Parametrization of the deuteron wave function obtained within a dispersion approach

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We present a convenient analytical parametrization of the deuteron wave function obtained previously within a certain dispersion technique. We fit the numerical results with a discrete superposition of Yukawa-type functions in both configuration and momentum spaces.

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Recently, in Ref. [1], it was shown that the deuteron tensor polarization component $T_{20}(Q^2)$ provides a crucial test of deuteron wave functions in the range of momentum transfers available in recent experiments. The calculation [1] shows that the most popular model wave functions (see, e.g., Refs. [2,3]) do not give an adequate description of $T_{20}(Q^2)$ and should be avoided in favor of those obtained in the dispersion potential-less inverse scattering approach with no adjustable parameters [4] (see also Ref. [5]) and which give the best description. Some time ago this function (which we call the MT wave function, following Ref. [6]) was used in the calculation of the neutron charge form factor [7]. The results of the calculation (providing 12 new points) are compatible with existing values of this form factor from other authors. A fit is obtained for the whole set (36 points) taking into account the data for the slope of the form factor at $Q^2 = 0$. These results will be used in the neutrino-scattering experiments at Fermilab [8].

The aim of this Brief Report is to present a conventional algebraic parametrization (as a discrete superposition of Yukawa-type terms) of the deuteron MT wave function calculated within a dispersion approach.

The first article on this approach was published by Shirokov [9], where the discontinuity of the wave function through the right (kinematical) cut was given in terms of the discontinuity of the Jost matrix. In Ref. [10] the important role of left cuts ($\approx$25%) was shown. The characteristics of these cuts can be found using the unitarity condition [11–13]. The difficult problem of constructing the Jost matrix in the case of mixing of channels was solved in Ref. [14], where the results of Ref. [15] were generalized. Let us note briefly the main characteristic features of these wave functions, which are obtained in the frame of the potential-less approach to the inverse scattering problem.

An important feature of these wave functions is the fact that they are "almost model independent:" no form of any *NN* interaction Hamiltonian is used. Instead, the MT wave functions are given by the dispersion-type integral directly and in terms of the experimental scattering phases and the mixing parameter for *NN* scattering in the $^3S_1$-$^3D_1$ channel [16]. To describe the scattering phases at high energies, the Regge-pole model for *NN*-scattering amplitudes was used. Parameters of the Regge-pole fit were determined from data on the total and differential *NN* cross sections, on the ratio of the real and imaginary parts of the forward scattering amplitude, and on the slope of the differential cross section [17].

It is worth noting that the MT wave functions were obtained using quite general assumptions about analytical properties of quantum amplitudes such as the validity of the Mandelstam representation for the deuteron electrodisintegration amplitude. These wave functions have no fitting parameters and can be altered only along with a modification of the *NN*-scattering phase analysis.

As noted above, in the potential-less approach to the inverse scattering problem, one constructs the wave function (and not the potential). In principle, one can then construct the interaction potential corresponding to this wave function. However, two problems arise in this approach. The first one is how to choose the class of functions from which the potential is to be obtained (local, nonlocal, separable, etc.). The second problem, a more difficult one, is familiar to inverse scattering problem in general: the problem of potential stability against small variations of the wave function. Because we are focusing on the wave functions, both these problems remain outside the scope of this Brief Report.

Let us note that the process of constructing these wave functions is closely related to the equations obtained in the framework of the dispersion approach based on the analytic properties of the scattering amplitudes [18,19] (see also Ref. [20] and especially the detailed version in Ref. [21]). This approach applies the dispersion technique using integrals over composite-system masses.

The dispersion problem of the wave-function reconstruction from the scattering phases is stable, that is small changes in phases result in small changes of the wave functions. In the dispersion approach, the wave function in the coordinate representation is related to the scattering phases through a double integral formula: The phase is integrated to obtain the Jost matrix, which is further integrated to obtain the wave function. This results in a suppression of the uncertainties

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TABLE I. Deuteron properties.

	MT model	Empirical	Ref.
Binding energy ε_d (MeV)	2.224996	2.224575(9)	[23]
Asymptotic D/S state η	0.0253	0.0256(4)	[24]
Matter radius r_d (fm)	1.972	1.971(6)	[25]
Quadrupole moment Q_d (fm^2)	0.2731	0.2859(3)	[26]
D-state probability P_D (%)	6.2		

in individual measurements of the scattering phases that are taken into account in the construction of the fitting function.

