Relativistic dynamics of nonideal fluids: Viscous and heat-conducting fluids. I. General aspects and 3+1 formulation for nuclear collisions

Azwinndini Muronga
Centre for Theoretical Physics & Astrophysics, Department of Physics, University of Cape Town, Rondebosch 7701, South Africa, and UCT-CERN Research Centre, Department of Physics, University of Cape Town, Rondebosch 7701, South Africa
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Relativistic nonideal fluid dynamics is formulated in 3+1 space-time dimensions. The equations governing dissipative relativistic hydrodynamics are given in terms of the time and three-space quantities which correspond to those familiar from nonrelativistic physics. Dissipation is accounted for by applying the causal theory of relativistic dissipative fluid dynamics. As a special case, we consider a fluid without viscous/heat couplings in the causal system of transport/relaxation equations. For the study of physical systems, we consider pure (1+1)-dimensional expansion in planar geometry, (1+1)-dimensional spherically symmetric (*fireball*) expansion, (1+1)-dimensional cylindrically symmetric expansion, and a (2+1)-dimensional expansion with cylindrical symmetry in the transverse plane (*firebarrel* expansion). The transport/relaxation equations are given in terms of the spatial components of the dissipative fluxes, since these are not independent. The choice for the independent components is analogous to the nonrelativistic equations.

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I. INTRODUCTION

The study of nonequilibrium properties of matter produced at relativistic high energy nuclear collisions is important to unraveling the underlying interactions between the constituents of the system. Nonequilibrium effects are of central importance to the space-time description of these collisions.

In real fluids, there are internal processes that result in the transport of momentum and energy from one fluid element to another on a microscopic level. The momentum transport mechanisms give rise to internal frictional forces (viscous forces) that enter directly into the equations of motion and that also produce frictional energy dissipation in the flow. The energy transport mechanisms lead to energy conduction from one point in the flow to another.

In this work, we use equations of fluid flow that explicitly account for the processes described above. We adopt a continuum view here and leave the microscopic kinetic-theory view to paper II of this work [1]. In paper II, we recover essentially the same set of equations, but now with a much clearer understanding of the underlying physics. The alternative approach also allows us to evaluate explicitly, for a given molecular model, the transport coefficients that are introduced on empirical grounds in the macroscopic equations. For a review of the nonequilibrium models for these collisions, see Ref. [2].

To investigate the space-time evolution of high energy relativistic nuclear collisions in a more complete 3+1 space-time formalism by means of dissipative fluid dynamics requires the use of numerical schemes to solve the fluid dynamical equations of motion and the transport (evolution or relaxation) equations. The existing numerical schemes for solving relativistic ideal fluid dynamics can be used provided the relativistic dissipative fluid dynamics equations, in their covariant structure, are cast in a more transparent form.

The dynamics of relativistic nonideal fluids is of considerable interest in high energy heavy ion collisions [3–13]. In particular, the recent application of causal relativistic fluid dynamics to heavy ion collisions [4] has proven to be excellent in explaining the spectra of particles produced at the BNL Relativistic Heavy Ion Collider (RHIC) [12,13].

The modeling of relativistic matter produced in these collisions is most conveniently performed within the 3+1 formalism, where the four-vectors, tensors, and equations of motion are decomposed with respect to a general space-time coordinate, allowing one to express space-time derivatives in a more transparent way as in nonrelativistic mechanics. The 3+1 representation of the equations for relativistic ideal fluids has been discussed by many authors (see, e.g., Ref. [14]), in the context of applications to nuclear collisions.

Modeling dissipative processes requires nonequilibrium fluid dynamics and irreversible thermodynamics. Standard dissipative fluid dynamics derived from standard irreversible thermodynamics was first extended from nonrelativistic to relativistic fluids by Eckart [15]. However, as in its nonrelativistic counterpart, in the Eckart theory and a variation thereof by Landau and Lifshitz [16], dissipative fluctuations may propagate at an infinite speed (causality problem). In addition, small perturbations of equilibria driven by dissipative processes are unstable [17], and, finally, no well-posed initial value problem exists [3]. These problems in standard dissipative fluid dynamics originate from the description of nonequilibrium states via the local equilibrium states alone; i.e., it is assumed that local thermodynamic equilibrium is established on an infinitely short time scale. This is a consequence of the assumption that the entropy four-current includes only terms linear in dissipative quantities.

Causal theories of dissipative fluids based on Grad's 14-moment method [18] were formulated by Müller [19] and generalized relativistically by Israel and Stewart [20]. They were formulated to remedy some of the problems encountered in standard theories, in particular the causality problem. They are based upon extended irreversible thermodynamics, where the entropy four-current vector of standard thermodynamics is extended by including terms that are quadratic in the dissipative quantities. Hence the causal theory is also referred to as the second-order theory of dissipative fluid dynamics. In causal theories of dissipative fluids, the set of thermodynamic variables is extended to include the dissipative variables. The resulting transport equations and hence the equations of motion are causal and have a well-posed initial value problem for realistic equations of state, transport coefficients, relaxation, and coupling coefficients. Besides the Müller-Israel-Stewart causal theory of relativistic nonideal fluids, another causal relativistic theory for dissipative fluids exists, namely, a theory by Liu *et al.* [21] which is of the divergence type, in which the dissipative fluxes are subject to a conservation equation. In this work, we will follow the former.

Causal theories of relativistic dissipative fluids provide the easiest extension of standard theory toward finite signal speeds and are therefore appealing for nuclear collision modeling. Extended fluid theories are still rarely applied in the simulation of nuclear collisions, which is also a result of the lack of an appropriate formulation. However, there are developments in this direction [4,10]. The purpose of this paper is to provide such a formulation for dissipative relativistic fluid dynamics.

This paper provides a complete set of equations for dissipative fluid dynamics in their (3+1)-dimensional representation, using a causal description of thermodynamics. Having at hand an appropriate 3+1 space-time formulation for dissipative relativistic fluid theories, we can then employ them in modeling relativistic nuclear collisions. In Sec. II, the basic elements of causal dissipative hydrodynamics are presented in their covariant compact form. In Sec. III, we give the causal system of transport equations in their covariant compact form. In Sec. IV, we present the weak coupling limit of the transport equations and the linearized form of the complete system of the 14 equations. In Sec. V, we show the 3+1 representation of relativistic dissipative fluid dynamics. As special cases for application to the study of nuclear collision dynamics, we specify the system of equations given in Sec. V to the case of (i) (1+1)-dimensional expansion in planar geometry, (ii) (1+1)-dimensional cylindrically symmetric expansion, and (iii) (2+1)-dimensional expansion with cylindrical symmetry in the transverse plane (firebarrel expansion) in Sec. VI. The (1+1)-dimensional spherically symmetric expansion is left for Appendix A.

Throughout this article, we adopt the units $\hbar = c = k_{\rm B} = 1$. The signature of the metric tensor is always taken to be $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Upper Greek indices are contravariant, lower Greek indices covariant. The Greek indices used in four-vectors go from 0 to 3 (t, x, y, z) and the Roman indices used in three-vectors go from 1 to 3 (x, y, z). The scalar product of two four-vectors a^μ, b^μ is denoted by $a^\mu g_{\mu\nu} b^\nu \equiv a^\mu b_\mu$. The scalar product of two three-vectors is denoted by bold type, namely, $\mathbf{a}, \mathbf{b}, \mathbf{a}\cdot\mathbf{b}$. The notations $A^{(\alpha\beta)} \equiv (A^{\alpha\beta} + A^{\beta\alpha})/2$ and $A^{[\alpha\beta]} \equiv (A^{\alpha\beta} - A^{\beta\alpha})/2$ denote symmetrization and antisymmetrization, respectively. The four-derivative is denoted by $\partial_\alpha \equiv \partial/\partial x^\alpha$. The dot product between two tensors will be denoted $A^{\mu\nu}B_{\mu\nu}$; while for spatial components of the tensor, $\mathbf{A}\cdot\mathbf{B} \equiv \sum_k A^{ik}B^{kj}$; and for the double scalar product between tensors, $\mathbf{A} : \mathbf{B} \equiv \sum_{i,k} A^{ik}B^{ki}$. The notation $\mathbf{b}\otimes\mathbf{b} = b^i b^j$ and similarly $\mathbf{b}\otimes(\mathbf{B}\cdot\mathbf{a}) = b^i \sum_k B^{jk}a^k$ denotes the multiplication of two vectors.

II. NONIDEAL RELATIVISTIC FLUID DYNAMICS

Nonideal fluids, like ideal fluids, are described by conservation laws for the particle current vector and the energy-momentum stress tensor. A thermodynamic equilibrium state of a nonideal fluid dynamics theory can be characterized by the five independent dynamical fields which describe a reversible thermodynamic process in an ideal fluid dynamics, and one often chooses two scalar equilibrium state variables and the three independent components of the preferred four-velocity. In high energy heavy ion collisions, we generally take the net baryon number density and energy density as state variables. A general nonequilibrium state or irreversible thermodynamic process is to be characterized by 14 independent dynamical fields. These are, in addition to the ideal fields, the projections of the stress-energy tensor parallel and orthogonal to the preferred velocity (dissipative fluxes), e.g., the trace and trace-free part of the spatial stress tensor, and the spatial heat flux vector.

We choose the average four-velocity u^μ such that the particle flux in the associated rest frame vanishes. This is the Eckart frame [15] and is the natural frame in systems where there exists some conserved net charge (see Ref. [20] for the alternative energy frame description). The state of the fluid is assumed to be close to a thermodynamic equilibrium state, characterized by the local thermodynamic equilibrium scalars such as the equilibrium net charge density $n_{\rm eq}$, equilibrium energy density $\varepsilon_{\rm eq}$, and local equilibrium velocity $u^\mu_{\rm eq}$ which in the Eckart frame can be chosen such that only the pressure p deviates from the local equilibrium pressure $p_{\rm eq}$ by the bulk viscous pressure $\Pi = p - p_{\rm eq}$, whereas $n = n_{\rm eq}$ and $\varepsilon = \varepsilon_{\rm eq}$. Other thermodynamic quantities such as the temperature, chemical potential, and entropy are obtained from the equation of state and thermodynamic relations.

We consider a fluid that consists of a single component. The variables of concern are the net charge four-current N^μ, the energy-momentum stress tensor $T^{\mu\nu}$, and the entropy flux S^μ. The divergence of the energy-momentum tensor and net charge four-current vanishes locally. That is, the energy-momentum and net charge are conserved locally. However, in general, the divergence of the entropy four-current does not vanish. The second law of thermodynamics requires that it be a positive and nondecreasing function. In equilibrium, the entropy is maximum and the divergence of the entropy four-current vanishes. Thus,

$$\partial_\mu N^\mu = 0, \tag{2.1}$$

$$\partial_\nu T^{\mu\nu} = 0, \tag{2.2}$$

$$\partial_\mu S^\mu \geqslant 0, \tag{2.3}$$

where

$$N^{\mu} = nu^{\mu}, \tag{2.4}$$

$$T^{\mu\nu} = (\varepsilon + p + \Pi)u^{\mu}u^{\nu} - (p + \Pi)g^{\mu\nu} + 2q^{(\mu}u^{\nu)} + \pi^{\mu\nu}, \tag{2.5}$$

$$S^{\mu} = su^{\mu} + \beta q^{\mu} - \tfrac{1}{2}\beta u^{\mu}(\beta_0\Pi^2 - \beta_1 q^{\nu}q_{\nu} + \beta_2\pi^{\lambda\nu}\pi_{\lambda\nu}) - \beta(\alpha_0 q^{\mu}\Pi - \alpha_1 q_{\nu}\pi^{\mu\nu}). \tag{2.6}$$

In the local rest frame defined by $u^{\mu} = (1, \mathbf{0})$, the quantities appearing in the decomposed tensors take their actual meanings: $n \equiv u_{\mu}N^{\mu}$ is the net charge density, $\varepsilon \equiv u_{\mu}T^{\mu\nu}u_{\nu}$ is the energy density, $p + \Pi \equiv -\frac{1}{3}\triangle_{\mu\nu}T^{\mu\nu}$ is the local isotropic pressure plus bulk pressure, $q^{\mu} \equiv u_{\nu}T^{\nu\lambda}\triangle^{\mu}_{\lambda}$ is the heat flow, $\pi^{\mu\nu} \equiv T^{\langle\mu\nu\rangle}$ is the shear stress tensor, and $s \equiv u_{\mu}S^{\mu}$ is the entropy density. The angular bracket notation, representing the symmetrized spatial and traceless part of the tensor, is defined by $A^{\langle\mu\nu\rangle} \equiv [\frac{1}{2}(\triangle^{\mu}_{\sigma}\triangle^{\nu}_{\tau} + \triangle^{\mu}_{\tau}\triangle^{\nu}_{\sigma}) - \frac{1}{3}\triangle^{\mu\nu}\triangle_{\sigma\tau}]A^{\sigma\tau}$. The space-time derivative decomposes into $\partial^{\mu} = u^{\mu}D + \nabla^{\mu}$ with $u^{\mu}\nabla_{\mu} = 0$. In this space-time derivative decomposition $D \equiv u^{\mu}\partial_{\mu}$ is the convective time derivative, and $\nabla^{\mu} \equiv \triangle^{\mu\nu}\partial_{\nu}$ is the gradient operator. The projection onto the three-space $\triangle^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu} \equiv \triangle^{\nu\mu}$ is orthogonal to u^{μ}, that is, $\triangle^{\mu\nu}u_{\nu} = 0$ and u^{μ} is the hydrodynamical four-velocity of the net charge and is to be normalized such that $u^{\mu}u_{\mu} = 1$ and therefore $u^{\mu}\partial_{\nu}u_{\mu} = 0$. Here, $\beta \equiv 1/T$ is the inverse temperature. The $\alpha_i(\varepsilon, n)$ and $\beta_i(\varepsilon, n)$ in Eq. (2.6) are the second-order coefficients which are expressed in terms of thermodynamic integrals and therefore are given by the equation of state. These second-order coefficients are presented in more details in paper II of this work [1].

