Bohr Hamiltonian with different mass coefficients for the ground- and γ bands from experimental data

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Based on experimental data for axially symmetric well deformed nuclei it is shown that the "Grodzins product" of the energy of the 2^+_γ state with the $B(E2; 2^+_\gamma \to 0^+_{\text{g.s.}})$ for the transition from the 2^+_γ state to the ground state is a relatively smooth function of Z and A. It is shown also that a ratio of the mass coefficients for the γ-motion and for the rotational ground band extracted from the experimental data take the values in the limits 3–5. The implied large difference between the two mass coefficients means that the mass tensor of the Bohr Hamiltonian cannot be reduced to a scalar as it usually assumed.

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I. INTRODUCTION

In the description of collective nuclear dynamics the mass coefficient plays as important role as the potential energy. However, much less is known about the mass coefficient in comparison with the numerous calculations of the potential energy. Frequently, it is assumed that the mass coefficient is a constant and in the case of the well deformed nuclei the same mass coefficient is used for description of the rotational motion of the ground band and of the γ-vibrations. However, an information about the mass coefficient can be extracted from the experimental data on excitation energies and electric quadrupole transition probabilities. In the framework of the Bohr-Mottelson model and if the well-deformed axially symmetric nuclei are considered the two "Grodzins products" $E(2^+_1)B(E2; 2^+_1 \to 0^+_{\text{g.s.}})$ and $E(2^+_\gamma)B(E2; 2^+_\gamma \to 0^+_{\text{g.s.}})$ are inversely proportional to the corresponding mass coefficients. The same is true for the first 2^+ state in the vibrational limit.

In a previous paper [1] we have used the experimental values of the "Grodzins products" to obtain information about mass coefficients for the γ- and rotational motion. Our analysis has been based on the Bohr Hamiltonian [2] admitting the use of different values of the mass coefficient for different bands of eigenstates. In fact different Hamiltonians were used for a description of different rotational bands. It was found that the mass coefficient for the γ-motion is significantly larger than that for the rotational motion. However, it was assumed that the mass coefficient is a scalar function of the shape variables.

In this paper we will use one Hamiltonian for all bands. This Hamiltonian in the intrinsic frame is derived from a Hamiltonian in the laboratory frame of the Gneuss-Greiner type [3–5] which has a more general kinetic energy term than is usually assumed (however, see [3–5]). In this respect our present analysis is different from the analysis presented in [1]. We will furthermore compare with a larger set of data including all data on axially symmetric and beta-rigid nuclei in the rare earth and actinide regions.

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II. COLLECTIVE HAMILTONIAN

To analyze the experimental data we use the Bohr Hamiltonian with a kinetic energy term having a general form. In the laboratory frame this term looks as

$$\hat{T} = -\frac{\hbar^2}{2} \sum_{L=0,2,4} \sum_{M,\mu,\mu'} \sqrt{(5)} C^{LM}_{2\mu 2\mu'} \frac{\partial}{\partial \alpha_{2\mu}} (B^{-1})^{\text{lab}}_{LM} \frac{\partial}{\partial \alpha_{2\mu'}}, \tag{1}$$

where $(B^{-1})^{\text{lab}}_L$ is an inverted mass tensor. The usual form of $\hat{T}$ takes only a scalar component ($L = 0$) into account and it assumes that $(B^{-1})^{\text{lab}}_{00}$ is independent on β and γ. Then the kinetic energy term Eq. (1) is approximated by the expression

$$\hat{T} = -\frac{\hbar^2}{2B} \sum_{\mu} (-1)^{\mu} \frac{\partial}{\partial \alpha_{2\mu}} \frac{\partial}{\partial \alpha_{2-\mu}}. \tag{2}$$

In this paper we use, however the more general form of the kinetic energy given by Eq. (1). In our previous paper [1] we used only the scalar part of the mass tensor but assumed that it depends on the shape variables β and γ and being averaged over the wave functions of the 2^+_1 and 2^+_γ states it takes different values for the ground and gamma band. Below we use an alternative approach, namely, we start in Eq. (1) with a general expression for the mass tensor. Then using the expression Eq. (A2) for the derivative $\partial/\partial\alpha_{2\mu}$ and the representation Eq. (A3) for the components of the mass tensor in terms of the intrinsic variables and the Euler angles we obtain the kinetic energy term of the Bohr Hamiltonian with different mass coefficients for different terms. Following our main assumption of small amplitudes of the β- and γ-oscillations around $\beta = \beta_0$ and $\gamma = 0$ it follows that B_β, B_γ, and B_{rot} are constants and we obtain after lengthy but straightforward calculations

$$\hat{T} = \hat{T}_\beta - \frac{\hbar^2}{2B_\gamma \beta_0^2} \frac{1}{\gamma} \frac{\partial}{\partial \gamma} \gamma \frac{\partial}{\partial \gamma} + \frac{\hbar^2}{6B_{\text{rot}}\beta_0^2} \left(\hat{L}_1^2 + \hat{L}_2^2\right) + \frac{\hbar^2}{2B_\gamma} \frac{\hat{L}_3^2}{4\beta_0^2 \gamma^2}, \tag{3}$$

