasma limit, and $G_a^{\mu\nu}$ the field tensor, defined as

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc} A_b^\mu A_c^\nu, \tag{2}$$

in terms of four-vector potentials A_a^μ and where ϵ_{abc} is the Levi-Civita tensor, the structure constant of our $SU(2)$ YM system. I_a are the dynamical color charges that obey Wong's equation,

$$\frac{dI_a}{d\tau} = -g\epsilon_{abc} u^\mu A_{\mu b} I_c. \tag{3}$$

The vector potentials are obtained from the Yang-Mills field equation,

$$\partial_\mu G_a^{\mu\nu} + g\epsilon_{abc} A_{\mu b} G_c^{\mu\nu} = J_a^\nu, \tag{4}$$

where J_a^ν is the four-vector color current produced by various species in plasma with color charges, such as quarks, antiquarks, and gluons. For simplicity, here in our analysis, we consider quark-antiquark plasma and the current density is given by

$$J_a^\nu = g \sum_{\text{species}} n I_a u^\nu, \tag{5}$$

where n is the density of each species, determined by the continuity equation,

$$\partial_\mu (n u^\mu) = 0. \tag{6}$$

Because our main goal here is to look for non-Abelian features in QGP, we have neglected thermal effects by taking the cold plasma limit, for simplicity. On the other hand, one must

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include pressure tensor terms, which involves new features like pressure gradients, and viscosity terms in the equation of motion and an additional equation, the equation of state. Of course, if one proceeds with kinetic theory, [9], then all these thermal effects are included, which is a separate problem, and we know from the study of plasma physics [10] that major properties of plasma may be explained by the study of fluid theory. We also know that by taking different moments, with respect to particle momentum, p^μ, color charge, Q_a, etc., we may derive the set of fluid equations as shown in the Appendix. The lowest order two moments of the distribution function with respect to p^μ leads to density and fluid momentum. Further moments of the distribution function with respect to p^μ and Q_a may be used to define the fluid dynamical color charge, which rightly reproduces the Wong's equation for color charge (3). With these defined fluid variables one gets the expression for J_a^ν, Eq. (5), as shown in the Appendix.

In general, Eqs. (1) to (6) are a set of very complicated, coupled, nonlinear equations to be solved and hence one goes for approximations, such as moving frame ansatz [11], space-homogeneous solutions, and so on, to look for special solutions. Following Blaizot and Iancu [3], let us consider the homogeneous solutions of our set of equations, a few of them may be easily solved. The continuity equation for each species may be integrated and we get

$$n(t)u^0(t) = \text{constant} = n_0 u_0^0, \tag{7}$$

where n_0 and u_0^0 are the density and zero-component of fluid velocity at equilibrium. We also chose a gauge $A_a^0 = 0$ and the spatial part of the equation of motion may be easily integrated to get

$$u^j = -\frac{g I_a A_a^j}{m}, \tag{8}$$

with the assumption that, at equilibrium, the plasma is at rest. The zero-component fluid velocity is given by

$$u^0 = \sqrt{1 + u_j^2}, \tag{9}$$

and hence $u_0^0 = 1$. Similarly, the spatial part of the field equations gives

$$\ddot{A}_a^i + g^2\left[\left(A_b^j A_b^j\right) A_a^i - \left(A_b^j A_a^j\right) A_b^i\right] = g \sum n_0 I_a \frac{u^i}{u^0}, \tag{10}$$

and the temporal component gives

$$\sum n_0 I_a = \epsilon_{abc} A_b^i \dot{A}_c^i, \tag{11}$$

where the dot means differentiation with respect to time. Finally, the color dynamics equation reduces to

$$\dot{I}_a = g\epsilon_{abc}\frac{u^i}{u^0} A_b^i I_c. \tag{12}$$

For further simplification, let us use hedgehog ansatz where the color directions are taken to be along the spatial direction and redefine variables as

$$X \equiv \frac{g I_0 A_{x1}}{m}; \quad Y \equiv \frac{g I_0 A_{y2}}{m}; \quad Z \equiv \frac{g I_0 A_{z3}}{m}; \tag{13}$$

and rescaling time and color charges as

$$t \to \frac{m}{I_0} t \quad \text{and} \quad I_a \to \frac{I_a}{I_0}, \tag{14}$$

where I_0 is introduced to normalize $I_a I_a = 1$, which is one of the constants of motion as can be seen from the equation for color dynamics. Further, from Eq. (11), I_a of second species (antiquarks) is opposite to that of first species (quarks) and hence $I_{2a} = -I_{1a} \equiv -I_a$. In terms of redefined variables, our simplified set of equations becomes