From the entropy four-current (2.6), one sees that the entropy density and flux are, respectively, given by

$$s = u_{\mu}S^{\mu} = s(\varepsilon, n) - \tfrac{1}{2}\beta(\beta_0\Pi^2 - \beta_1 q_{\nu}q^{\nu} + \beta_2\pi_{\lambda\nu}\pi^{\lambda\nu}), \tag{2.7}$$

$$\Phi^{\mu} = \triangle^{\mu\nu}S_{\nu} = \beta q^{\mu} - \beta(\alpha_0\Pi q^{\mu} - \alpha_1\pi^{\mu\nu}q_{\nu}). \tag{2.8}$$

Note that the entropy density is independent of α_i, while the entropy flux is independent of β_i. The negative sign of the nonequilibrium contributions reflects the fact that the entropy density is maximum in equilibrium. The thermodynamic coefficients $\beta_i(\varepsilon, n) \geqslant 0$ in Eq. (2.7) model deviations of the physical entropy density from s due to scalar/vector/tensor dissipative contributions to S^{μ}. The $\alpha_i(\varepsilon, n)$ in Eq. (2.8) model contributions due to viscous/heat coupling, which do not influence the physical entropy density.

III. RELAXATION TRANSPORT EQUATIONS

As we mentioned before, we shall use the Müller-Israel-Stewart second-order phenomenological theory for dissipative fluids [19,20]. Essentially the extended irreversible thermodynamics theory rests on two hypothesis: (1) The dissipative flows, heat flow and viscous pressures, are considered as independent variables. Hence the entropy function depends not only on the classical variables, net baryon density and energy density, but on these dissipative flows as well. (2) At equilibrium, the entropy function is maximum. Moreover, its flow depends on all dissipative flows, and its production rate is semipositive definite. As a consequence, the bulk pressure Π, heat flow q^{μ}, and traceless shear viscous tensor $\pi^{\mu\nu}$ obey the evolution equations [20]

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\Theta - \frac{1}{2}\zeta T\Pi\partial_{\mu}\left(\frac{\tau_{\Pi}u^{\mu}}{\zeta T}\right) + l_{\Pi q}\nabla_{\mu}q^{\mu}, \tag{3.1}$$

$$\tau_q\triangle^{\mu}_{\nu}\dot{q}^{\nu} + q^{\mu} = \kappa T\left(\frac{\nabla^{\mu}T}{T} - a^{\mu}\right) + \frac{1}{2}\kappa T^2 q^{\mu}\partial_{\nu}\left(\frac{\tau_q u^{\nu}}{\kappa T^2}\right) - l_{q\pi}\nabla_{\nu}\pi^{\mu\nu} - l_{q\Pi}\nabla^{\mu}\Pi + \tau_q\omega^{\mu\nu}q_{\nu}, \tag{3.2}$$

$$\tau_{\pi}\triangle^{\mu\alpha}\triangle^{\nu\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \frac{1}{2}\eta T\pi^{\mu\nu}\partial_{\lambda}\left(\frac{\tau_{\pi}u^{\lambda}}{\eta T}\right) + l_{\pi q}\nabla^{\langle\mu}q^{\nu\rangle} + 2\tau_{\pi}\pi^{\alpha(\mu}\omega^{\nu)}_{\alpha}, \tag{3.3}$$

where $\omega^{\mu\nu} = \triangle^{\mu\alpha}\triangle^{\nu\beta}\partial_{[\beta}u_{\alpha]}$ is the vorticity, $\Theta = \nabla_{\mu}u^{\mu}$ is the expansion scalar, $\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle} = \nabla^{(\mu}u^{\nu)} - \frac{1}{3}\triangle^{\mu\nu}\nabla_{\lambda}u^{\lambda}$ is the shear tensor, and $a^{\mu} \equiv \dot{u}^{\mu}$ is the acceleration four-vector. The local enthalpy density is $w = \varepsilon + p$. Overdot denotes $\dot{A}_{\alpha\beta} = u^{\lambda}\partial_{\lambda}A_{\alpha\beta}$. The transport coefficients κ, ζ, and η denote the thermal conductivity and the bulk and shear viscous coefficients, respectively. The quantities

$$\tau_{\Pi} = \zeta\beta_0, \quad \tau_q = \kappa T\beta_1, \quad \tau_{\pi} = 2\eta\beta_2, \tag{3.4}$$

$$l_{\Pi q} = \zeta\alpha_0, \quad l_{q\Pi} = \kappa T\alpha_0, \quad l_{q\pi} = \kappa T\alpha_1, \quad l_{\pi q} = 2\eta\alpha_1, \tag{3.5}$$

are the relaxation times for the bulk pressure τ_{Π}, the heat flux τ_q, and the shear tensor τ_{π}, and the relaxation lengths for coupling between heat flux and bulk pressure ($l_{\Pi q}, l_{q\Pi}$) and between heat flux and shear tensor ($l_{q\pi}, l_{\pi q}$). The α_i and β_i are presented in Ref. [1].

IV. THE WEAK COUPLING LIMIT OF CAUSAL TRANSPORT SYSTEM

The complexity of the full evolution equations (3.1)–(3.3) makes their applications tractable only if certain simplifications are made. A particularly simple set of evolution equations results from the assumption that there is no viscous/heat coupling (specifically, $\alpha_0 = \alpha_1 = 0$). We will also drop the term with the one-half factor by assuming that the thermodynamic gradients are small. Finally we make a further assumption that there is no coupling between acceleration and dissipative fluxes. Such is the case in systems where the gradients in thermodynamic quantities are small and in systems where there is no coupling of velocity components. In relativistic heavy ion collisions, such cases are, for example, the scaling solution [22], effective (1+1)-dimensional expansions (expansions in planar geometry, in cylindrical/spherical geometry with symmetry—pure radial expansions). The evolution equations resulting from Eqs. (3.1)–(3.3) under the above assumptions are

$$\dot{\Pi} = \frac{1}{\tau_\Pi}(\Pi_E - \Pi), \tag{4.1}$$

$$\dot{q}^\mu = \frac{1}{\tau_q}\left(q_E^\mu - q^\mu\right), \tag{4.2}$$

$$\dot{\pi}_{\mu\nu} = \frac{1}{\tau_\pi}\left(\pi_E^{\mu\nu} - \pi^{\mu\nu}\right), \tag{4.3}$$

where

$$\Pi_E = -\zeta\Theta, \tag{4.4}$$

$$q_E^\mu = \kappa T\left(\frac{\nabla^\mu T}{T} - \dot{u}^\mu\right), \tag{4.5}$$

$$\pi_E^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{4.6}$$

are the standard Eckart thermodynamic fluxes. Equations (4.1)–(4.3) are of covariant relativistic Maxwell-Cattaneo form. In contrast to the algebraic constraint equations (4.4)–(4.6), the evolution equations (4.1)–(4.3) are first-order partial differential equations, which ensure that in the local rest frame the viscous bulk/shear stresses and the heat flux relax toward their standard limits $\{\Pi_E, q_E^\mu, \pi_E^{\mu\nu}\}$ on time scales τ_A. The relaxation times τ_A follow in principle from kinetic theory [1].

The conservation laws, that is, the net charge conservation equation, the equations of motion, and the energy equation, together with the evolution equations (4.1)–(4.3) constitute a complete system of hyperbolic first-order partial differential equations for the solution vector of 14 (=1+1+3+1+3+5) dynamical variables $\{n,\ \varepsilon,\ u^\mu,\ \Pi,\ q^\mu,\ \pi^{\mu\nu}\}$. This system represents a 14-field theory for relativistic dissipative fluids which is causal and stable for the appropriate equation of state, initial conditions, and transport and relaxation coefficients [3].

The equations of nonideal fluid dynamics and in particular those of relativistic dissipative fluid dynamics are in general rather complicated. It is therefore useful to have a simple means for judging both the relative importance of various phenomena that occur in a flow and the flow's qualitative nature. This is most easily done in terms of a set of dimensionless numbers which provides a convenient characterization of the dominant physical processes in the flow. Flows whose physical properties are such that they produce the same values of these numbers can be expected to be qualitatively similar even though the value of any one quantity—say velocity or characteristic length/time—may be substantially different from one flow to another. Thus the assumption made in this section (the weak coupling limit) must be thoroughly checked on a case-by-case basis for the problem under consideration. This is the subject of current investigation and will be presented elsewhere.

The analysis of the instantaneous dynamics of a relativistic fluid can be summarized by the identity

$$\partial^\nu u^\mu = a^\mu u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \tfrac{1}{3}\theta\triangle^{\mu\nu}, \tag{4.7}$$

which is a generalization of the Cauchy-Stokes decomposition theorem. It shows that at each space-time point, a fluid is accelerated along its proper time axis and experiences shear, rotation (vorticity), and expansion along its local space axes. Explicit expressions for $\triangle^{\mu\nu}$, a^μ, θ, $\omega^{\mu\nu}$, and $\sigma^{\mu\nu}$ in terms of the ordinary velocity $\mathbf{v}$ and laboratory frame space and time derivatives are given in Sec. V. Note that from $u_\mu\triangle^{\mu\nu} = 0$ we have $u_\mu\sigma^{\mu\nu} = u^\mu\sigma_{\mu\nu} = 0$ and $u_\mu\omega^{\mu\nu} = u^\mu\omega_{\mu\nu} = 0$, and from $\triangle^{\mu\nu} \equiv \triangle^{\mu\alpha}\triangle_\alpha^\nu$ and $\triangle^{\mu\nu}\triangle_{\mu\nu} = 3$ we have $\triangle_{\mu\nu}\sigma^{\mu\nu} = \triangle^{\mu\nu}\sigma_{\mu\nu} = 0$. From the shear tensor we define a shear scalar σ from

$$\sigma_{\mu\nu}\sigma^{\mu\nu} = \tfrac{1}{6}\sigma. \tag{4.8}$$

V. RELATIVISTIC DISSIPATIVE FLUID DYNAMICS IN 3+1 FORMULATION

A. 3+1 space-time formulation

To find a numerical solution to the relativistic dissipative fluid dynamics equations, we must first cast the equations into a suitable form. The equations are usually written in a more compact covariant form. The equations in this form are not well suited for solving on a computer. First it is unclear what information to specify on the boundary of the four-dimensional space-time in order to get a well-posed problem. Second, this form of the equations does not allow us to pose the type of physical questions we often wish to ask. Typically we want to be able to specify a physical system at some initial time and find the development of this system as time evolves. Because of these two issues, we break the covariance of the equations in two steps: First, we split the four-dimensional space-time coordinates into three-dimensional spatial coordinates plus a 1-dimensional time coordinate. Second, we break a second-rank tensor into 1+3+6(00, 0i, ij) independent components. A four-vector is broken into 1+3(0, i) components, while a scalar is unchanged. The numerical problem of constructing the 3+1 space-time is to (i) determine a set of initial conditions that satisfy conservation equations and transport equations and (ii) evolve this set of initial conditions forward in time.

In this section, we will give explicit expressions for the net charge four-current N^μ, energy-momentum stress tensor $T^{\mu\nu}$, net charge conservation $\partial_\mu N^\mu = 0$, energy-momentum conservation $\partial_\mu T^{\mu\nu}$ in terms of the ordinary velocity and laboratory frame space and time derivatives. In addition, we give the explicit expressions for the transport/relaxation equations for the dissipative quantities, namely, the bulk pressure equation, heat flux equation, and shear stress tensor equation, in terms of the ordinary velocity and laboratory frame space and time derivatives.