where $\hat{L}_i$ is the projection of the angular momentum on the intrinsic axis i and $3B_{\rm rot}\beta_0^2$ is the moment of inertia. The detailed form of the kinetic energy term of β vibrations is unnecessary for the discussion below but we give it for completeness

$$\hat{T}_\beta = -\frac{\hbar^2}{2}\left(\frac{1}{B_\beta}\frac{\partial^2}{\partial\beta^2} + \frac{2}{B_\gamma}\frac{1}{\beta}\frac{\partial}{\partial\beta} + \frac{2}{B_{\rm rot}}\frac{1}{\beta}\frac{\partial}{\partial\beta}\right). \quad (4)$$

It is interesting to note that the mass coefficient in the fourth term of Eq. (3) coincides with that of the second term and differs from the mass coefficients in the third term. This result was obtained before in [6], however, for the less general kinetic energy term with a cubic anharmonicity only. The reason for the coincidence of the mass coefficient in the second and fourth terms of Eq. (3) is related to the following. This is seen from the expression for the derivative $\partial/\partial\alpha_{2\mu}$ shown in Eq. (A2). The term producing the rotation around axis 3 changes K by two units as is also the case for the term producing γ-vibrations. In contrast, rotations around the first and second axes change K by one unit only.

The full Hamiltonian

$$H = T + U \quad (5)$$

is obtained by adding the kinetic energy term Eq. (3) and the potential U which in the case of the well deformed axially symmetric nuclei takes the form

$$U = \frac{1}{2}C_\beta(\beta-\beta_0)^2 + \frac{1}{2}\frac{C_\gamma}{\beta_0^2}\gamma^2. \quad (6)$$

Since in these nuclei the γ-motion is separated from rotation the second and the fourth terms in Eq. (3) together with the corresponding potential energy term form a sub-Hamiltonian describing the γ-vibrations. We stress once more that the mass coefficients B_β, B_γ, and $B_{\rm rot}$ and the parameters C_β and C_γ can be taken as constants in the case of the well deformed nuclei. This is an advantage of using the intrinsic frame [see Eq. (A2)]. The expressions for B_β, B_γ, and $B_{\rm rot}$ are given in Appendix A.

The normalized eigenfunctions of the states of the ground and γ bands of the Bohr Hamiltonian Eq. (5) are

$$\Psi_{gs}(IM) = \sqrt{\frac{2I+1}{8\pi^2}}D^I_{M0}\psi_0(\beta)\left(\frac{2\sqrt{B_\gamma C_\gamma}}{\hbar}\right)^{1/2} \times \exp\left(-\frac{\sqrt{B_\gamma C_\gamma}}{2\hbar}\gamma^2\right), \quad (7)$$

$$\Psi_\gamma(IM) = \sqrt{\frac{2I+1}{16\pi^2}}\left(D^I_{M2}+D^I_{M-2}\right)\psi_0(\beta)\sqrt{\frac{2B_\gamma C_\gamma}{\hbar^2}}\,\gamma \times \exp\left(-\frac{\sqrt{B_\gamma C_\gamma}}{2\hbar}\gamma^2\right). \quad (8)$$

Here $\psi_0(\beta)$ describes β oscillations around $\beta=\beta_0$ and its detailed knowledge is not needed here. The excitation energy of the 2_1^+ state is equal to

$$E(2_1^+) = \frac{\hbar^2}{B_{\rm rot}\beta_0^2}. \quad (9)$$

The excitation energy of the 2_γ^+ state is

$$E(2_\gamma^+) = \frac{1}{\beta_0^2}\hbar\sqrt{\frac{C_\gamma}{B_\gamma}} + \frac{\hbar^2}{3B_{\rm rot}\beta_0^2}. \quad (10)$$

Taking the general quadrupole transition operator given by Bohr and Mottelson [7] and following our main assumption that the amplitudes of the β- and γ-oscillations around average values are small we obtain the following approximate expression:

$$Q_{2\mu} = q\left(D^2_{\mu0}\beta_0 + D^2_{\mu0}(\beta-\beta_0)\right) + \frac{q}{\sqrt{2}}\left(D^2_{\mu2}+D^2_{\mu-2}\right)\beta_0\,\gamma. \quad (11)$$

Using the wave functions Eqs. (7) and (8) we get the relations

$$E(2_1^+)B(E2;0^+_{\rm g.s.}\to 2_1^+) = \frac{\hbar^2 q^2}{B_{\rm rot}}, \quad (12)$$

$$E(2_\gamma^+)B(E2;0^+_{\rm g.s.}\to 2_\gamma^+) = \frac{\hbar^2 q^2}{B_\gamma}, \quad (13)$$

$$\frac{E(2_\gamma^+)B(E2;0^+_{\rm g.s.}\to 2_\gamma^+)}{E(2_1^+)B(E2;0^+_{\rm g.s.}\to 2_1^+)} = \frac{B_{\rm rot}}{B_\gamma}, \quad (14)$$

where we have neglected a small contribution [second term in Eq. (11) of the rotational energy term of the Hamiltonian in the energy of the 2_γ^+ state]. The expressions of Eqs. (10) and (14) are different from those obtained in [1] because of the different forms of the Hamiltonians used.