$$\ddot{X} + (Y^2 + Z^2)X = -\epsilon I_x^2 \frac{X}{\sqrt{1 + (I_x X)^2 + (I_y Y)^2 + (I_z Z)^2}} \tag{15}$$

and

$$\dot{I}_x = -\frac{I_y I_z (Y^2 - Z^2)}{\sqrt{1 + (I_x X)^2 + (I_y Y)^2 + (I_z Z)^2}}, \tag{16}$$

and similar equations for y and z components that may be obtained by cyclic change among x, y, and z. The parameter $\epsilon \equiv \frac{2\omega_p^2 I_0^2}{m^2}$, where the plasma frequency $\omega_p^2 \equiv \frac{n_0 g^2 I_0^2}{m}$. The above set of equations has two immediate constants of motion, namely, $I_a I_a = 1$ and

$$(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)/2 + (X^2Y^2 + Y^2Z^2 + Z^2X^2)/2 + \epsilon\sqrt{1 + (I_x X)^2 + (I_y Y)^2 + (I_z Z)^2} = E, \tag{17}$$

the energy. This approximate set of equations retains all the important aspects of QGP such as YM nonlinearity, plasma nonlinearity, and color dynamics nonlinearity. In the earlier calculations of Blaizot and Iancu [3], the plasma nonlinearity is neglected and in Ref. [5] the YM nonlinearity is dropped out.

To extract the results of Blaizot and Iancu [3], let us assume that color charge I_a is constant, and then Eq. (15) reduces to

$$\ddot{X} + (Y^2 + Z^2)X = -\epsilon\frac{1}{3}\frac{X}{\sqrt{1 + (X^2 + Y^2 + Z^2)/3}}, \tag{18}$$

where the square root term is the plasma nonlinearity, coming from the relativistic treatment just like in Ref. [11]. Further, expanding the plasma nonlinearity term up to 3rd order in vector potential gives

$$\ddot{X} + \left(1 - \frac{\epsilon}{18}\right)(Y^2 + Z^2)X - \frac{\epsilon}{18}X^3 + \frac{\epsilon}{3}X = 0, \tag{19}$$

which is similar to that of Blaizot and Iancu [3], except with a few new terms containing a $(-\frac{\epsilon}{18})$. This new terms may lead to additional new features like chaotic scattering [12]. This model without these new additional terms was studied extensively by Matinyan, Savviddy, and Ter-Arutyunyan-Savvidy [2] and by Blaizot and Iancu [3].

Let us look for some other new solutions of our model Eq. (18). For example, a special solution with $Z = 0$ leads to

$$\ddot{X} + Y^2 X = -\epsilon\frac{1}{3}\frac{X}{\sqrt{1 + (X^2 + Y^2)/3}}, \tag{20}$$

for X and a similar equation for Y with X and Y interchanged. It differs from similar work by Matinyan, Savvidy, and Ter-Arutyunyan-Savvidy [2] and Blaizot and Iancu [3] because we have kept the plasma nonlinearity also. Our numerical study shows that the plasma nonlinearity enhances the chaos and therefore increases the order-to-chaos transition parameter defined in Ref. [2], which will be discussed later.

Next, let us look at another special solution with $X = Y = Z$ of our general equation Eq. (15) and we get

$$\ddot{X} + 2X^3 = -\epsilon \frac{1}{3} \frac{X}{\sqrt{1 + X^2}}, \tag{21}$$

which describes a nonlinear oscillation. It is a more general nonlinear oscillation, including the plasma nonlinearity, than the elliptic functions Cn discussed in Refs. [2,3]. It is easy to see that on neglecting the plasma nonlinearity, we get back Cn or Sn, depending on the strength of the non-Abelian parameter compared to the plasma frequency. It is interesting to note that the above mode is an exact solution of QGP because, for $X = Y = Z$, the color dynamics equation shows that the color charges are constant.