The four-velocity, metric tensor, and projection tensor are decomposed as

$$u^\mu = (\gamma, \mathbf{u}), \quad \mathbf{u} = \gamma\mathbf{v}, \quad \gamma = (1 - \mathbf{v}\cdot\mathbf{v})^{1/2}, \tag{5.1}$$

$$g^{\mu\nu} = (1, \mathbf{0}, \mathbf{g}), \quad \mathbf{0} \equiv \{g^{0i}\}, \quad \mathbf{g} \equiv \{g^{ij}\} = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{5.2}$$

$$\triangle^{\mu\nu} = -(\gamma^2 v^2, \gamma\mathbf{u}, -\Delta), \quad \Delta = \mathbf{g} - \mathbf{u}\otimes\mathbf{u}, \tag{5.3}$$

where $\gamma = (1 - \mathbf{v}^2)^{-1/2}$ and $\mathbf{v}$ is the three-velocity of the fluid. The space-time gradient is decomposed as

$$\partial^\mu = (\partial_t, \boldsymbol{\partial}) \equiv u^\mu D + \nabla^\mu, \tag{5.4}$$

$$D = (\gamma\partial_t, \mathrm{D}), \quad \mathrm{D} = \mathbf{u}\cdot\boldsymbol{\partial}, \quad \boldsymbol{\partial} = \{\partial_i\}, \tag{5.5}$$

$$\nabla^\mu = -(\gamma^2 v^2\partial_t + \gamma\mathbf{u}\cdot\boldsymbol{\partial}, \gamma\mathbf{u}\partial_t - \boldsymbol{\nabla}), \quad \boldsymbol{\nabla} = \Delta\cdot\boldsymbol{\partial}, \tag{5.6}$$

In the irreducible decomposition

$$\boldsymbol{\partial} \otimes \mathbf{u} = \boldsymbol{\sigma} + \boldsymbol{\omega} + \tfrac{1}{3}\vartheta\Delta + \mathbf{a} \otimes \mathbf{u}, \tag{5.7}$$

the kinematic properties of the fluid, i.e., expansion scalar, acceleration, shear, and vorticity tensors, are given by

$$\Theta = \partial_t \gamma + \boldsymbol{\partial} \cdot \mathbf{u}, \tag{5.8}$$
$$\mathbf{a} = \gamma \partial_t \mathbf{u} + \mathrm{D}\mathbf{u}, \tag{5.9}$$
$$\boldsymbol{\sigma} = (\boldsymbol{\partial} \otimes \mathbf{u}) - (\mathbf{a} \otimes \mathbf{u}) - \tfrac{1}{3}\Theta\Delta, \tag{5.10}$$
$$\boldsymbol{\omega} = -[\boldsymbol{\partial}\mathbf{u}] + [\mathbf{a} \otimes \mathbf{u}]. \tag{5.11}$$

These kinematic quantities can be written in terms of time derivatives and spatial ones

$$\Theta = \partial_t \gamma + \vartheta, \tag{5.12}$$
$$\mathbf{a} = \gamma \partial_t \mathbf{u} + \mathbf{a}_s, \tag{5.13}$$
$$\boldsymbol{\sigma} = -\left\{\gamma(\mathbf{u} \otimes \partial_t \mathbf{u}) + \tfrac{1}{3}\Delta \partial_t \gamma - \boldsymbol{\sigma}_s\right\}, \tag{5.14}$$
$$\boldsymbol{\omega} = \gamma[\mathbf{u}\partial_t \otimes \mathbf{u}] + \boldsymbol{\omega}_s, \tag{5.15}$$

where

$$\boldsymbol{\sigma}_s \equiv (\boldsymbol{\partial} \otimes \mathbf{u}) - (\mathbf{a}_s \otimes \mathbf{u}) - \tfrac{1}{3}\Delta\vartheta, \tag{5.16}$$
$$\boldsymbol{\omega}_s \equiv [\boldsymbol{\partial} \otimes \mathbf{u}] + [\mathbf{a}_s \otimes \mathbf{u}], \tag{5.17}$$
$$\mathbf{a}_s \equiv \mathrm{D}\mathbf{u}, \tag{5.18}$$
$$\vartheta \equiv \boldsymbol{\partial} \cdot \mathbf{u}. \tag{5.19}$$

A complete split involves 3+1 representations for the fields N^μ, $T^{\mu\nu}$, and S^μ, together with the equations governing their evolution. The representation of $T^{\mu\nu}$ and S^μ involves the 3+1 formulation of dissipative quantities, i.e., Π, q^0, and $\mathbf{q} = \{q^i\}$, as well as π^{00}, π^{0i}, and $\boldsymbol{\pi} = \pi^{ij}$ of dissipative fluxes $(\Pi, q^\mu, \pi^{\mu\nu})$. Not all of them are independent. Orthogonality to the fluid velocity field, $q^\mu u_\mu = \pi^{\mu\nu} u_\nu = 0$, together with $\pi^\nu_\nu = 0$, yields three and five independent components of heat flux and shear tensor, respectively. Analogy with nonrelativistic physics suggests to generally eliminate q^0 in favor of $\mathbf{q}$, as well as π^{00} and π^{0i} in favor of $\boldsymbol{\pi}$, though different choices might be more appropriate in particular applications. For a discussion on the different choices of the dissipative fluxes, see Ref. [10]. Orthogonality also naturally holds for dissipative forces $\nabla^\mu \ln T - a^\mu$ and $\sigma^{\mu\nu}$. The 3+1 representation of the orthogonality relations for q^μ and $\pi^{\mu\nu}$ is

$$0 = \gamma(q^0 - \mathbf{q} \cdot \mathbf{v}) \Longrightarrow q^0 = \mathbf{q} \cdot \mathbf{v}, \tag{5.20}$$
$$0 = \gamma(\pi^{00} - \pi^{0i} \cdot \mathbf{v}) \Longrightarrow \pi^{00} = \pi^{0i} \cdot \mathbf{v} = (\boldsymbol{\pi} \cdot \mathbf{v}) \cdot \mathbf{v}, \tag{5.21}$$
$$\mathbf{0} = \gamma(\pi^{0i} - \boldsymbol{\pi} \cdot \mathbf{v}) \Longrightarrow \pi^{0i} = \boldsymbol{\pi} \cdot \mathbf{v}, \tag{5.22}$$

where Eq. (5.22) was used in the last equality of Eq. (5.21). Finally, $\pi^\nu_\nu = 0$ becomes

$$\pi^{00} - \mathrm{tr}(\boldsymbol{\pi}) = 0 \Longrightarrow \mathrm{tr}(\boldsymbol{\pi}) = (\boldsymbol{\pi} \cdot \mathbf{v}) \cdot \mathbf{v}, \tag{5.23}$$

and allows one to eliminate one component of $\boldsymbol{\pi}$. After substituting q^0, π^{00}, and π^{0i} in this way, it remains that expressions be provided for one scalar function, the bulk viscous pressure Π, three components of the heat flux (i.e., $\mathbf{q}$), and five components of the viscous stress tensor (i.e., $\boldsymbol{\pi}$), as in nonrelativistic physics.

B. Conservation laws

The local net charge density N^0 is subject to the 3+1 representation of Eq. (2.1), and the 3+1 representation of Eq. (2.2) yields conservation laws for both the local total energy density T^{00} and the local momentum density T^{0i},

$$\partial_\mu N^\mu \equiv \partial_t \mathcal{N} + \boldsymbol{\partial} \cdot \{\mathcal{N}\mathbf{v}\} = 0, \tag{5.24}$$
$$\begin{aligned}\partial_\mu T^{\mu 0} \equiv\ & \partial_t E + \boldsymbol{\partial} \cdot \{(E + \mathcal{P})\mathbf{v} + \gamma(\mathbf{q} - (\mathbf{q} \cdot \mathbf{v})\mathbf{v}) \\ & + \boldsymbol{\pi} \cdot \mathbf{v} - ((\boldsymbol{\pi} \cdot \mathbf{v}) \cdot \mathbf{v})\mathbf{v}\} = 0,\end{aligned} \tag{5.25}$$
$$\begin{aligned}\partial_\mu T^{\mu i} \equiv\ & \partial_t \mathbf{M} + \boldsymbol{\partial} \cdot \{\mathbf{M} \otimes \mathbf{v} - \mathcal{P}\mathbf{g} + \gamma(\mathbf{v} \otimes \mathbf{q} \\ & - (\mathbf{q} \cdot \mathbf{v})(\mathbf{v} \otimes \mathbf{v})) + \boldsymbol{\pi} - (\boldsymbol{\pi} \cdot \mathbf{v}) \otimes \mathbf{v}\} = \mathbf{0},\end{aligned} \tag{5.26}$$

Thus the charge four-current vector N^μ and the stress-energy tensor $T^{\mu\nu}$ are represented by $N^\mu \equiv (\mathcal{N}, \mathbf{N})$ and $T^{\mu\nu} \equiv (E, \mathbf{M}, \mathbf{P})$ where

$$\mathcal{N} \equiv N^0 = n\gamma, \tag{5.27}$$
$$\mathbf{N} \equiv n\gamma\mathbf{v} = \mathcal{N}\mathbf{v}, \tag{5.28}$$
$$\begin{aligned}E \equiv\ & T^{00} = (\varepsilon + \mathcal{P})\gamma^2 - \mathcal{P} + 2\gamma\mathbf{q} \cdot \mathbf{v} \\ & + (\boldsymbol{\pi} \cdot \mathbf{v}) \cdot \mathbf{v},\end{aligned} \tag{5.29}$$
$$\begin{aligned}\mathbf{M} \equiv\ & \{T^{0i}\}_{i=x,y,z} = (\varepsilon + \mathcal{P})\gamma^2\mathbf{v} \\ & + \gamma(\mathbf{q} + (\mathbf{q} \cdot \mathbf{v})\mathbf{v}) + \boldsymbol{\pi} \cdot \mathbf{v},\end{aligned} \tag{5.30}$$
$$\begin{aligned}\mathbf{P} \equiv\ & \{T^{ij}\}_{i,j=x,y,z} = (\varepsilon + \mathcal{P})\gamma^2\mathbf{v} \otimes \mathbf{v} - \mathcal{P}\mathbf{g} \\ & + 2\gamma^{(}\mathbf{v} \otimes \mathbf{q}^{)} + \boldsymbol{\pi},\end{aligned} \tag{5.31}$$

with $\mathcal{P} = (p + \Pi)$ the effective pressure. The physical meaning of the components of the energy-momentum stress tensor is as follows: T^{00} is the total energy density of the fluid, T^{0i} is the energy flux density in the ith direction of the flow, T^{i0} is the momentum density in the ith direction, and T^{ij} is the rate of transport of the ith component of the momentum per unit volume through a unit area oriented perpendicular to the jth coordinate axis. For a fluid at rest, every element of area experiences only a force normal to the surface of the element, and this force is independent of the orientation of the element. If the fluid is ideal, then it is nonviscous and will not support tangential stress even when the fluid is in motion. The stress acting across a surface in an ideal fluid is thus always normal to the surface. A nonideal fluid will not support tangential stress when it is at rest, but can do so when it is in motion. The energy-momentum stress tensor has the *viscous stress tensor*. The pressure tensor T^{ij} gives the normal component of the surface force (in the j direction) acting on a surface element that is oriented perpendicular to the ith coordinate axis. The spatial diagonal components of $T^{\mu\nu}$ are the normal stresses, while the off-diagonal spatial components are the tangential stresses (or shearing stresses).

From Eqs. (5.29)–(5.31), we can write $\mathbf{M}$ and $\mathbf{P}$ as

$$\begin{aligned}\mathbf{M} =\ & (E + \mathcal{P})\mathbf{v} + \gamma(\mathbf{q} - (\mathbf{q} \cdot \mathbf{v})\mathbf{v}) \\ & + \boldsymbol{\pi} \cdot \mathbf{v} - ((\boldsymbol{\pi} \cdot \mathbf{v}) \cdot \mathbf{v})\mathbf{v},\end{aligned} \tag{5.32}$$
$$\begin{aligned}\mathbf{P} =\ & \mathbf{M} \otimes \mathbf{v} - \mathcal{P}\mathbf{g} + \gamma(\mathbf{v} \otimes \mathbf{q} - (\mathbf{q} \cdot \mathbf{v})(\mathbf{v} \otimes \mathbf{v})) \\ & + \boldsymbol{\pi} - (\boldsymbol{\pi} \cdot \mathbf{v}) \otimes \mathbf{v},\end{aligned} \tag{5.33}$$

and the local net charge density and energy density are found from

$$\varepsilon = E - (\mathbf{M} + \gamma^{-1}\mathbf{q}) \cdot \mathbf{v}, \tag{5.34}$$

$$n = (1 - \mathbf{v}\cdot\mathbf{v})^{1/2}\mathcal{N}. \tag{5.35}$$

In ideal fluids, one notices that **M** and **v** are parallel, and one can find **v** from the simple expressions [14]. In nonideal fluids, **M** and **v** are in general not parallel. Only in special cases where the fluid velocity components decouple can one still find **M** and **v** parallel and one can still solve for velocity [4]. In the case where there is strong coupling between the velocity components, one has to solve for **v** iteratively from Eqs. (5.32) or from Eqs. (5.34) and (5.35).

C. Maxwell-Cateneo limit of the transport equations in the 3+1 formalism

Because of orthogonality, it suffices to provide transport equations only for spatial components of the dissipative fluxes, i.e., $\{\Pi, \mathbf{q}, \boldsymbol{\pi}\}$. For a particular set, Eqs. (4.1)–(4.3), this yields

$$\gamma\partial_t\Pi + \gamma\mathbf{v}\cdot\boldsymbol{\partial}\Pi = \frac{1}{\tau_\Pi}(\Pi_E - \Pi), \tag{5.36}$$

$$\gamma\partial_t\mathbf{q} + \gamma\mathbf{v}\cdot\boldsymbol{\partial}\mathbf{q} = \frac{1}{\tau_q}(\mathbf{q}_E - \mathbf{q}), \tag{5.37}$$

$$\gamma\partial_t\boldsymbol{\pi} + \gamma\mathbf{v}\cdot\boldsymbol{\partial}\boldsymbol{\pi} = \frac{1}{\tau_\pi}(\boldsymbol{\pi}_E - \boldsymbol{\pi}). \tag{5.38}$$

The complete 3+1 representation of the constraints, Eqs. (4.4)–(4.6) can be written with the help of the orthogonality relations Eqs. (5.20)–(5.23) as

$$\Pi_E = -\zeta\Theta, \tag{5.39}$$

$$\mathbf{q}_E = -\lambda T(\gamma^2\mathbf{v}\partial_t \ln T - \boldsymbol{\nabla}\ln T + \mathbf{a}), \tag{5.40}$$

$$\boldsymbol{\pi}_E = 2\eta\boldsymbol{\sigma}. \tag{5.41}$$

Note that the weak coupling limit of the causal transport equations can be further simplified for special cases such as the scaling solution case. In that case, the last term (acceleration coupled to heat or shear flux) on the right hand side of Eqs. (5.37) and (5.38) vanishes. This is also the case in simple (1+1)-dimensional problems, as we will see in the following sections.