The relations Eqs. (12) and (13) can be derived also using a double commutator of the corresponding transition operator with the Hamiltonian as in a derivation of the energy weighted sum rules [8]. The electric quadrupole transition operator Eq. (11) consists of the three terms

$$Q_{2\mu} = Q^{\rm rot}_{2\mu} + Q^\beta_{2\mu} + Q^\gamma_{2\mu}, \quad (15)$$

where

$$Q^{\rm rot}_{2\mu} = q\beta_0 D^2_{\mu0},$$
$$Q^\beta_{2\mu} = q(\beta-\beta_0)D^2_{\mu0}, \quad (16)$$
$$Q^\gamma_{2\mu} = q\beta_0\frac{1}{\sqrt{2}}\left(D^2_{\mu2}+D^2_{\mu-2}\right)\gamma.$$

The operator $Q^{\rm rot}_{2\mu}$ produces transitions only inside the rotational bands, $Q^\beta_{2\mu}$ excites only the states of the β-band acting on the states of the ground state band, $Q^\gamma_{2\mu}$ excites only the states of the γ-band acting on the states of the ground state band. Let us derive the relation Eq. (13). A derivation of the relation Eq. (12) can be done in the same way. Using the expression Eq. (3) for the kinetic energy term and the expression Eq. (16) for $Q^\gamma_{2\mu}$ we obtain (see also [9])

$$\sum_{\mu,\mu'} C^{00}_{2\mu2\mu'}\left[\left[H, Q^\gamma_{2\mu}\right], Q^\gamma_{2\mu'}\right] = -\frac{2}{\sqrt{5}}\frac{\hbar^2 q^2}{B_\gamma} - \frac{2}{\sqrt{5}}\frac{\hbar^2 q^2}{3B_{\rm rot}}\gamma^2, \quad (17)$$

where both the γ-vibrational and rotational terms in H contribute to the right hand side of Eq. (17). Averaging this

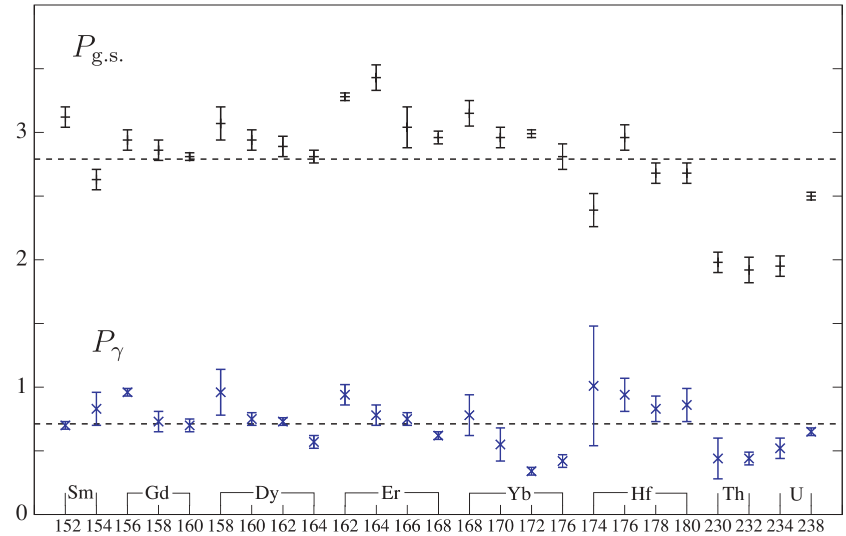

FIG. 1. (Color online) Evolution of the values $P_{g.s.} = E(2_1^+)\, B(E2; 0_1^+ \rightarrow 2_1^+)\, Z^{-2}\, A^{2/3}$ and $P_\gamma = E(2_\gamma^+)\, B(E2; 0_1^+ \rightarrow 2_\gamma^+)\, Z^{-2}\, A^{2/3}$ given in the units keV $e^2\, b^2$ over a range of isotopes. The average values $\langle P_g \rangle = 2.790$ and $\langle P_\gamma \rangle = 0.712$ over all considered isotopes are plotted as dotted lines. The experimental data are taken from [16].

expression over the ground state wave function we get

$$E(2_\gamma^+)B(E2; 0_{g.s.}^+ \rightarrow 2_\gamma^+) = \frac{\hbar^2 q^2}{B_\gamma} + \frac{\hbar^2 q^2}{B_{rot}} \langle 0_{g.s.}^+ | \gamma^2 | 0_{g.s.}^+ \rangle. \quad (18)$$

Neglecting by a small term proportional to $\langle 0_{g.s.}^+ | \gamma^2 | 0_{g.s.}^+ \rangle$ we obtain the relation Eq. (13).