In principle one expects that, in the dispersion approach, the wave functions at small distances (and the electromagnetic deuteron form factors at large momentum transfer) should depend strongly on the behavior of the scattering phases at high energies (including those for which experimental data are not available). The corresponding estimate was performed in Ref. [22] for the example of the deuteron charge form factor. However, a good description of the deuteron electromagnetic form factors and of $T_{20}(Q^2)$ in the model (see Refs. [1,7]) suggests that the phase reconstruction by the Regge-pole analysis of the scattering amplitudes gives an adequate description of the scattering phases for the energy ranges discussed above.

A nonrelativistic calculation of the low-energy properties of the deuteron gives the values presented in Table I together with the experimental values. An update on the wave functions [4] taking account of recent data on the phase analysis in the $^3S_1$-$^3D_1$ channel is an interesting task to be performed. Usually, the deuteron wave functions are approximated by finite sets of Yukawa-type functions. So, we present here a simple parametrization of the deuteron function as a superposition of Yukawa-type terms (in the spirit of Ref. [27] for the Paris potential; see also the fit in Ref. [28] for the CD-Bonn wave function).

We consider the deuteron wave functions $\varphi_l(r)$ in the states with orbital momentum $l = 0, \varphi_0(r) = u(r)$ and $l = 2, \varphi_2(r) = w(r)$. The ansatz for the analytic versions of the r-space wave functions, denoted by $u_a(r)$ and $w_a(r)$, is

$$
\begin{aligned}
u_a(r) &= \sum_{j=1}^{n_u} C_j e^{-m_j r}, \\
w_a(r) &= \sum_{j=1}^{n_w} D_j e^{-m_j r} \left(1 + \frac{3}{m_j\, r} + \frac{3}{(m_j\, r)^2}\right), \qquad (1) \\
m_j &= \alpha + m_0\,(j-1),
\end{aligned}
$$

where the coefficients C_j, D_j, the maximal value of the index j, and m_0 are defined by the condition of the best fit. Furthermore, $\alpha = \sqrt{M\varepsilon_d}$, where M is average nucleon mass and ε_d is the binding energy of the deuteron.

These wave functions are normalized according to

$$
\int_0^\infty dr \left\{[u(r)]^2 + [w(r)]^2\right\} = 1\ . \qquad (2)
$$

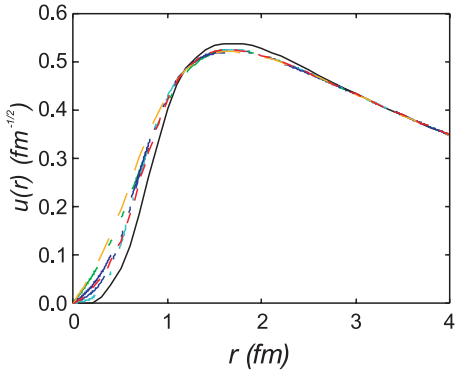

FIG. 1. (Color online) S-wave deuteron wave function in various models: MT [4,5], solid; Paris [27], short-dashed; CD-Bonn [28], middle-dashed; Nijmegen-I [29], dash-dot-dotted; Nijmegen-II [29], dash-dotted; Nijmegen-93 [29], long-dashed; Argonne V18 [30], dash-dot-dot-dotted lines. Some of the curves are indistinguishable.

The conventional boundary conditions at small r,

$$
u(r) \sim r,\ w(r) \sim r^3, \qquad (3)
$$

lead to one condition for C_j and three constraints for D_j, as follows:

$$
\sum_{j=1}^{n_u} C_j = 0, \quad \sum_{j=1}^{n_w} D_j = \sum_{j=1}^{n_w} D_j m_j^2 = \sum_{j=1}^{n_w} \frac{D_j}{m_j^2} = 0. \qquad (4)
$$

Using the form (2) it is easy to describe the standard behavior of the deuteron wave functions at $r \to \infty$. The asymptotic behavior of the S state is

$$
u(r) \sim A_S e^{-\alpha r}, \qquad (5)
$$

and that of the D state is

$$
w(r) \sim \eta\, A_S \left[1 + \frac{3}{\alpha\, r} + \frac{3}{(\alpha\, r)^2}\right] e^{-\alpha r}. \qquad (6)
$$

Here A_S and $A_D = \eta A_S$ are the asymptotic S-state and D-state normalizations, respectively, and η is the asymptotic D/S state ratio. In our calculation of the MT wave function we use $\alpha = 0.231625$ fm^{-1}.