VI. PHYSICAL PROBLEMS

In this section, we consider special cases with emphasis on different geometries (coordinate systems). These cases represent some of the various scenarios one can find in high energy nuclear collisions. In this regard the longitudinal direction is taken to be the z axis in all scenarios. We will consider a pure (1+1)-dimensional expansion along the longitudinal z axis in planar geometry, a (1+1)-dimensional cylindrically symmetric expansion along the radial direction (r axis), and a (2+1)-dimensional expansion along the longitudinal direction (z axis) as well as along the transverse direction with cylindrical symmetry (r axis).

To evaluate the conservation laws $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ in a particular coordinate system, we calculate N^μ and $T^{\mu\nu}$ components in that coordinate system and convert these components into physical components using Eqs. (B2), (B3), (B4), and (B5) from Appendix B. Then we apply Eqs. (B17) and (B10) from Appendix B.

For (1+1)-dimensional expansion along the z axis in planar geometry, derivatives with respect to (x, y) must be identically zero by symmetry. So we need to calculate only terms containing derivatives in (t, z). Similarly, for (1+1)-dimensional spherical symmetric flow, the terms in $(\partial/\partial\theta)$ and $(\partial/\partial\phi)$ vanish identically; while for (1+1)-dimensional cylindrical symmetric flow, the terms in $(\partial/\partial\phi)$ and $(\partial/\partial z)$ also vanish identically. So we need to calculate only terms containing $(\partial/\partial t)$ and $(\partial/\partial r)$.

A. (1+1)-dimensional expansion in planar geometry

In high energy nuclear collisions, one often considers (1+1)-dimensional expansion of the produced matter along the beam or collision axis taken to be the z axis. In this case, there is only one nonvanishing spatial component of the four-velocity. In relativistic ideal fluid dynamics, it is sufficient to decompose the net charge four-current and the energy-momentum stress tensor using only one four-vector, namely, the four-velocity, in addition to the metric tensor. However in relativistic nonideal fluid dynamics due to additional dissipative fluxes that are orthogonal to the four-velocity, the decomposition of the net charge four-current and the energy-momentum stress tensor requires additional four-vectors in order for the decomposed quantities to be expressed in terms of the four-vectors, the metric tensor, and the local scalar quantities. Since in the local rest frame the components of the dissipative fluxes are spatial, the additional four-vectors are spacelike while the four-velocity is timelike. In (1+1)-dimensional expansion, we need only one more four-vector in addition to the four-velocity. Note that the additional four-vectors are not needed if one wants to keep the dissipative fluxes in their laboratory frame tensorial form. Only if we want to write them in their local rest frame scalar form in the energy-momentum stress tensor will we need additional four-vectors. This is possible for simple cases such as (1+1)-dimensional expansions.

From the line element

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \tag{6.1}$$

the Minkowski metric is $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. For a local observer comoving with the fluid, the two four-vectors are

$$\widehat{u}^\mu(t, z, x, y) = (1, 0, 0, 0), \tag{6.2}$$

$$\widehat{m}^\mu(t, z, x, y) = (0, 1, 0, 0), \tag{6.3}$$

in the local rest frame: one is the four-velocity which is timelike, $\widehat{u}_\mu\widehat{u}^\mu = 1$, and the other one is spacelike, $\widehat{m}_\mu\widehat{m}^\mu = -1$, and it is orthogonal to the four-velocity, $\widehat{m}^\mu\widehat{u}_\mu = 0$. Hence the local rest frame is defined by $(\widehat{u}^\mu = \delta^\mu_t, \widehat{m}^\mu = \delta^\mu_x)$. Note that in this simple (1+1)-dimensional case, there is only one independent component of the heat flux, taken to be the z component. In the local rest frame, the heat flux is $\widehat{q}^\mu = (0, \mathcal{Q}^z, 0, 0) = q\delta^\mu_x = q\widehat{m}^\mu$ which we simply denote by q, the heat flow. In the local rest frame, the shear tensor takes the form

$$\widehat{\pi}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau^{zz} & 0 & 0 \\ 0 & 0 & \tau^{xx} & 0 \\ 0 & 0 & 0 & \tau^{yy} \end{pmatrix}, \tag{6.4}$$

where $\tau^{\mu\nu}$ represent the shear stress tensor components in the local rest frame of the fluid. There is also one independent component of the shear tensor in this simple (1+1)-dimensional problem, taken to be the zz component $\widehat{\pi}^{zz} = \tau^{zz} = \pi$, which we simply denote by π. The tracelessness property of the shear stress tensor then implies that in Eq. (6.4), $\widehat{\pi}^{xx} = \tau^{xx} = \widehat{\pi}^{yy} = \tau^{yy} = -\frac{\pi}{2}$. Note also that the bulk viscous pressure enters the nonideal part of the energy-momentum tensor as $-\Pi\Delta^{\mu\nu}$; and in the local rest frame, it is $-\Pi\widehat{\Delta}^{\mu\nu}$, where $\widehat{\Delta}^{\mu\nu} = g^{\mu\nu} - \widehat{u}^{\mu}\widehat{u}^{\nu}$.

The net charge four-current in the local rest frame is

$$\widehat{N}^{\mu} = (n, 0, 0, 0). \tag{6.5}$$

The energy-momentum tensor in the local rest frame is given by

$$\widehat{T}^{\mu\nu} = \begin{pmatrix} \varepsilon & q & 0 & 0 \\ q & \mathcal{P}_{\perp} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{\perp} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{z} \end{pmatrix}, \tag{6.6}$$

where $\mathcal{P}_z = p + \pi + \Pi$, $\mathcal{P}_{\perp} = p - \pi/2 + \Pi$. That is, the net charge four-current and the local rest frame energy-momentum tensor can be decomposed as

$$\widehat{N}^{\mu} = n\widehat{u}^{\mu}, \tag{6.7}$$

$$\widehat{T}^{\mu\nu} = (\varepsilon + \mathcal{P}_{\perp})\widehat{u}^{\mu}\widehat{u}^{\nu} - \mathcal{P}_{\perp}g^{\mu\nu} + (\mathcal{P}_z - \mathcal{P}_{\perp})\widehat{m}^{\mu}\widehat{m}^{\nu} + 2q\widehat{m}^{(\mu}\widehat{u}^{\nu)}, \tag{6.8}$$

The dynamics of the system can be studied by applying a Lorentz boost with v_z in the z direction. Thus the net charge four-current and the energy-momentum stress tensor as measured by an observer with velocity v_z with respect to the fluid configuration are given by

$$N^{\mu} = nu^{\mu}, \tag{6.9}$$

$$T^{\mu\nu} = (\varepsilon + \mathcal{P}_{\perp})u^{\mu}u^{\nu} - \mathcal{P}_{\perp}g^{\mu\nu} + (\mathcal{P}_z - \mathcal{P}_{\perp})m^{\mu}m^{\nu} + 2q\, m^{(\mu}u^{\nu)}, \tag{6.10}$$

where u^{μ} and m^{μ} are given by

$$u^{\mu}(t, z, x, y) = (\gamma, \gamma v_z, 0, 0), \tag{6.11}$$

$$m^{\mu}(t, z, x, y) = (\gamma v_z, \gamma, 0, 0), \tag{6.12}$$

and v_z is the fluid three-velocity in the z direction and $\gamma = (1 - v_z^2)^{-1/2}$. Note that $q^{\mu}u_{\mu} = 0$ and $q = \sqrt{-q^{\mu}q_{\mu}}$. Note also that $m^{\mu} = q^{\mu}/q$. Explicit components of the net charge four-current and the energy-momentum tensor read

$$N^0 = \gamma n, \tag{6.13}$$

$$N^z = N^0 v_z, \tag{6.14}$$

$$T^{00} = \mathcal{W}\gamma^2 - \mathcal{P}_z + 2q\gamma^2 v_z, \tag{6.15}$$

$$T^{0z} = \mathcal{W}\gamma^2 v_z + q\gamma^2(1 + v_z^2), \tag{6.16}$$

$$T^{zz} = \mathcal{W}\gamma^2 v_z^2 + \mathcal{P}_z + 2q\gamma^2 v_z, \tag{6.17}$$

$$T^{xx} = T^{yy} = \mathcal{P}_{\perp}, \tag{6.18}$$

with $\mathcal{W} \equiv \varepsilon + \mathcal{P}_z$. Using Eqs. (6.13), (6.15), and (6.16), the local velocity, energy density, and net charge density can be obtained from

$$N^0 = \gamma n, \quad \varepsilon = T^{00} - (T^{0z} + q)v_z, \qquad T^{0z} = (T^{00} + \mathcal{P}_z)v_z + q. \tag{6.19}$$

That is,

$$v_z = \frac{T^{0z} - q}{T^{00} + \mathcal{P}_z}, \tag{6.20}$$

$$\varepsilon = T^{00} - \frac{T^{0z2} - q^2}{T^{00} + \mathcal{P}_z}, \tag{6.21}$$

$$n = N^0\sqrt{1 - v_z^2}. \tag{6.22}$$

The net charge and the energy-momentum conservation equations can be written as

$$\partial_{\mu}N^{\mu} \equiv 0 \Longrightarrow \partial_t N^0 + \partial_z(N^0 v_z) = 0. \tag{6.23}$$

$$\partial_{\mu}T^{\mu 0} \equiv 0 \Longrightarrow \frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial z}\{(T^{00} + \mathcal{P}_z)v_z + q\} = 0, \tag{6.24}$$

$$\partial_{\mu}T^{\mu z} \equiv 0 \Longrightarrow \frac{\partial}{\partial t}T^{0z} + \frac{\partial}{\partial z}\{(T^{0z} + q)v_z + \mathcal{P}_z\} = 0, \tag{6.25}$$

The Maxwell-Cattaneo transport equations in simple one-dimensional expansion take the form

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\theta, \tag{6.26}$$

$$\tau_q\dot{q} + q = -\kappa T\left(\frac{T'}{T} + a\right), \tag{6.27}$$

$$\tau_{\pi}\dot{\pi} + \pi = -2\cdot 2\eta\sigma, \tag{6.28}$$

where

$$\dot{f} \equiv \left[\gamma\frac{\partial}{\partial t} + \gamma v_z\frac{\partial}{\partial z}\right]f, \tag{6.29}$$

$$f' \equiv \gamma\left[\gamma v_z\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial z}\right]f, \tag{6.30}$$

$$\theta \equiv \frac{\partial\gamma}{\partial t} + \frac{\partial\gamma v_z}{\partial z}, \tag{6.31}$$

$$\sigma = \sigma^{xx} = \frac{1}{3}\theta. \tag{6.32}$$

The calculational frame components of the shear tensor $\pi^{\mu\nu}$ are related to the local rest frame components $\tau^{\mu\nu}$ as

$$\pi^{zz} = \gamma^2\tau^{zz} = \gamma^2\pi, \tag{6.33}$$

$$\pi^{xx} = \pi^{yy} = \tau^{xx} = \tau^{yy} = -\frac{\pi}{2}, \tag{6.34}$$

which can be read directly from Eqs. (6.8) and (6.10). Thus, because of its simplistic structure, we numerically evolve the transverse components (π^{xx} or π^{yy}) since they do not change under the boost and thus they do not involve the Lorentz γ

factors. Thus we numerically solve Eq. (6.28). Then we can obtain the longitudinal component π^{zz} from which the local rest frame quantity τ^{zz} follows from Eq. (6.33). The local rest frame quantity τ^{zz} can also be obtained from the tracelessness of $\pi^{\mu\nu}$, which implies that

$$\tau^{zz} = -(\tau^{xx} + \tau^{yy}). \tag{6.35}$$

B. (2+1)-dimensional axisymmetric expansions

(2+1)-dimensional axisymmetric expansions are (2+1)-dimensional in Cartesian coordinates or planar geometry (the flow four-vectors such as the four-velocity are functions of time and two spatial coordinates), but they are (1+1)-dimensional in one-dimensional cylindrically/spherically symmetric expansions. In this section, we consider systems that exhibit cylindrical symmetry, though the results can be generalized to include spherical (fireball) symmetry for radial expansions. In cylindrical coordinates, all derivatives with respect to the axial and azimuthal directions are zero, and the four-vectors are functions of time and radial coordinates only. We consider the case where the azimuthal and axial components of the four-flow vectors are zero and the only spatial component is the radial component.

Note that since this is effectively a (1+1)-dimensional problem as a result of symmetry considerations, there is only one independent component of heat flux taken to be the radial component which in the local rest frame is $\widehat{q}^{\mu} = (0, \mathcal{Q}^r \boldsymbol{e}_r, 0) = (0, q\boldsymbol{e}_r, 0)$, and there are two independent components of the shear stress tensor taken to be the zz and $\phi\phi$ components in cylindrical polar coordinates. Here, $\boldsymbol{e}_r = (\cos\phi, \sin\phi)$. We will derive the equations for the case with cylindrical symmetry, and then restate important results so that they also apply to spherical symmetry and Cartesian coordinates.