III. THE "GRODZINS PRODUCTS" AND THE MASS COEFFICIENTS

Equations (12) and (13) indicate that the "Grodzins products" for the ground band and the gamma band are inversely proportional to the mass coefficients which in hydrodynamic approach are slowly varying functions of the mass number A and the charge number Z. This is well known for the "Grodzins product" for the ground band. For the ground band an empirical dependence of the product proportional to $Z^2 A^{-2/3}$ was established by Raman [10]. In the following we put this dependence in the definition of a "modified Grodzins products" P_g and P_γ:

$$P_g = E(2_1^+)B(E2; 0_{g.s.}^+ \rightarrow 2_1^+) \cdot Z^{-2} A^{2/3}, \quad (19)$$

$$P_\gamma = E(2_\gamma^+)B(E2; 0_{g.s.}^+ \rightarrow 2_\gamma^+) \cdot Z^{-2} A^{2/3}. \quad (20)$$

The experimental data for the products Eqs. (19) and (20) are given in Fig. 1 and in Table I. Let us consider at first the results presented in Fig. 1. It is seen that the value of the product of the excitation energy and $B(E2)$ for the γ-band demonstrates a smooth dependence on the charge Z and mass A numbers as the value for the 2_1^+ state although the relative fluctuations are larger. It is surprising that the modified product $E(2_\gamma^+)B(E2; 0_{g.s.}^+ \rightarrow 2_\gamma^+) \cdot Z^{-2} \cdot A^{2/3}$ is approximately a constant. In this modified product for the 2_γ^+ state we use the same dependence on Z and A as was suggested in [10] for the 2_1^+ state. This constancy of the considered product is well known for the 2_1^+ state and was considered by Grodzins [11] and Raman [10]. However, for the 2_γ^+ [12] state there are much less data and the "Grodzins product" was not considered (see, however, [1]). The observed larger fluctuations of the product corresponding to the 2_γ^+ state as compared to the 2_1^+ state can be explained by a smaller collectivity of the 2_γ^+ state. The relation Eq. (14) can be used to obtain the ratio B_γ/B_{rot} from the experimental data on the energies and the $E2$ transition probabilities.

TABLE I. Experimental values of $P_{g.s.} = E(2_1^+)\, B(E2; 0_{g.s.}^+ \rightarrow 2_1^+)\, Z^{-2}\, A^{2/3}$ and $P_\gamma = E(2_\gamma^+)\, B(E2; 0_{g.s.}^+ \rightarrow 2_\gamma^+)\, Z^{-2}\, A^{2/3}$ given in the units keV $e^2\, b^2$ and the values of the ratio B_γ/B_{ror} calculated using the Eq. (14). The experimental data are taken from [16].

Nucleus	$P_{g.s.}$	P_γ	B_γ/B_{rot}
^{152}Sm	3.12(8)	0.70(3)	4.46
^{154}Sm	2.63(8)	0.83(13)	3.17
^{156}Gd	2.94(8)	0.96(3)	3.06
^{158}Gd	2.86(8)	0.73(8)	3.92
^{160}Gd	2.81(3)	0.70(5)	4.01
^{158}Dy	3.07(13)	0.96(18)	3.2
^{160}Dy	2.94(8)	0.75(5)	3.92
^{162}Dy	2.89(8)	0.73(3)	3.96
^{164}Dy	2.81(5)	0.57(5)	4.93
^{162}Er	3.28(3)	0.94(8)	3.49
^{164}Er	3.43(10)	0.78(8)	4.40
^{166}Er	3.04(16)	0.75(5)	4.05
^{168}Er	2.96(5)	0.62(3)	4.77
^{168}Yb	3.15(10)	0.78(16)	4.04
^{170}Yb	2.96(8)	0.55(13)	5.38
^{172}Yb	2.99(3)	0.34(3)	8.79
^{176}Yb	2.81(10)	0.42(5)	6.69
^{174}Hf	2.39(13)	1.01(47)	2.37
^{176}Hf	2.96(10)	0.94(13)	3.15
^{178}Hf	2.68(8)	0.83(10)	3.23
^{180}Hf	2.68(8)	0.86(13)	3.12
^{230}Th	1.98(8)	0.44(16)	4.5
^{232}Th	1.92(10)	0.44(5)	4.36
^{234}U	1.95(8)	0.52(8)	3.75
^{238}U	2.50(3)	0.65(3)	3.85

The large deviation of the ratio $B_\gamma/B_{\rm rot}$ from unity was already discussed in our previous paper [1], however, a smaller set of data has been used there and also a different form of the kinetic energy term of the total Hamiltonian has been used. As a consequence the values of the ratio $B_\gamma/B_{\rm rot}$ presented in Table I are different from those given in [1].