III. RESULTS

The most general set of equations of our model comprises nonlinear, coupled equations and may not be easy to solve. So we have made an approximation, known as hedgehog ansatz, and reduced the number of equations to be solved, but having all non-Abelian and nonlinear features. From this simplified set of equations, we may get the results of all other earlier works in this field. For example, in Fig. 1, we plotted the Poincare section of our model with the approximation that the dynamical color charges are constant and $Z = 0$, Eq. (18). Figures 1(a) and 1(b) are for the system without plasma nonlinearity and show that the regular orbits seen in Fig. 1(a) for $\epsilon = 5$ disappear at the critical value of $\epsilon = 2$, Fig. 1(b), and hence are chaotic. Similar figures with plasma nonlinearity show changes from ordered orbit islands for $\epsilon = 8.15$, Fig. 1(c), into chaotic motion for $\epsilon = 6$. Therefore, the critical value of ε for the order-to-chaos transition is higher with plasma nonlinearity. The chaos seen with $\epsilon = 6.0$ with plasma nonlinearity develops islands of ordered motion without plasma nonlinearity and we need smaller ε ($\epsilon = 2$) to have chaos. Hence the plasma nonlinearity enhances the chaos, which is an additional new feature compared to the results of Blaizot and Iancu [3]. Another special solution of our model with I_a = constant is $X = Y = Z$, Eq. (21), which is not an elliptic function as in Ref. [3], but little more general nonlinear oscillation.

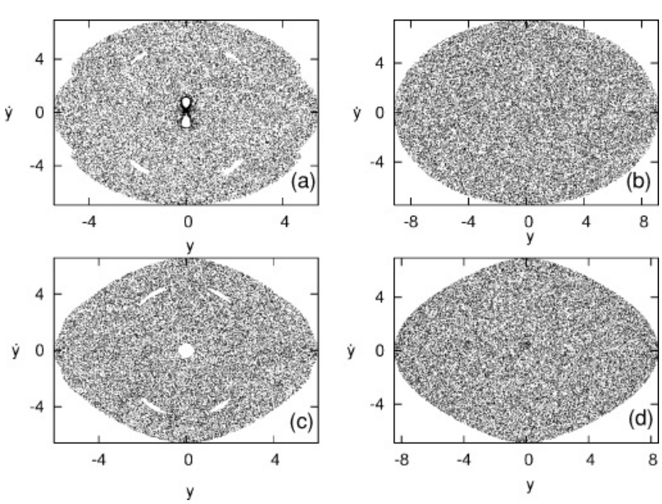

FIG. 1. Poincare sections of our model (with $Z = 0$ and I_a = constant) without plasma nonlinearity [(a) $\epsilon = 5$ and (b) $\epsilon = 2$)] and with plasma nonlinearity [(c) $\epsilon = 8.15$ and (d) $\epsilon = 6$)].

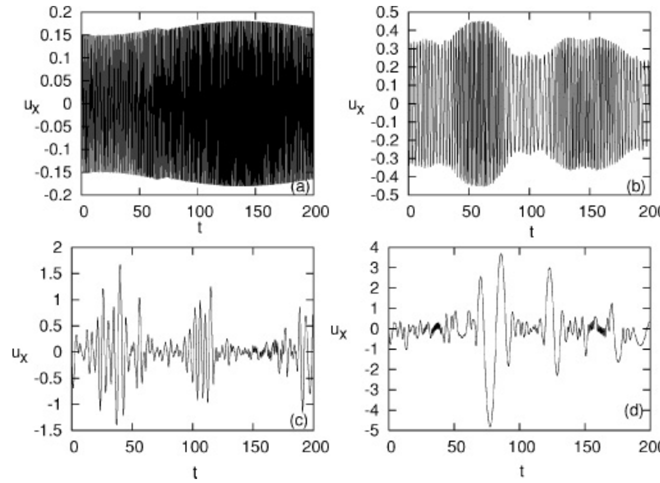

FIG. 2. Exact numerical solutions of our model, as an example u_x, for different values of ε with the same initial conditions [(a) $\epsilon = 100$, (b) $\epsilon = 20$, (c) $\epsilon = 2$, and (d) $\epsilon = 0$)].

Next, in Fig. 2, we plotted the general solutions of our model, for example, u_x [Eq. (8)], with hedgehog ansatz for different values of ε with the same initial conditions and we see that as the ε decreases the system becomes more and more chaotic, which is, qualitatively, similar to the results of Gupta, Kaw, and Parikh [5]. For a large ε, say, $\epsilon = 100$ [Fig. 1(a)], the amplitude of oscillations is small and the YM nonlinearity and plasma nonlinearity may be negligible and hence it is just the Abelian oscillations, modulated by color dynamics. As ε decreases, amplitude increases and all nonlinearities due to YM fields, color dynamics, and plasma nonlinearity come into play and drive the system to chaotic motion as can be seen from Figs. 2(b) and 2(c) with intermittent oscillations. For $\epsilon = 0$, Fig. 2(d), the chaotic oscillations are mainly due to YM nonlinearity. Similar behavior is also seen in the other components of velocity.