Starting from the planar geometry (where the four-velocity has two nonvanishing spatial components, namely, the x and y components), the local rest frame shear stress tensor is transformed into the cylindrical coordinate form

$$\widehat{\pi}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau^{xx} & \tau^{xy} & 0 \\ 0 & \tau^{yx} & \tau^{yy} & 0 \\ 0 & 0 & 0 & \tau^{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & [\tau^{rr} - \tau^{\phi\phi}]\cos^2\phi + \tau^{\phi\phi} & [\tau^{rr} - \tau^{\phi\phi}]\cos\phi\sin\phi & 0 \\ 0 & [\tau^{rr} - \tau^{\phi\phi}]\cos\phi\sin\phi & [\tau^{rr} - \tau^{\phi\phi}]\sin^2\phi + \tau^{\phi\phi} & 0 \\ 0 & 0 & 0 & \tau^{zz} \end{pmatrix} \tag{6.36}$$

The local rest frame net charge four-current and energy-momentum stress tensor takes the form

$$\widehat{N}^{\mu} = (n, 0, 0, 0), \tag{6.37}$$

$$\widehat{T}^{\mu\nu} = \begin{pmatrix} \varepsilon & \mathcal{Q}^x & \mathcal{Q}^y & 0 \\ \mathcal{Q}^x & \mathcal{P}_{xx} & \mathcal{P}_{xy} & 0 \\ \mathcal{Q}^y & \mathcal{P}_{yx} & \mathcal{P}_{yy} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{zz} \end{pmatrix} \tag{6.38}$$

$$= \begin{pmatrix} \varepsilon & \mathcal{Q}^r\cos\phi & \mathcal{Q}^r\sin\phi & 0 \\ \mathcal{Q}^r\cos\phi & [\mathcal{P}_r - \mathcal{P}_\phi]\cos^2\phi + \mathcal{P}_\phi & [\mathcal{P}_r - \mathcal{P}_\phi]\cos\phi\sin\phi & 0 \\ \mathcal{Q}^r\sin\phi & [\mathcal{P}_r - \mathcal{P}_\phi]\cos\phi\sin\phi & [\mathcal{P}_r - \mathcal{P}_\phi]\sin^2\phi + \mathcal{P}_\phi & 0 \\ 0 & 0 & 0 & \mathcal{P}_z \end{pmatrix}, \tag{6.39}$$

where $\mathcal{P}_r = p + \Pi + \tau^{rr}$, $\mathcal{P}_\phi = p + \Pi + \tau^{\phi\phi}$, $\mathcal{P}_z = p + \Pi + \tau^{zz}$, and $\mathcal{P}_{ij} = p + \Pi + \tau_{ij}$. The pressure tensor is defined such that $\mathcal{P}^{ij}$ is the net rate of transport, per unit area of a surface oriented perpendicular to the jth coordinate axis, of the ith component of momentum.

Since in this (t, x, y, z) coordinate system we have two nonvanishing spatial components of four-velocity, there are now more vectors required to construct a complete set of tensors with respect to which $T^{\mu\nu}$ should be decomposed. The four-vector fields that we need are u^μ, l^μ, and m^μ, which in the local rest frame take the forms

$$\widehat{u}^{\mu}(t, x, y, z) = (1, 0, 0, 0), \tag{6.40}$$

$$\widehat{l}^{\mu}(t, x, y, z) = (0, 0, 0, 1), \tag{6.41}$$

$$\widehat{m}^{\mu}(t, x, y, z) = (0, \boldsymbol{e}_r, 0), \tag{6.42}$$

with the properties $\widehat{u}_\mu\widehat{u}^\mu = 1, \widehat{l}_\mu\widehat{l}^\mu = \widehat{m}^\mu\widehat{m}_\mu = -1$, and $\widehat{u}_\mu\widehat{l}^\mu = \widehat{u}_\mu\widehat{m}^\mu = \widehat{l}_\mu\widehat{m}^\mu = 0$. One vector is timelike and the other two are spacelike. The spacelike vectors are all orthogonal to each other and to the timelike vector. The two additional spacelike four-vectors are now required, since we have flows in two spatial directions. Also, unlike in the ideal fluid dynamics we now have the coupling of the two nonvanishing velocity components as a result of dissipative effects. Thus in the local rest frame, the net charge four-current and the energy-momentum tensor can be decomposed as

$$\widehat{N}^{\mu} = n\widehat{u}^{\mu}, \tag{6.43}$$

$$\widehat{T}^{\mu\nu} = (\varepsilon + \mathcal{P}_{\phi})\widehat{u}^{\mu}\widehat{u}^{\nu} + \mathcal{P}_{\phi}g^{\mu\nu} + (\mathcal{P}_z - \mathcal{P}_{\phi})\widehat{l}^{\mu}\widehat{l}^{\nu} + (\mathcal{P}_r - \mathcal{P}_{\phi})\widehat{m}^{\mu}\widehat{m}^{\nu} + 2\mathcal{Q}^r\widehat{u}^{(\mu}\widehat{m}^{\nu)}. \tag{6.44}$$

Relativistic dynamics of the viscous heat-conducting fluid is accomplished by a Lorentz boost in the radial direction. For an observer moving with velocity v_r in the radial direction with respect to the fluid configuration, the net charge four-current and the energy momentum as measured by such an observer are

$$N^{\mu} = nu^{\mu}, \tag{6.45}$$

$$T^{\mu\nu} = (\varepsilon + \mathcal{P}_{\phi})u^{\mu}u^{\nu} + \mathcal{P}_{\phi}g^{\mu\nu} + (\mathcal{P}_z - \mathcal{P}_{\phi})l^{\mu}l^{\nu} + (\mathcal{P}_r - \mathcal{P}_{\phi})m^{\mu}m^{\nu} + 2\mathcal{Q}^r u^{(\mu}m^{\nu)}, \tag{6.46}$$

where

$$u^{\mu}(t,x,y,z) = (\gamma, \gamma v_r \boldsymbol{e}_r), \tag{6.47}$$

$$l^{\mu}(t,x,y,z) = (\gamma v_r, 0, 0, \gamma), \tag{6.48}$$

$$m^{\mu}(t,x,y,z) = (\gamma v_r, \gamma \boldsymbol{e}_r). \tag{6.49}$$

The relationship between the local rest frame and calculational frame components of the net charge four-current and the energy-momentum stress tensor can be read off from Eqs. (6.44) and (6.46). The components of the energy-momentum tensor in the projected coordinate system are

$$T^{00} = \mathcal{W}\gamma^2 - \mathcal{P}_r + 2\mathcal{Q}^r\gamma^2 v_r, \tag{6.50}$$

$$T^{0r} = \mathcal{W}\gamma^2 v_r + \mathcal{Q}^r\gamma^2\left(1 + v_r^2\right), \tag{6.51}$$

$$T^{rr} = \mathcal{W}\gamma^2 v_r^2 + \mathcal{P}_r + 2\mathcal{Q}^r\gamma^2 v_r, \tag{6.52}$$

$$T^{\phi\phi} = \mathcal{P}_{\phi}, \tag{6.53}$$

$$T^{zz} = \mathcal{P}_z, \tag{6.54}$$

with $\mathcal{W} \equiv \varepsilon + \mathcal{P}_r$, and the usual components are obtained in terms of these as

$$T^{xx} = T^{rr}\cos^2\phi + T^{\phi\phi}\sin^2\phi, \tag{6.55}$$

$$T^{yy} = T^{rr}\sin^2\phi + T^{\phi\phi}\cos^2\phi, \tag{6.56}$$

$$T^{xy} = [T^{rr} - T^{\phi\phi}]\cos\phi\sin\phi, \tag{6.57}$$

$$T^{0x} = T^{0r}\cos\phi, \quad T^{0y} = T^{0r}\sin\phi, \tag{6.58}$$

$$T^{xz} = T^{rz}\cos\phi, \quad T^{yz} = T^{rz}\sin\phi. \tag{6.59}$$

The components of four-vectors and second-rank energy-momentum stress tensor presented here are physical components and are identical to the components measured with respect to an orthonormal tetrad in a curvilinear, in this case cylindrical, coordinate system. The contravariant or covariant components can be obtained from the transformation rules given in Appendix B starting from cylindrical coordinates.

The local velocity, energy density, and net charge density can be obtained from

$$\varepsilon = T^{00} - (T^{0r} + \mathcal{Q}^r)v_r, \qquad T^{0r} = (T^{00} + \mathcal{P}_r)v_r + \mathcal{Q}^r, \quad N^0 = \gamma n, \tag{6.60}$$

that is,

$$v_r = \frac{T^{0r} - \mathcal{Q}^r}{T^{00} + \mathcal{P}_r}, \tag{6.61}$$

$$\varepsilon = T^{00} - \frac{T^{0r\,2} - \mathcal{Q}^{r\,2}}{T^{00} + \mathcal{P}_r}, \tag{6.62}$$

$$n = \left(1 - v_r^2\right)^{1/2}N^0. \tag{6.63}$$

For the space-time derivatives, we use Eq. (B17). The net charge conservation $\partial_{\mu}N^{\mu} = 0$ and the energy-momentum conservation $\partial_{\mu}T^{\mu\nu} = 0$ can be written as

$$\partial_{\mu}N^{\mu} \equiv 0 \Longrightarrow \partial_t N^0 + \frac{1}{r^{\alpha}}\partial_r(r^{\alpha}N^0 v_r) = 0, \tag{6.64}$$

$$\partial_{\mu}T^{\mu 0} \equiv 0 \Longrightarrow \partial_t T^{00} + \frac{1}{r^{\alpha}}\partial_r\{r^{\alpha}T^{r0}\} = 0, \tag{6.65}$$

$$\partial_{\mu}T^{\mu r} \equiv 0 \Longrightarrow \partial_t T^{0r} + \frac{1}{r^{\alpha}}\partial_r\{r^{\alpha}T^{rr}\} - \frac{\alpha}{r}T^{\phi\phi} = 0, \tag{6.66}$$

where $\alpha = 1, 2$ for cylindrical and spherical geometry, respectively. These equations can be simplified and written in a better way, suitable for numerical purposes, as

$$\frac{\partial}{\partial t}N^0 + \frac{\partial}{\partial r}(N^0 v_r) = -\frac{\alpha}{r}N^0 v_r, \tag{6.67}$$

$$\frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial r}\{(T^{00} + \mathcal{P}_r)v_r + q\} = -\frac{\alpha}{r}(T^{00} + \mathcal{P}_r)v_r - \frac{\alpha}{r}q, \tag{6.68}$$

$$\frac{\partial}{\partial t}T^{0r} + \frac{\partial}{\partial r}\{(T^{0r} + q)v_r + \mathcal{P}_r\} = -\frac{\alpha}{r}(T^{0r} + q)v_r - \frac{\alpha}{r}(\mathcal{P}_r - \mathcal{P}_{\phi}). \tag{6.69}$$

Note that the structure of these equations is similar to pure (1+1)-dimensional expansion except for the geometric terms on the right hand side. These terms result from the transformation using Eqs. (B17) and (B10).

For the transport equations, recall that there is only one independent component of the heat flux and two of the shear stress tensor. For the heat flux, this translates into the equation for q, the heat flow. For the shear stress tensor, we choose the zz and $\phi\phi$ components. This is just for convenience, since these choices make the calculations more tractable. The simplicity comes from the fact that these transverse components of the shear stress tensor do not change under the boost and hence they do not pick up the Lorentz γ factors which would otherwise render the solution more difficult. Even though not all the equations are independent, we shall present them all, since depending on the problem under consideration, it may be more advantageous using one set instead of the other. The weak coupling limit of the causal transport equations with the cylindrical symmetry takes the form

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\Theta, \tag{6.70}$$

$$\tau_q \dot{q}^r + q^r = -\kappa T\left(\frac{T'}{T} + a\right), \quad (6.71)$$

$$\tau_\pi \dot{\pi}^{rr} + \pi^{rr} = -2\eta\sigma^{rr}, \quad (6.72)$$

$$\tau_\pi \dot{\pi}^{\phi\phi} + \pi^{\phi\phi} = -2\eta\sigma^{\phi\phi}, \quad (6.73)$$

$$\tau_\pi \dot{\pi}^{zz} + \pi^{zz} = -2\eta\sigma^{zz}, \quad (6.74)$$

where

$$\Theta \equiv \theta + \alpha\frac{\gamma v_r}{r}, \quad (6.75)$$

$$\theta \equiv \frac{\partial}{\partial t}\gamma + \frac{\partial}{\partial r}\gamma v_r, \quad (6.76)$$

$$\dot{f} \equiv \left[\gamma\frac{\partial}{\partial t} + \gamma v_r\frac{\partial}{\partial r}\right] f, \quad (6.77)$$

$$f' \equiv \gamma\left[\gamma v_r\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial r}\right] f, \quad (6.78)$$

$$\sigma^{rr} = -\gamma^2\left[-\alpha\frac{\gamma v_r}{r} + \frac{2}{3}\Theta\right], \quad (6.79)$$

$$\sigma^{\phi\phi} = -\alpha\frac{\gamma v_r}{r} + \frac{1}{3}\Theta, \quad (6.80)$$

$$\sigma^{zz} = \tfrac{1}{3}\Theta. \quad (6.81)$$

The relationship between the calculational frame and the local rest frame components of the shear tensor can be read off from Eqs. (6.46) and (6.44). In particular,

$$q^r = \gamma\mathcal{Q}^r = \gamma q, \quad (6.82)$$

$$\pi^{rr} = \gamma^2\tau^{rr}, \quad (6.83)$$

$$\pi^{zz} = \tau^{zz}, \quad (6.84)$$

$$\pi^{\phi\phi} = \tau^{\phi\phi}. \quad (6.85)$$

The orthogonality and tracelessness of the shear stress tensor imply the following relations, respectively,

$$\pi^{rr}/\gamma_r^2 + \pi^{\phi\phi} + \pi^{zz} = 0, \quad (6.86)$$

$$\tau^{rr} + \tau^{\phi\phi} + \tau^{zz} = 0. \quad (6.87)$$

Including the bulk equation, we only have three transport equations. Again these equations have the same structure as in the (1+1)-dimensional case, except for geometrical terms. Note that due to symmetry in the problem, it can be shown that in Eqs. (3.1)–(3.3),

$$\tau_q\omega^{r\nu}q_\nu = \tau_q\Delta^{r\alpha}\Delta^{\nu\beta}\partial_{[\beta}u_{\alpha]} = \tau_q\Delta^{r\alpha}\partial_{[\beta}u_{\alpha]}q^\beta = 0, \quad (6.88)$$

$$2\eta\pi^{\alpha(\mu}\omega_\alpha^{\nu)} = 0. \quad (6.89)$$

This is also the case with spherically symmetric expansions. One can deduce that in (1+1)-dimensional cylindrically symmetric flow, it is not possible to choose a scalar viscous pressure as is the case in (1+1) planar geometry and (1+1)-dimensional spherically symmetric flows. This is because viscous effects originate from a tensor, which is not, in general, isotropic even for (1+1)-dimensional flows.