To get some explanation of the fact that the ratio $B_\gamma/B_{\rm rot}$ deviate so strongly from unity we have estimated the value of this ratio based on the cranking model expressions for the mass coefficients. We used a technique suggested by Migdal in his paper on the nuclear moment of inertia [13]. Details of calculations are given in Appendix B. We have obtained the value 4.3 for the average ratio $B_\gamma/B_{\rm rot}$ which agrees with the values presented in Fig. 1 and Table I. The main sources of a deviation of $B_\gamma/B_{\rm rot}$ from unity are the rather large value of the frequency of γ-vibrations and the difference in the characteristic energies of the most important transitions produced by the operators j_x and $\partial/\partial a_{22}$.

IV. SUMMARY

Considering the experimental data for the energies and the $E2$ reduced transition probabilities for the 2_1^+ and 2_γ^+ states in the well deformed axially symmetric nuclei we have shown that the mass coefficients for the rotational and γ-vibrational motion are significantly different. A simplified estimate of the ratio $B_\gamma/B_{\rm rot}$ obtained using the cranking model expression for the mass coefficients is in agreement with the experimental data. Thus, even for nuclei in the middle of the rare earth region which are axially symmetric and have small fluctuations in β and γ the usual assumption of a single constant mass coefficient does not work. These nuclei can be described correctly, however, with a form of Bohr Hamiltonian with different but constant values for mass coefficients for the ground and γ bands [see Eqs. (3) and (4)]. An analysis of the kinetic energy term for the Bohr Hamiltonian given in the intrinsic frame shows that an inclusion into consideration of not only scalar but also other components of the mass tensor can explain the difference in the values of the mass coefficients for the γ-vibrations and the ground state rotations.

The description of the β-band must be still considered as an open problem in this respect. It is shown on the basis of the experimental data that the product $E(2_\gamma^+)B(E2;0_{\rm g.s.}^+\to 2_\gamma^+)\cdot Z^{-2}A^{2/3}$ which is the analog of the Grodzins product for the first 2^+ state is nearly a constant as it was found for the 2_1^+ state. However, the existing set of experimental data for the 2_γ^+ state is much smaller than for the 2_1^+ state. By using modern equipment like radioactive beams and or advanced spectrometers experiments should be feasible which would extend our knowledge about a Grodzins product for the γ-band from 20 to about 40 nuclei in the rare earth and from 4 to about 20 nuclei in the actinide regions.

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APPENDIX A

In order to obtain the expressions for the mass coefficients corresponding to β, γ and rotational motion we should transform the kinetic energy term Eq. (1) into the intrinsic frame. In terms of β, γ and the Euler angles the collective coordinate $\alpha_{2\mu}$ is expressed as

$$\alpha_{2\mu} = \beta\left(\left(D^2_{\mu 0}\right)^*\cos\gamma + \frac{1}{\sqrt{2}}\left(\left(D^2_{\mu 2}\right)^* + \left(D^2_{\mu -2}\right)^*\right)\sin\gamma\right). \tag{A1}$$

Using several relations from [6,14] we obtain after lengthy but straightforward calculations the following expressions for $\partial/\partial\alpha_{2\mu}$ in terms of the intrinsic variables:

$$\begin{aligned}\frac{\partial}{\partial\alpha_{2\mu}} =& \left(D^2_{\mu 0}\cos\gamma + \frac{1}{\sqrt{2}}\left(D^2_{\mu 2}+D^2_{\mu -2}\right)\sin\gamma\right)\frac{\partial}{\partial\beta}\\ &+\frac{1}{\beta}\left(-D^2_{\mu 0}\sin\gamma + \frac{1}{\sqrt{2}}\left(D^2_{\mu 2}+D^2_{\mu -2}\right)\cos\gamma\right)\frac{\partial}{\partial\gamma}\\ &+\frac{1}{2\beta\sin\left(\gamma+\frac{\pi}{3}\right)}\frac{1}{\sqrt{2}}\left(D^2_{\mu 1}+D^2_{\mu -1}\right)\hat{L}_1\\ &+\frac{\imath}{2\beta\sin\left(\gamma-\frac{\pi}{3}\right)}\frac{1}{\sqrt{2}}\left(D^2_{\mu 1}-D^2_{\mu -1}\right)\hat{L}_2\\ &+\frac{1}{2\beta\sin\gamma}\frac{1}{\sqrt{2}}\left(D^2_{\mu 2}-D^2_{\mu -2}\right)\hat{L}_3,\end{aligned} \tag{A2}$$

where $\hat{L}_i$ is the projection of the angular momentum on the intrinsic axis i. Substituting Eq. (A2) into Eq. (1) and using the following representation for the components of the mass tensor:

$$\begin{aligned}(B^{-1})^{\rm lab}_{LM} =& D^L_{M0}(B^{-1})^{\rm int}_{L0} + \frac{1}{\sqrt{2}}\left(D^L_{M2}+D^L_{M-2}\right)(B^{-1})^{\rm int}_{L2}\\ &+\frac{1}{\sqrt{2}}\left(D^L_{M4}+D^L_{M-4}\right)(B^{-1})^{\rm int}_{L4},\end{aligned} \tag{A3}$$

where $(B^{-1})^{\rm int}_{LK}$ in general case depends on β and γ, we obtain an expression for the kinetic energy term which is more complicated than the one with a constant mass coefficient and contains in addition some terms linear in derivatives $\partial/\partial\beta$ and $\partial/\partial\gamma$. Taking the coefficients at the second order derivative $\partial^2/\partial\gamma^2$ and at $\hat{L}_1^2, \hat{L}_2^2, \hat{L}_3^2$ we obtain the expressions for the mass coefficients corresponding to γ-motion and rotations. For the case of an axial symmetry, when we put $\gamma = 0$, they are

$$\frac{1}{B_\beta} = (B^{-1})^{\rm int}_{00} - \sqrt{\frac{10}{7}}(B^{-1})^{\rm int}_{20} + 3\sqrt{\frac{2}{7}}(B^{-1})^{\rm int}_{40}, \tag{A4}$$

$$\frac{1}{B_\gamma} = (B^{-1})^{\rm int}_{00} + \sqrt{\frac{10}{7}}(B^{-1})^{\rm int}_{20} + \frac{1}{2}\sqrt{\frac{2}{7}}(B^{-1})^{\rm int}_{40}, \tag{A5}$$

$$\frac{1}{B_{\rm rot}} = (B^{-1})^{\rm int}_{00} - \frac{1}{2}\sqrt{\frac{10}{7}}(B^{-1})^{\rm int}_{20} - 2\sqrt{\frac{2}{7}}(B^{-1})^{\rm int}_{40}. \tag{A6}$$

If only a scalar $(B^{-1})^{\rm int}_{00}$ is taken into account then all three mass coefficients coincide.

APPENDIX B

The cranking model expressions for the moment of inertia $\Im$ and for the mass coefficient of the γ-motion corrected by taking into account a finite value of ω_γ are

$$\Im = 2\hbar^2 \sum_k \frac{|\langle k|[H, j_x]|gs\rangle|^2}{(E_k - E_{\rm g.s.})^3}, \tag{B1}$$

$$B_\gamma = 2\hbar^2 \sum_k \frac{|\langle k|[H, \partial/\partial a_{22}]|{\rm g.s.}\rangle|^2 (E_k - E_{\rm g.s.})}{\left((E_k - E_{\rm g.s.})^2 - \hbar^2\omega_\gamma^2\right)^2}, \tag{B2}$$

where $|k\rangle$ is an excited state, $(E_k - E_{\rm g.s.})$ is the excitation energy of this state and a_{22} is a collective variable describing a deviation from axial symmetry. Approximating H by a single quasiparticle Hamiltonian

$$H = H_{\rm spher} - \beta\hbar\omega_0 r^2 Y_{20} - a_{22}\hbar\omega_0 r^2 \frac{1}{\sqrt{2}}(Y_{22} + Y_{2-2}) - \Delta(A^+_{00} + A_{00}), \tag{B3}$$

where $H_{\rm spher}$ is the spherically symmetric shell model Hamiltonian, Δ is the energy gap and $A^+_{00}(A_{00})$ is the nucleon pair creation (annihilation) operator, we obtain

$$[H, j_x] = \beta\hbar\omega_0\sqrt{3} r^2 \frac{1}{\sqrt{2}}(Y_{21} + Y_{2-1}), \tag{B4}$$

$$[H, \partial/\partial a_{22}] = \hbar\omega_0 r^2 \frac{1}{\sqrt{2}}(Y_{22} + Y_{2-2}). \tag{B5}$$

Substituting Eqs. (B4) and (B5) into Eqs. (B1) and (B2) and using a hydrodynamic expression for the moment of inertia $\Im = 3B_{\rm rot}\beta^2$ we get

$$B_{\rm rot} = 2\hbar^2(\hbar\omega_0)^2 \sum_k \frac{|\langle k|\frac{1}{\sqrt{2}}(q_{21} + q_{2-1})|gs\rangle|^2}{(E_k - E_{\rm g.s.})^3}, \tag{B6}$$

$$B_\gamma = 2\hbar^2(\hbar\omega_0)^2 \sum_k \frac{|\langle k|\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})|{\rm g.s.}\rangle|^2 (E_k - E_{\rm g.s.})}{\left((E_k - E_{\rm g.s.})^2 - \hbar^2\omega_\gamma^2\right)^2}. \tag{B7}$$