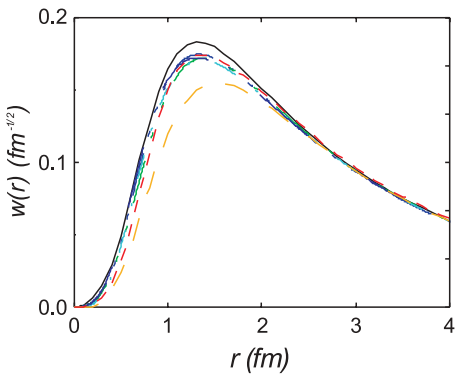

FIG. 2. (Color online) D-wave deuteron wave function. Legend is the same as in Fig. 1.

The Fourier transforms of wave functions $\psi_l(k)$, $l = 0, 2$ in the momentum representation in r-space are given by

$$\frac{\varphi_l(r)}{r} = \sqrt{\frac{2}{\pi}} \int_0^\infty k^2 dk j_l(kr)\psi_l(k), \tag{7}$$

where $j_l(kr)$ is the spherical Bessel function.

The normalization condition for these wave functions is given by

$$\int_0^\infty k^2 dk \left\{[\psi_0(k)]^2 + [\psi_2(k)]^2\right\} = 1. \tag{8}$$

The expression of the momentum space wave functions, following from (2) and (7) are given by

$$\begin{aligned} \psi_0^a(k) &= \sqrt{\frac{2}{\pi}} \sum_j \frac{C_j}{\left(k^2 + m_j^2\right)}, \\ \psi_2^a(k) &= \sqrt{\frac{2}{\pi}} \sum_j \frac{D_j}{\left(k^2 + m_j^2\right)}. \end{aligned} \tag{9}$$

The calculated coefficients in Eqs. (2) and (9) are listed in Table II; a value of $m_0 = 0.9$ fm^{-1} was used.

The asymptotic behavior at $r \to \infty$ yields, for the obtained fits of the MT wave functions, the following asymptotic D/S state ratio:

$$\eta = \frac{D_1}{C_1} = 0.02531511. \tag{10}$$

This value is in a good agreement with that of Ref. [2].

The accuracy of our parametrization is illustrated by the magnitudes of the integrals:

$$\left\{\int_0^\infty dr\,[u(r) - u_a(r)]^2\right\}^{1/2} = 4.1 \times 10^{-3}, \tag{11}$$

$$\left\{\int_0^\infty dr\,[w(r) - w_a(r)]^2\right\}^{1/2} = 2.2 \times 10^{-3}. \tag{12}$$

The use of the analytical expressions obtained in this work results in negligible (∼0.2%) changes in the deuteron low-energy properties as compared to those obtained with a numerical wave function. The change in the deuteron matter radius is 0.5%, whereas the deuteron tensor polarization component $T_{20}(Q^2)$ changes by ∼2%.

TABLE II. Coefficients for the parametrized deuteron wave function calculated within a dispersion approach. The last C_j and last three D_j are to be computed from Eq. (4) ($n_u = 16$, $n_w = 12$).

j	C_j (fm$^{-1/2}$)	D_j (fm$^{-1/2}$)
1	0.87872995	$0.22245143 \times 10^{-1}$
2	−0.50381047	−0.41548258
3	0.28787196×10^{2}	$-0.18618515 \times 10^{1}$
4	$-0.82301294 \times 10^{3}$	0.21987598×10^{2}
5	0.12062383×10^{5}	$-0.16885413 \times 10^{3}$
6	$-0.10574260 \times 10^{6}$	0.76001430×10^{3}
7	0.59534957×10^{6}	$-0.22287203 \times 10^{4}$
8	$-0.22627706 \times 10^{7}$	0.43330023×10^{4}
9	0.59953379×10^{7}	$-0.54072021 \times 10^{4}$
10	$-0.11282284 \times 10^{8}$	Eq. (4)
11	0.15181681×10^{8}	Eq. (4)
12	$-0.14519973 \times 10^{8}$	Eq. (4)
13	0.96491938×10^{7}	
14	$-0.42403857 \times 10^{7}$	
15	0.11092702×10^{7}	
16	Eq. (4)	

So, we present a convenient analytical parametrization of the deuteron wave function calculated previously within a dispersion approach as a discrete superposition of Yukawa-type functions. This function is plotted in Figs. 1 and 2. The wave functions [27–30] are also given for comparison. These high-precision wave functions [27–30] are now widely used (see, e.g., [31]). Note that the MT wave functions differ from those of other models up to $r \lesssim 2.7$ fm, i.e., even in the domain where the one-pion exchange presumably dominates. This is related to the fact that no additional restrictions (except the assumption that the Mandelstam representation is applicable for the amplitudes) were imposed when the inverse problem was solved by the dispersion method. In particular, no restrictions were imposed on locality properties and on the form of the NN interaction.

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