In going from the Cartesian coordinate system to the projected coordinate system, one notices that this is similar to having started with the pure (t, r, ϕ, z) cylindrical coordinate system in which the dynamics of the system are obtained by applying a Lorentz boost in the radial direction with

$$u^\mu(t, r, \phi, z) = (\gamma, \gamma v_r, 0, 0), \quad (6.90)$$

$$m^\mu(t, r, \phi, z) = (\gamma v_r, \gamma, 0, 0), \quad (6.91)$$

as done in Appendix A for the spherical symmetry case using the mixed tensor T_ν^μ. Note that since in this case we have only one nonvanishing spatial component of four-velocity, we now need only one additional spatial four-vector for energy-momentum stress tensor decomposition.

C. (3+1)-dimensional axisymmetric expansion: Firebarrel expansion

We now consider the relativistic dynamics of nonideal fluids in firebarrel geometry. (3+1)-dimensional axisymmetric flows are (3+1)-dimensional with respect to Cartesian coordinates, i.e., the velocity components are functions of time and all three spatial coordinates, but they are only (2+1)-dimensional in cylindrical coordinate system. All derivatives with respect to azimuthal direction are zero, and all four-vector components such as the four-velocity components are functions of only the time and axial and radial coordinates (t, z, r). In the case without swirl, the azimuthal velocity component is zero everywhere. As it is much easier to work with 2+1 independent variables than with 3+1, for axisymmetric flows, it makes sense to work in a cylindrical coordinate rather than a Cartesian one.

Let us consider a system undergoing expansion along the longitudinal direction taken to be the z axis and at the same time expanding along the transverse (radial) direction taken to be the r axis. The four-velocity in cylindrical polar coordinates takes the form

$$u^\mu(t, x, y, z) = (\gamma, \gamma v_r\cos\phi, \gamma v_r\sin\phi, \gamma v_z), \quad (6.92)$$

with

$$\gamma = (1 - v^2)^{-1/2}, \quad (6.93)$$

$$v^2 = v_z^2 + v_r^2. \quad (6.94)$$

The net charge density, energy density, and momentum density components read

$$N^0 = \gamma n, \quad (6.95)$$

$$T^{00} = (\varepsilon + \mathcal{P})\gamma^2 - \mathcal{P} + 2\gamma(q^z v_z + q^r v_r) + \pi^{rr} + \pi^{\phi\phi} + \pi^{zz}, \quad (6.96)$$

$$T^{0z} = (\varepsilon + \mathcal{P})\gamma^2 v_z + \gamma q^z\left(1 + v_z^2\right) + q^r\gamma v_r v_z + \pi^{zz}v_z + \pi^{zr}v_r, \quad (6.97)$$

$$T^{0r} = (\varepsilon + \mathcal{P})\gamma^2 v_r + \gamma q^r\left(1 + v_r^2\right) + q^z\gamma v_z v_r + (\pi^{rr} + \pi^{\phi\phi})v_r + \pi^{rz}v_z, \quad (6.98)$$

while the other components needed in the conservation equations can be written in terms of the above equations as

$$N^z = N^0 v_z, \quad N^r = N^0 v_r, \quad (6.99)$$

$$T^{zz} = (\varepsilon + \mathcal{P})\gamma^2 v_z^2 + \mathcal{P} + 2q^z\gamma v_z + \pi^{zz}, \quad (6.100)$$

$$T^{rr} = (\varepsilon + \mathcal{P})\gamma^2 v_r^2 + \mathcal{P} + 2q^r\gamma v_r + \pi^{rr}, \quad (6.101)$$

$$T^{rz} = (\varepsilon + \mathcal{P})\gamma^2 v_r v_z + \gamma(q^z v_r + q^r v_z) + \pi^{rz}, \quad (6.102)$$

$$T^{\phi\phi} = \mathcal{P} + \pi^{\phi\phi}, \tag{6.103}$$
$$T^{xx} = [T^{rr} - T^{\phi\phi}]\cos^2\phi + T^{\phi\phi}, \tag{6.104}$$
$$T^{yy} = [T^{rr} - T^{\phi\phi}]\sin^2\phi + T^{\phi\phi}, \tag{6.105}$$
$$T^{xy} = [T^{rr} - T^{\phi\phi}]\cos\phi\sin\phi, \tag{6.106}$$
$$T^{0x} = T^{0r}\cos\phi, \quad T^{0y} = T^{0r}\sin\phi, \tag{6.107}$$
$$T^{xz} = T^{rz}\cos\phi, \quad T^{yz} = T^{rz}\sin\phi. \tag{6.108}$$

Note that we have replaced π^{00} and π^{0i} by their spatial components using the orthogonality and the tracelessness of the shear tensor $\pi^{\mu\nu}$. In particular,

$$\pi^{00} = \pi^{xx} + \pi^{yy} + \pi^{zz} = \pi^{rr} + \pi^{\phi\phi} + \pi^{zz}, \tag{6.109}$$
$$\pi^{0z} = \pi^{zz}v_z + \pi^{zr}v_r, \tag{6.110}$$
$$\pi^{0r} = [\pi^{rr} + \pi^{\phi\phi}]v_r + \pi^{rz}v_z, \tag{6.111}$$

where the first equation for π^{00} comes from the tracelessness of $\pi^{\mu\nu}$ ($\pi^{\nu}_{\nu} = 0$) and the others come from the orthogonality condition ($\pi^{\mu\nu}u_\nu = 0$). Note also that in the projected coordinate system, one could replace π^{00} by the spatial components using the orthogonality relations giving

$$\pi^{00} = \pi^{rr}v_r^2 + 2\pi^{rz}v_r v_z + \pi^{zz}v_z^2, \tag{6.112}$$

from which the tracelessness conditions now becomes

$$\frac{\pi^{rr} + \pi^{\phi\phi}}{\gamma_r^2} - 2\pi^{rz}v_r v_z + \frac{\pi^{zz}}{\gamma_z^2} = 0. \tag{6.113}$$

For simplicity and in the rest of this section, we will now consider the dynamics of a relativistic viscous non-heat-conducting system in this firebarrel geometry. For numerical purposes, we rewrite the momentum components of the energy-momentum stress tensor in terms of the energy component and the spatial components in terms of the momentum components, that is,

$$T^{0z} = (T^{00} + \mathcal{P} - [\pi^{rr} + \pi^{\phi\phi}])v_z + \pi^{zr}v_r, \tag{6.114}$$
$$T^{0r} = (T^{00} + \mathcal{P} - \pi^{zz})v_r + \pi^{rz}v_z, \tag{6.115}$$
$$T^{rr} = T^{0r}v_r + \mathcal{P} + \pi^{rr}/\gamma_r^2 - \pi^{\phi\phi}v_r^2 - \pi^{rz}v_r v_z, \tag{6.116}$$
$$T^{zz} = T^{0z}v_z + \mathcal{P} + \pi^{zz}/\gamma_z^2 - \pi^{zr}v_z v_r, \tag{6.117}$$
$$T^{zr} = T^{0z}v_r + \pi^{zr}/\gamma_r^2 + \pi^{zz}v_z v_r, \tag{6.118}$$
$$T^{rz} = T^{0r}v_z + \pi^{rz}/\gamma_z^2 - [\pi^{rr} + \pi^{\phi\phi}]v_r v_z. \tag{6.119}$$

In differential form, the (2+1)-dimensional conservation equations for net charge density, energy density, and momentum density written in the cylindrical coordinate system read

$$\partial_\mu N^\mu \equiv \partial_t N^0 + \partial_z N^z + \frac{1}{r}\partial_r\{rN^r\}, \tag{6.120}$$
$$\partial_\mu T^{\mu 0} \equiv \partial_t T^{00} + \partial_z T^{0z} + \frac{1}{r}\partial_r\{rT^{0r}\} = 0, \tag{6.121}$$
$$\partial_\mu T^{\mu z} \equiv \partial_t T^{z0} + \partial_z T^{zz} + \frac{1}{r}\partial_r\{rT^{zr}\} = 0, \tag{6.122}$$
$$\partial_\mu T^{\mu r} \equiv \partial_t T^{r0} + \partial_z T^{rz} + \frac{1}{r}\partial_r\{rT^{rr}\} - \frac{1}{r}T^{\phi\phi} = 0. \tag{6.123}$$

For numerical purposes, we write the above set of conservation equations in terms of the conserved quantities in the calculational frame, namely, the net charge density, energy density, and momentum densities, that is,

$$\partial_t N^0 = -\partial_z(N^0 v_z) - \frac{1}{r}\partial_r\{rN^0 v_r\} = 0, \tag{6.124}$$
$$\partial_t T^{00} = -\partial_z\{T^{00}v_{sz}\} - \partial_z\{T^{00}v_{sr}\} - \frac{1}{r}T^{r0}, \tag{6.125}$$
$$\begin{aligned}\partial_t T^{z0} = &-\partial_z\{(T^{z0}v_z + \mathcal{P} + \pi^{zz}/\gamma_z^2 - \pi^{zr}v_z v_r\} \\ &- \partial_r\{T^{z0}v_r + \pi^{zr}\gamma_r^2 + \pi^{zz}v_z v_r\} \\ &- \frac{1}{r}\{T^{0z}v_r + \pi^{zr}/\gamma_r^2 + \pi^{zz}v_z v_r\},\end{aligned} \tag{6.126}$$
$$\begin{aligned}\partial_t T^{r0} = &-\partial_z\{T^{r0}v_z + \pi^{rz}/\gamma_z^2 - [\pi^{rr} + \pi^{\phi\phi}]v_r v_z\} \\ &- \partial_r\{(T^{r0}v_r + \mathcal{P} + \pi^{rr}/\gamma_r^2 - \pi^{\phi\phi}v_r^2 - \pi^{rz}v_r v_z\} \\ &- \frac{1}{r}\{(T^{r0}v_r + \mathcal{P} + [\pi^{rr} + \pi^{\phi\phi}]/\gamma_r^2 - \pi^{rz}v_r v_z\},\end{aligned} \tag{6.127}$$

where

$$v_{sz} = \frac{T^{0z}}{T^{00}}, \quad v_{sr} = \frac{T^{0r}}{T^{00}}. \tag{6.128}$$

The local velocities can be obtained from simultaneously solving Eqs. (6.114) and (6.115), while the local net charge density and energy density are obtained from

$$n = \mathcal{N}(1 - v^2)^{1/2}, \tag{6.129}$$
$$\varepsilon = T^{00} - T^{0z}v_z - T^{0r}v_r, \tag{6.130}$$

For the transport equations, we need the bulk pressure equation and the equations for the nonvanishing shear stress tensor components that are needed in the conservation equations

$$\partial_t\Pi + [v_z\partial_z + v_r\partial_r]\Pi = -\frac{1}{\gamma\tau_\Pi}(\zeta\Theta - \Pi), \tag{6.131}$$
$$\partial_t\pi^{rr} + [v_z\partial_z + v_r\partial_r]\pi^{rr} = \frac{1}{\gamma\tau_\pi}(2\eta\sigma^{rr} - \pi^{rr}), \tag{6.132}$$
$$\partial_t\pi^{zz} + [v_z\partial_z + v_r\partial_r]\pi^{zz} = \frac{1}{\gamma\tau_\pi}(2\eta\sigma^{zz} - \pi^{zz}), \tag{6.133}$$
$$\partial_t\pi^{rz} + [v_z\partial_z + v_r\partial_r]\pi^{zr} = \frac{1}{\gamma\tau_\pi}(2\eta\sigma^{zr} - \pi^{zr}), \tag{6.134}$$
$$\partial_t\pi^{\phi\phi} + [v_z\partial_z + v_r\partial_r]\pi^{\phi\phi} = \frac{1}{\gamma\tau_\pi}(2\eta\sigma^{\phi\phi} - \pi^{\phi\phi}), \tag{6.135}$$

where

$$\Theta = \partial_t\gamma + \partial_z(\gamma v_z) + \partial_r(\gamma v_r) + \frac{\gamma v_r}{r}, \tag{6.136}$$
$$\sigma^{rr} = \nabla^r(\gamma v_r) + \frac{1}{3}\left(2 + \gamma^2 v_r^2\right)\Theta, \tag{6.137}$$
$$\sigma^{zz} = \nabla^z(\gamma v_z) + \frac{1}{3}\left(1 + \gamma^2 v_z^2\right)\Theta, \tag{6.138}$$
$$\sigma^{rz} = \frac{1}{2}[\nabla^r(\gamma v_z) + \nabla^z(\gamma v_r)] + \frac{1}{3}\gamma^2 v_r v_z\Theta, \tag{6.139}$$
$$\sigma^{\phi\phi} = -\frac{\gamma v_r}{r} + \frac{1}{3}\Theta, \tag{6.140}$$

with

$$\nabla_z = [\partial_z + \gamma^2 v_z(\partial_t + v_z\partial_z + v_r\partial_r)] = -\nabla^z, \tag{6.141}$$
$$\nabla_r = [\partial_r + \gamma^2 v_r(\partial_t + v_z\partial_z + v_r\partial_r)] = -\nabla^r. \tag{6.142}$$