Let $|k\rangle$ be a two-quasiparticle state $|st\rangle$ where s and t are quantum numbers of the single quasiparticle states. Following the procedure outlined in [15] (p. 14) we obtain

$$\langle st|\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})|gs\rangle = \langle s|\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})|t\rangle (u_s v_t + (-1)^\mu v_s u_t), \tag{B8}$$

where u_s, v_s are coefficients of the u, v Bogoliubov transformation. Substituting Eq. (B8) and the expressions for u_s, v_s into Eqs. (B6) and (B7) we obtain

$$B_{\rm rot} = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \frac{\left|\langle s|\frac{1}{\sqrt{2}}(q_{21} + q_{2-1})|t\rangle\right|^2 (E_s E_t - (\epsilon_s - \lambda)(\epsilon_t - \lambda) - \Delta^2)}{2E_s E_t (E_s + E_t)^3}, \tag{B9}$$

$$B_\gamma = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \frac{\left|\langle s|\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})|t\rangle\right|^2 (E_s E_t - (\epsilon_s - \lambda)(\epsilon_t - \lambda) + \Delta^2)(E_s + E_t)}{2E_s E_t \left((E_s + E_t)^2 - \hbar^2\omega_\gamma^2\right)^2}, \tag{B10}$$

where E_s is a single-quasiparticle energy and $(\epsilon_s - \lambda)$ is a single–particle energy. Let us introduce into consideration the following function [13]:

$$\mathcal{L}^{(\pm)}_\omega(\epsilon_s - \lambda, \epsilon_t - \lambda) = \frac{(E_s + E_t)(E_s E_t - (\epsilon_s - \lambda)(\epsilon_t - \lambda) \pm \Delta^2)}{2E_s E_t((E_s + E_t)^2 - \hbar^2\omega^2)^2}. \tag{B11}$$

With this function we have

$$B_{\rm rot} = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \left|\langle s|\frac{1}{\sqrt{2}}(q_{21} + q_{2-1})|t\rangle\right|^2 \times \mathcal{L}^{(-)}_{\omega=0}(\epsilon_s - \lambda, \epsilon_t - \lambda), \tag{B12}$$

$$B_\gamma = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \left|\langle s|\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})|t\rangle\right|^2 \times \mathcal{L}^{(+)}_{\omega_\gamma}(\epsilon_s - \lambda, \epsilon_t - \lambda). \tag{B13}$$

At fixed $\epsilon_s - \epsilon_t = d$ $\mathcal{L}^{(\pm)}_\omega(\epsilon_s - \lambda, \epsilon_t - \lambda)$ has a maximum when $(\epsilon_s - \lambda) + (\epsilon_t - \lambda) = 0$ of a width of an order of Δ. This restrict the values of ϵ_s and ϵ_t contributing to the sum. It is assumed below that there is enough single-particle states satisfying this condition. For $\mathcal{L}^{(-)}$ the value $d = 0$ should be excluded. The most important single-particle transitions contributing to $B_{\rm rot}$ have $\epsilon_s - \epsilon_t = \hbar(\omega_2 - \omega_3) = \delta\hbar\omega_0$ [7] corresponding to a shift of the oscillator quantum number from a direction of the third to the direction of the second axis because $\Delta K = 1$ and

$$\frac{1}{\sqrt{2}}(q_{21} + q_{2-1}) \sim \left(\left(b^+_z b_y + b^+_y b_z\right) + \left(b^+_z b^+_y + b_y b_z\right)\right). \tag{B14}$$

The single-particle operator $\frac{1}{\sqrt{2}}(q_{22} + q_{2-2})$ has the following structure in terms of oscillator boson creation and annihilation operators:

$$\frac{1}{\sqrt{2}}(q_{22}+q_{2-2}) \sim \left(\left(b_x^+ b_x - b_y^+ b_y\right) + \tfrac{1}{2}\left(b_x^+ b_x^+ - b_y^+ b_y^+ + b_x b_x - b_y b_y\right)\right). \quad \text{(B15)}$$

Therefore the most important single-particle transitions contributing to B_γ are related to the first term in Eq. (B15) and have $(\epsilon_s - \epsilon_t) \approx 0$. As it is seen from Eq. (B14) the large matrix elements of $\frac{1}{\sqrt{2}}(q_{21}+q_{2-1})$ also connect states with $(\epsilon_s - \epsilon_t) = 2\hbar\omega_0$ but in this case large energy denominators significantly decrease a contribution of these matrix elements. In that part of the single-particle space which is restricted by the maxima of the functions $\mathcal{L}^{(\pm)}$ we can fix a value of ϵ_t in the argument of $\mathcal{L}^{(\pm)}$.