IV. CONCLUSIONS

We have studied fully nonlinear, non-Abelian excitations of quark-antiquark plasma using relativistic fluid theory. It exhibits new features like a special nonlinear oscillation, different from elliptic functions, and enhancement of chaos. Further, we have found that by neglecting color dynamics and plasma nonlinearity we get back the results of Blaizot and Iancu [3] and by neglecting YM field nonlinearity and plasma nonlinearity, we obtain the results of Gupta, Kaw, and Parikh [5]. Hence, we have the most general nonlinear, non-Abelian modes of QGP. In general, all three nonlinearities are always there in the system. For small amplitude excitations ($|X|$, $|Y|$, and $|Z|$ are $\ll 1$), YM field nonlinearity and plasma nonlinearity may be negligible like in Ref. [5]. At present, we don't know the appropriate limit to get the results of Ref. [3] where the role of Wong's color dynamics is not explicit. It is based on quantum kinetic theory [7] whereas our model is based on classical fluid theory which may be derived from classical kinetic theory [9]. By neglecting all non-Abelian nonlinearities our model resembles the model of laser propagation through relativistic plasma [11].

APPENDIX

The set of fluid equations, Eqs. (1), (3) and (6) may be also derived from the kinetic theory [10]. Following Kelly *et al.* [9], the Boltzmann equation is

$$p^\mu \left[\frac{\partial}{\partial x^\mu} - g Q_a G^a_{\mu\nu} \frac{\partial}{\partial p_\nu} - g f_{abc} A_{\mu b} Q_c \frac{\partial}{\partial Q_a} \right] f(x, p, Q) = 0, \tag{A1}$$

where x^μ, p^μ are the four-vector coordinates, and momenta and Q_a are the color charges. $f(x, p, Q)$ is the distribution function and f_{abc} are the structure constants of the group. On taking the lowest moment of the above equation, i.e., just integrate the above equation, Eq. (A1), with respect to momenta and color charges, we get the continuity equation

$$\partial_\mu (n P^\mu) = 0, \tag{A2}$$

where the fluid density, n, is defined as

$$n \equiv \int dp\, dQ f(x, p, Q), \tag{A3}$$

and the fluid four-momentum, P^μ, is defined through the relation

$$n P^\mu \equiv \int dp\, dQ p^\mu f(x, p, Q), \tag{A4}$$

where dp and dQ are the volume elements of momentum and color space, respectively. It reduces to Eq. (6) for $P^\mu = m u^\mu$, where m is the rest mass and u^μ is the fluid 4-velocity. Taking the next moment of the Boltzmann equation with respect to p^σ, we get

$$n P^\mu \partial_\mu P^\sigma = g G^{\mu\sigma}_a I_{\mu a} + \partial_\mu \tilde{T}^{\mu\sigma}, \tag{A5}$$

where $\tilde{T}^{\mu\sigma}$ is the pressure tensor term. I^μ_a is defined as

$$I^\mu_a \equiv \int dp\, dQ p^\mu Q_a f(x, p, Q), \tag{A6}$$

which may be used to define the fluid dynamical color charge, through the relation

$$n P^\mu I_a \equiv I^\mu_a, \tag{A7}$$

just like the definition of P^μ. In cold plasma limit we neglect the pressure tensor term and Eq. (A5) reduces to Eq. (1) using the continuity equation and the relation $\frac{d}{d\tau} = u^\mu \partial_\mu$. Next, taking the moments with respect to Q_a, we get

$$\partial_\mu I^\mu_a = -g f_{abc} A^\mu_b I_{\mu c}, \tag{A8}$$

which on using the definition of fluid color charge, Eq. (A7), we get the Wong's equation

$$P^\mu \partial_\mu I_a = -g f_{abc} A^\mu_b P_\mu I_c. \tag{A9}$$

All fluid variables, n, P^μ, and I_a, are functions of x^μ, four-vector coordinates. Thus we get all fluid equations, Eqs. (1), (3) and (6) from the kinetic theory by taking various moments and using the definition of n, P^μ, and I_a as given by Eqs. (A3), (A4), and (A7) and using the relation $P^\mu = m u^\mu$. The current density may be defined as

$$J^\mu_a \equiv g \sum_{\text{species}} \int dp\, dQ \frac{p^\mu}{m} Q_a f(x, p, Q) = g \sum_{\text{species}} n u^\mu I_a, \tag{A10}$$

using the definition of fluid variables, and leads to Eq. (5).

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