Note that

$$\sigma^{xx} = \sigma^{rr}\cos^2\phi + \sigma^{\phi\phi}\sin^2\phi, \quad (6.143)$$

$$\sigma^{yy} = \sigma^{rr}\sin^2\phi + \sigma^{\phi\phi}\cos^2\phi, \quad (6.144)$$

$$\nabla^x u^x + \nabla^y u^y = \nabla^r \gamma v_r - \frac{\gamma v_r}{r}. \quad (6.145)$$

The local rest frame dissipative fluxes are obtained from the above evolved quantities as

$$\pi^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \widehat{\pi}^{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \tau^{\alpha\beta}. \quad (6.146)$$

Note that due to symmetry considerations in our problem, we only need to solve the bulk pressure equation and the three independent components of the shear stress tensor. In the nonrelativistic limit, the equations for mass density and energy and momentum conservation are

$$\partial_t\rho + \partial_z(\rho v_z) + \frac{1}{r}\partial_r(r\rho v_r) = 0, \quad (6.147)$$

$$\partial_t E + \partial_z\{(E + \mathcal{P}_z)v_z + \pi^{zr}v_r\} + \frac{1}{r}\partial_r\{r[(E + \mathcal{P}_r)v_r + \pi^{rz}v_z]\} = 0, \quad (6.148)$$

$$\partial_t \mathcal{M}^z + \partial_z\{\mathcal{M}^z v_z + \mathcal{P}_z\} + \frac{1}{r}\partial_r\{r(\mathcal{M}^z v_r + \pi^{zr})\} = 0, \quad (6.149)$$

$$\partial_t \mathcal{M}^r + \partial_z\{\mathcal{M}^r v_z + \pi^{rz}\} + \frac{1}{r}\partial_r\{r[\mathcal{M}^r v_r + \mathcal{P}_r]\} - \frac{1}{r}\mathcal{P}_\phi = 0, \quad (6.150)$$

where

$$E = \epsilon + \frac{1}{2}\rho\left(v_z^2 + v_r^2\right), \quad (6.151)$$

$$\epsilon = \frac{1}{\Gamma - 1}p, \quad (6.152)$$

$$\mathcal{M}^z = \rho v_z, \quad (6.153)$$

$$\mathcal{M}^r = \rho v_r, \quad (6.154)$$

$$\mathcal{P}_r = p + \Pi + \pi^{rr}, \quad (6.155)$$

$$\mathcal{P}_z = p + \Pi + \pi^{zz}, \quad (6.156)$$

$$\mathcal{P}_\phi = p + \Pi + \pi^{\phi\phi}. \quad (6.157)$$

The weak coupling limit of the causal transport equations in nonrelativistic limit takes the form

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\,\boldsymbol{\partial}\cdot\mathbf{v}, \quad (6.158)$$

$$\tau_q \dot{\mathbf{q}} + \mathbf{q} = -\lambda\boldsymbol{\partial}T, \quad (6.159)$$

$$\tau_\pi \dot{\boldsymbol{\pi}} + \boldsymbol{\pi} = -2\eta\boldsymbol{\sigma}, \quad (6.160)$$

where here the upper dot stands for the material or Lagrangian time derivative (i.e., $d/dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\partial}$). The explicit transport equations for the nonvanishing shear tensor components in the weak coupling limit are

$$\partial_t\pi^{zz} + [v_z\partial_z + v_r\partial_r]\pi^{zz} = -\frac{1}{\tau_\pi}(2\eta\sigma^{zz} + \pi^{zz}), \quad (6.161)$$

$$\partial_t\pi^{rr} + [v_z\partial_z + v_r\partial_r]\pi^{rr} = -\frac{1}{\tau_\pi}(2\eta\sigma^{rr} + \pi^{rr}), \quad (6.162)$$

$$\partial_t\pi^{\phi\phi} + [v_z\partial_z + v_r\partial_r]\pi^{\phi\phi} = -\frac{1}{\tau_\pi}(2\eta\sigma^{\phi\phi} + \pi^{\phi\phi}), \quad (6.163)$$

$$\partial_t\pi^{rz} + [v_z\partial_z + v_r\partial_r]\pi^{rz} = -\frac{1}{\tau_\pi}(2\eta\sigma^{rz} + \pi^{rz}), \quad (6.164)$$

with

$$\sigma^{zz} = \partial_z v_z - \tfrac{1}{3}\boldsymbol{\partial}\cdot\mathbf{v}, \quad (6.165)$$

$$\sigma^{rr} = \partial_r v_r - \tfrac{1}{3}\boldsymbol{\partial}\cdot\mathbf{v}, \quad (6.166)$$

$$\sigma^{\phi\phi} = -\frac{v_r}{r} - \tfrac{1}{3}\boldsymbol{\partial}\cdot\mathbf{v}, \quad (6.167)$$

$$\sigma^{zr} = \tfrac{1}{2}(\partial_r v_z + \partial_z v_r). \quad (6.168)$$

For numerical purposes, we write the conservation equations, Eqs. (6.147)–(6.150), as

$$\partial_t\mathcal{N} + \partial_z(\mathcal{N}v_z) + \partial_r(\mathcal{N}v_r) = -\frac{\mathcal{N}}{r}, \quad (6.169)$$

$$\partial_t T^{00} + \partial_z\{T^{00}v_{sz}\} + \partial_r\{T^{00}v_{sr}\} = -\frac{T^{0r}}{r}, \quad (6.170)$$

$$\partial_t T^{0z} + \partial_z\{T^{0z}v_z + \mathcal{P}_z\} + \partial_r\{T^{0z}v_r + \pi^{zr}\} = -\frac{(T^{0z}v_r - \pi^{zr})}{r}, \quad (6.171)$$

$$\partial_t T^{0r} + \partial_z\{T^{0r}v_z + \pi^{rz}\} + \partial_r\{T^{0r}v_r + \mathcal{P}_r\} = -\frac{[T^{0r}v_r + \mathcal{P}_r - \mathcal{P}_\phi]}{r}, \quad (6.172)$$

where

$$\mathcal{N} = \rho, \quad T^{00} = E, \quad T^{0z} = \mathcal{M}^z, \quad T^{0r} = \mathcal{M}^r, \quad (6.173)$$

$$v_{sz} = \frac{T^{0z}}{T^{00}}, \quad v_z = \frac{T^{0z}}{\mathcal{N}}, \quad (6.174)$$

$$v_{sr} = \frac{T^{0r}}{T^{00}}, \quad v_r = \frac{T^{0r}}{\mathcal{N}}. \quad (6.175)$$

In formulating the relativistic equations in this section, we decided to keep the laboratory frame quantities for the dissipative fluxes. Sometimes the local rest frame equations are more complicated than the laboratory frame equations. The local rest frame in this case of an effective (2+1)-dimensional flow contains Lorentz contraction/dilation factors and higher powers of velocity, which make the calculations and computations more difficult. The transport/relaxation equations are readily written in their laboratory frame, and if one wants the corresponding local rest frame quantities, one makes the necessary Lorentz transformation. In the nonrelativistic limit, the equations are cast in laboratory frame quantities.

VII. NUMERICAL ASPECTS OF RELATIVISTIC NONIDEAL FLUID DYNAMICS

The equations of relativistic nonideal fluid dynamics are truly formidable; indeed, they have never been solved for nontrivial problems except in simplified cases such as scaling solutions [3]. Analytic solutions of relativistic fluid dynamics are rare, and in most cases of interest we must use numerical methods [4].

The complicated structure of relativistic dissipative fluid dynamics equations makes it unattractive to use in modeling relativistic nuclear collisions. Unlike those of nonrelativistic dissipative fluid dynamics, the relativistic equations have time derivatives appearing in more than one place. This makes computation more expensive. Otherwise, the system of equations given here is readily solved by existing numerical schemes used to solve fluid dynamics problems. Knowledge of the equation of state, initial conditions, and transport coefficients is needed to completely solve nonideal fluid dynamics. Once these are obtained consistently, one would then be in a position to perform a thorough study of the effects of dissipation in nuclear collisions.

In this section, we discuss basic aspects of numerical solution schemes for relativistic fluid dynamics. For the sake of simplicity, we consider the case of one conserved charge only. With the definitions of Eqs. (5.27)–(5.31) and the conservation laws of Eqs. (5.24)–(5.26), the conservation equations can be solved numerically with any scheme that also solves the nonrelativistic conservation equations. There is, however, one fundamental difference between the nonrelativistic equations and the relativistic ones. To solve the latter for $\mathcal{N}$, E, and $\mathbf{M}$, the net charge density, energy density, and momentum density in the calculational frame, one has to know the equation of state $p(\varepsilon, n)$ and $\mathbf{v}$. The equation of state, however, depends on n and ε, the net charge density and energy density in the local rest frame of the fluid. One therefore has to locally transform from the calculational frame to the rest frame of the fluid in order to extract $n, \varepsilon, \mathbf{M}$ from $\mathcal{N}, E, \mathbf{v}$. In the nonrelativistic limit, there is no difference between n and $\mathcal{N}$ or between ε and E, and the equation of state can be employed directly in the conservation equations. Also, the momentum density of the fluid is related to the fluid velocity by a simple expression. The transformation between local rest frame and calculational frame quantities is described explicitly below.

A. Mathematical structure of the equations of nonideal fluid dynamics

In Sec. V, we formulated the equations of nonideal fluid dynamics for (1+1)-dimensional expansion in planar geometry, (1+1)-dimensional cylindrical symmetric expansion [(1+1)-dimensional spherical symmetric expansion is discussed in Appendix A], and (2+1)-dimensional expansion with cylindrical symmetry. We now ask how to solve them. In this connection, it is instructive to count the number of variables to be determined and the number of equations available to determine them.

In nonideal relativistic fluid dynamics, we must find six fluid variables: n, ε, p, and three components of the ordinary velocity $\mathbf{v}$. In addition, we must now find nine dissipative variables: the bulk pressure Π, the three components of heat flow $\mathbf{q}$, and the five independent (nonredundant) components of the shear tensor $\boldsymbol{\pi}$. These 14 variables are related by 14 partial differential equations governing the flow of relativistic nonideal fluid: the five conservation equations which are the net charge conservation equation, the energy conservation equation, and the three components of momentum conservation equation. In addition, we now have nine transport/relaxation equations for the dissipative variables, namely, the bulk pressure equation, the three components of the heat flux equation, and the five independent components of the shear tensor equation. We also have constitutive relations: we choose the equation of state to be $p \equiv p(n, \varepsilon)$.

For (1+1)-dimensional expansions, we need to determine only seven variables: n, ε, p, v, Π, q, and π from a total of six differential equations and an equation of state. The number of variables increases with the number of spatial dimensions. The system of 14 unknowns and 14 equations (the 14-field theory) can be solved for the spatial variations of all unknowns as a function of time once we are given a set of initial conditions that specify the state and motion of the fluid at a particular time, plus a set of boundary conditions that place constraints on the flow.

The techniques for solving these nonlinear equations are under investigation and will be presented elsewhere. Analytical methods can yield solutions for some simplified problems, for example, (1+1)-dimensional flow with scaling solution [3]. But in general, this approach is too restrictive, and we should resort to numerical methods for a more realistic description. One effective numerical method of solving equations of fluid dynamics is to replace the original differential equations by a set of finite difference equations that determine the physical properties of the fluid on discrete space and time meshes.

Given suitable initial and boundary conditions, we follow the evolution of the fluid by solving this discrete algebraic system at successive time steps. We must make sure that the finite difference equations are numerically stable and must find an efficient scheme for handling shocks, which can produce discontinuities in the solution or between mesh points. We need numerical techniques, which not only can handle the full nonlinear equations, but also are versatile and flexible enough to (1) permit a detailed description of the microphysics of the gas, (2) allow for structural complexities in the transient regimes, (3) be generalized easily to include various processes of dissipation, and (4) account for departures from local thermodynamic equilibrium. This is a subject of current investigation and will be presented elsewhere.