Using these facts we can write

$$B_{\rm rot} = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \left| \langle s| \frac{1}{\sqrt{2}}(q_{21}+q_{2-1})|t\rangle \right|^2 \times \mathcal{L}^{(-)}_{\omega=0}(\epsilon_s-\lambda, \epsilon_s-\lambda+\delta\hbar\omega_0), \quad \text{(B16)}$$

$$B_\gamma = 2\hbar^2(\hbar\omega_0)^2 \sum_{st} \left| \langle s| \frac{1}{\sqrt{2}}(q_{22}+q_{2-2})|t\rangle \right|^2 \times \mathcal{L}^{(+)}_{\omega_\gamma}(\epsilon_s-\lambda, \epsilon_s-\lambda+d). \quad \text{(B17)}$$

Summing over t in the limits of the width of maxima of the $\mathcal{L}^{(\pm)}$ we can rewrite Eqs. (B16) and (B17) approximately as

$$B_{\rm rot} \approx 2\hbar^2(\hbar\omega_0)^2 \sum_{s} \langle s| \left| \frac{1}{\sqrt{2}}(q_{21}+q_{2-1}) \right|^2 |s\rangle \times \mathcal{L}^{(-)}_{\omega=0}(\epsilon_s-\lambda, \epsilon_s-\lambda+\delta\hbar\omega_0), \quad \text{(B18)}$$

$$B_\gamma \approx 2\hbar^2(\hbar\omega_0)^2 \sum_{s} \langle s| \left| \frac{1}{\sqrt{2}}(q_{22}+q_{2-2}) \right|^2 |s\rangle \times \mathcal{L}^{(+)}_{\omega_\gamma}(\epsilon_s-\lambda, \epsilon_s-\lambda+d). \quad \text{(B19)}$$

Now we can use the relations $|\frac{1}{\sqrt{2}}(q_{21}+q_{2-1})|^2 = \frac{15}{4\pi}z^2y^2$ and $|\frac{1}{\sqrt{2}}(q_{22}+q_{2-2})|^2 = \frac{15}{16\pi}(x^2-y^2)^2$. For a rough estimate we neglect the deformation of the density distribution and assume a spherically symmetric radial density of the state s. Then

$$\left\langle \left| \frac{1}{\sqrt{2}}(q_{21}+q_{2-1}) \right|^2 \right\rangle \approx \left\langle \left| \frac{1}{\sqrt{2}}(q_{22}+q_{2-2}) \right|^2 \right\rangle \quad \text{(B20)}$$

and

$$\frac{B_\gamma}{B_{\rm rot}} = \frac{\sum_s \mathcal{L}^{(+)}_{\omega_\gamma}(\epsilon_s-\lambda, \epsilon_s-\lambda+d)}{\sum_s \mathcal{L}^{(-)}_{\omega=0}(\epsilon_s-\lambda, \epsilon_s-\lambda+\delta\hbar\omega_0)}. \quad \text{(B21)}$$

Substituting in Eq. (B21) $d=0$, $\delta\hbar\omega_0 = 2.3$ MeV, $\hbar\omega_\gamma = 1$ MeV and integrating over $(\epsilon_s - \lambda)$ instead of taking a sum with a realistic single-particle level scheme we obtain as a rough estimate $B_\gamma/B_{\rm rot} \approx 4.3$.

[1] R. V. Jolos and P. von Brentano, Phys. Rev. C **74**, 064307 (2006).
[2] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, no. 14, 1 (1952).
[3] G. Gneuss and W. Greiner, Nucl. Phys. **171**, 449 (1971).
[4] D. Troltenier, J. A. Maruhn, W. Greiner, V. Velazquez Aguilar, P. O. Hess, and J. H. Hamilton, Z. Phys. A **338**, 261 (1991).
[5] P. Petkov, A. Dewald, and W. Andrejtscheff, Phys. Rev. C **51**, 2511 (1995).
[6] J. M. Eisenberg and W. Greiner, *Nuclear Models*, Vol. 1 (North-Holland, Amsterdam, 1970).
[7] A. Bohr and B. Mottelson, *Nuclear Structure*, Vol. 2 (Benjamin, Reading, MA, 1975).
[8] P. Ring and P. Schuck, *The Nuclear Many Body Problem*, (Springer, New York, 1980).
[9] R. V. Jolos, P. von Brentano, and N. Pietralla, Phys. Rev. C **71**, 044305 (2005).
[10] S. Raman, C. W. Nestor, Jr., and P. Tikkanen, At. Data Nucl. Data Tables **78**, 1 (2001).
[11] L. Grodzins, Phys. Lett. **2**, 88 (1962).
[12] J. B. Gupta and K. Kavathekar, Indian Journal of Physics **61**, 167 (2003).
[13] A. B. Migdal, Nucl. Phys. **13**, 655 (1959).
[14] W. Greiner and A. Maruhn, *Kernmodelle* (Hurre Deutsch, Frankfurt, 1995).
[15] D. J. Rowe, *Nuclear Collective Motion* (Methuen and Co. Ltd., London, 1970).
[16] http://www.nndc.bnl.gov/nndc/ensdf/