B. Transformation between calculation frame and fluid rest frame

In principle, the transformation is explicitly given by Eqs. (5.27)–(5.31), i.e., by finding the roots of a set of 14 nonlinear equations [the nonlinearity enters through the equation of state $p(\varepsilon, n)$]. In numerical applications, however, this transformation has to be done several times in each time step and each cell. It is therefore advisable to reduce the complexity of the transformation problem. Boosting to a frame moving with velocity $-\mathbf{v}$ (the collective velocity of the fluid is $\mathbf{v}$), the calculational and local rest frame transformations for the heat flux and shear tensor are given by

$$\pi^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\widehat{\pi}^{\alpha\beta}, \tag{7.1}$$

$$q^{\mu} = \Lambda^{\mu}_{\alpha}\widehat{q}^{\alpha}, \tag{7.2}$$

where

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & \gamma \mathbf{v}^{\mathrm{T}} \\ \gamma \mathbf{v} & \mathrm{I} + (\gamma - 1) v^{-2} \mathbf{v}\mathbf{v}^{\mathrm{T}} \end{pmatrix}, \tag{7.3}$$

$$\widehat{q}^{\mu} = (0, \mathcal{Q}^x, \mathcal{Q}^y, \mathcal{Q}^z), \tag{7.4}$$

$$\widehat{\pi}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau^{xx} & \tau^{xy} & \tau^{xz} \\ 0 & \tau^{yx} & \tau^{yy} & \tau^{yz} \\ 0 & \tau^{zx} & \tau^{zy} & \tau^{zz} \end{pmatrix}, \tag{7.5}$$

while the bulk pressure Π transforms like the isotropic pressure p. In Eq. (7.3), $\mathbf{v}$ is a 3×1 column matrix, $\mathbf{v}^{\mathrm{T}}$ is the transpose of $\mathbf{v}$, and I is a 3×3 identity matrix.

First note that unlike the case in relativistic ideal fluid dynamics, $\mathbf{M}$ and $\mathbf{v}$ are in general no longer parallel. In the ideal case, $\mathbf{M}$ and $\mathbf{v}$ are in parallel and

$$\mathbf{M} \cdot \mathbf{v} \equiv M v = E - \varepsilon, \tag{7.6}$$

where $M \equiv |\mathbf{M}|$ and $v \equiv |\mathbf{v}|$. One then obtains the relationship between ε and E

$$\varepsilon = E - M v, \tag{7.7}$$

while the relationship between n and $\mathcal{N}$,

$$n = \mathcal{N}\sqrt{1 - v^2}, \tag{7.8}$$

is the same as for nonideal fluid dynamics, because we are using the Eckart definition of the four-velocity. With these equations, ε and n can be expressed in terms of $\mathcal{N}$, E, M, and v. The five-dimensional root search is therefore reduced to finding the modulus of v for given $\mathcal{N}$, E, and M, which is a simple one-dimensional problem. To solve this, one uses the definition of M,

$$M = (E + p) v. \tag{7.9}$$

This equation can be rewritten as a fixed point equation for v for given $\mathcal{N}$, E, M:

$$v = \frac{M}{E + p(E - Mv, \mathcal{N}\sqrt{1 - v^2})}. \tag{7.10}$$

The fixed point yields the modulus of the fluid velocity, from which one can reconstruct $\mathbf{v} = v\mathbf{M}/M$, and find ε and n via Eq. (7.7). The equation of state $p(\varepsilon, n)$ then yields the final unknown variable, the pressure p.

Alternatively, one can use Eq. (7.10) to rewrite ε and n as

$$\varepsilon = E - \frac{\mathbf{M}^2}{E + p}, \tag{7.11}$$

$$n = \mathcal{N}\left[1 - \frac{\mathbf{M}^2}{(E + p)^2}\right]^{1/2} = \mathcal{N}\left[\frac{\varepsilon + p}{E + p}\right]^{1/2}, \tag{7.12}$$

from which we can get ε and n by simultaneously solving

$$f(\varepsilon, n) = (E - \varepsilon)(E + p) - \mathbf{M}^2 = 0, \tag{7.13}$$

$$g(\varepsilon, n) = (\mathcal{N}^2 - n^2)(E + p)^2 - \mathcal{N}^2\mathbf{M}^2 = 0, \tag{7.14}$$

and an equation of state $p(\varepsilon, n)$ then yields the final unknown variable, the pressure p.

However, in the nonideal case, dissipative effects brings complications to the mathematical structure of the problem. But in some cases such as the (1+1)-dimensional expansions where the heat flux and the shear tensor can still be replaced by some scalar quantities, as done in Sec. VI, one can still generalize the above results. For example, in the 1+1 radial expansion,

$$v_r = \frac{M^r - \mathcal{Q}^r}{(E + \mathcal{P}_r)}, \tag{7.15}$$

$$\varepsilon = E - \frac{M^{r\,2}}{E + \mathcal{P}_r}, \tag{7.16}$$

$$n = \mathcal{N}\left[1 - \frac{M^{r\,2}}{(E + \mathcal{P}_r)^2}\right]^{1/2}, \tag{7.17}$$

from which we can get ε and n by simultaneously solving

$$f(\varepsilon, n) = (E - \varepsilon)(E + \mathcal{P}_r) - [M^{r\,2} - \mathcal{Q}^{r\,2}] = 0, \tag{7.18}$$

$$g(\varepsilon, n) = (\mathcal{N}^2 - n^2)(E + \mathcal{P}_r)^2 - \mathcal{N}^2[M^r - \mathcal{Q}^r]^2 = 0, \tag{7.19}$$

where $\mathcal{P}_r = p(\varepsilon, n) + \Pi + \tau^{rr}$, and equation of state $p(\varepsilon, n)$ then yields the final unknown variable, the pressure p. In addition, one needs the transformations for the heat flux and the shear stress tensor, which in this case are given by

$$q^r = \gamma \mathcal{Q}^r, \tag{7.20}$$

$$\pi^{rr} = \gamma^2 \tau^{rr}, \tag{7.21}$$

where the local rest frame heat flow is $\mathcal{Q}^r$ and the local rest frame shear tensor is τ^{rr}.

The complications begin to arise when one moves away from the simple cases such as the one space dimensional expansions. In two and three space dimensional expansions, there are now more than one nonvanishing spatial components of four-velocity flow. The dissipative fluxes, in particular the shear tensor, strongly couple the various components of the nonvanishing spatial components of the four-velocity, thus making the momentum components to be not in parallel with the corresponding nonvanishing spatial components of the four-velocity. In finding the solution to this problem numerically, the alternative method discussed above for finding the local velocity, energy density, and the net charge density is no longer plausible. One then uses the generalized first method. In this method, the local velocities are calculated iteratively from Eqs. (5.32) and then the local energy density and net charge density from Eqs. (5.34) and (5.35). In addition, the transformation of the heat flux and the shear tensor should be calculated from Eqs. (7.1) and (7.2). But the latter transformation is not necessary if the energy-momentum stress tensor is written in terms of the calculational frame dissipative fluxes, since their equations are already in this frame. In more complex situations, this will be preferred to avoid further difficulties. The calculational frame is then the natural frame for the dissipative fluxes. In the nonrelativistic limit, the equations should reduce to the laboratory frame equations of nonideal Newtonian fluids, as shown in this work (see Appendix C).

C. Numerical schemes

To handle the coupled system of the partial differential equations presented here, we replace the differential equations by suitable discrete approximations and solve these numerically. In general, to model a heavy ion collision with nonideal fluid dynamics requires solving the five conservation equations and nine transport/relaxation equations in three space dimensions. Since this is in general a formidable numerical task, one should resort to the so-called operator splitting method; i.e., the full three-dimensional solution is constructed by solving sequentially three one-dimensional problems. More explicitly, all conservation equations are of the type

$$\partial_t U + \sum_{i=x,y,z} \partial_i F_i(U) = 0, \qquad (7.22)$$

U being $\mathcal{N}, E$, or $\mathbf{M}$ and $F(U)$ being $\mathbf{N}$ [Eq. (5.27)], $\mathbf{M}$ [Eq. (5.32)], or $\mathbf{P}$ [Eq. (5.33)]. Such an equation is numerically solved on a space-time grid, and time and space derivatives are replaced by finite differences. The solution to the partial differential equation (7.22) in three space dimensions is obtained by solving a sequence of one space dimension partial differential equations

$$\partial_t U + \partial_i F_i(U) = 0, \qquad (7.23)$$

where $i = x$. The three-divergence operator in Eq. (7.22) is split into a sequence of three partial derivative operators. Physically speaking, in a given time step, one first propagates the fields in the x direction, then in the y direction, and then in the z direction. In actual numerical applications, it is advisable to permutate the order xyz to minimize systematic errors. The advantage of the operator splitting method is that there exists a variety of numerical algorithms that solve evolution equations of the type Eq. (7.23) in one space dimension (see, for instance, Ref. [23] and references therein).

The transport/relaxation equations on the other hand are of the convective type

$$\partial_t U + \sum_{i=x,y,z} v_i \partial_i U = F_i(U), \qquad (7.24)$$

U being Π, $\mathbf{q}$, or $\boldsymbol{\pi}$, and $F(U)$ being $(\Pi_E - \Pi)/\gamma\tau_\Pi$, $(\mathbf{q}_E - \mathbf{q})/\gamma\tau_q$, or $(\boldsymbol{\pi}_E - \boldsymbol{\pi})/\gamma\tau_\pi$ in the weak coupling limit of the causal transport/relaxation equations. These equations are of nonconservative convective type. They should be solved by numerical schemes which solve the advective equations of the form

$$\partial_t U + \mathbf{v}.\boldsymbol{\partial} U = \text{Sources} \qquad (7.25)$$

rather than the conservative form

$$\partial_t U + \boldsymbol{\partial}\{U\mathbf{v}\} = \text{Sources} \qquad (7.26)$$

unless the transport equations are recast in the form of Eq. (7.26). The actual numerical schemes for future calculations are the subject of current investigation and will be presented elsewhere.

VIII. SUMMARY AND DISCUSSION

We have provided a complete set of equations for relativistic dissipative hydrodynamics in their 3+1 representation. Causality has been accounted for by using the extended causal description of thermodynamics, relativistically formulated on a phenomenological level. In contrast to the conventionally used compressible Navier-Stokes description of nonideal hydrodynamics, the equations of extended causal thermodynamics guarantee finite propagation speeds of heat and viscous signals and yield stable local thermodynamic equilibria.

The extended theories provide a model for evolving dissipative fluids in a causal and stable manner. In addition, these theories allow one to describe physical systems which cannot be modeled by Navier-Stokes-Fourier equations, in particular, situations where the relaxation times are not equal to the microscopic collision time.

In their simplest form (Maxwell-Cattaneo limit), these transport equations describe the relaxation on finite time scales of dissipative fluxes toward the standard Navier-Stokes-Fourier equations, which guarantees finite signal propagation.

A causality preserving formulation is required whenever the thermodynamic time scale becomes comparable to the dynamical time scale, and therefore the assumption of local thermodynamic equilibrium is not justified. In many problems of interest, the inertia due to the dissipative contributions to the energy-momentum stress tensor can be neglected. The 3+1 formulation of the corresponding simplified set of equations is given.

The five conservation laws for particle number, energy, and momentum, and the nine evolution equations for the thermodynamic fluxes (i.e., the 14-field theory of relativistic nonideal fluid dynamics) form a hyperbolic system of first-order partial differential equations tractable by numerical methods.

Ideally one would like to be able to extract the transport coefficients and associated time and length scales from a particular model of an equation of state (interactions among the constituents), and then study the effects of dissipative nonequilibrium processes on the space-time evolution of the system and on the calculated distributions of particles. And finally one would like to compare the predicted distributions with those observed in experiments.

Finally it is noted that nonideal fluid dynamics may provide a better way of combining the hydrodynamic approach and the kinetic (sequential scattering) approach required for a realistic description of high energy nuclear collisions.

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APPENDIX A: SPHERICALLY SYMMETRIC FLOW: FIREBALL EXPANSION

For a one-dimensional, spherical symmetric flow, the terms in $(\partial/\partial\theta)$ and $(\partial/\partial\phi)$ vanish identically. When viewed by an observer moving relative to the new coordinates with proper velocity v_r in the radial direction, the physical content of space consists of an anisotropic fluid of energy density ε, radial pressure $\mathcal{P}_r$, tangential pressure $\mathcal{P}_\perp$, and radial heat flux q. Thus when viewed by this comoving observer, the covariant net charge four-current and the energy-momentum tensor in Minkowski coordinates are

$$\hat{N}^\mu = (n, 0, 0, 0), \tag{A1}$$

$$\hat{T}^{\mu\nu} = \begin{pmatrix} \varepsilon & q & 0 & 0 \\ q & \mathcal{P}_r & 0 & 0 \\ 0 & 0 & \mathcal{P}_\perp & 0 \\ 0 & 0 & 0 & \mathcal{P}_\perp \end{pmatrix}. \tag{A2}$$

Then a Lorentz transformation leads to

$$N^\mu = n u^\mu, \tag{A3}$$
$$T^\mu_\nu = \varepsilon u^\mu u_\nu - \mathcal{P}_{\rm eff}\Delta^\mu_\nu + (\mathcal{P}_r - \mathcal{P}_\perp)\big[m^\mu m_\nu + \tfrac{1}{3}\Delta^\mu_\nu\big] + q(m^\mu u_\nu + m_\nu u^\mu), \tag{A4}$$

with $\Delta^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu$, $\mathcal{P}_{\rm eff} = \frac{1}{3}(\mathcal{P}_r + 2\mathcal{P}_\perp)$, $\mathcal{P}_r = p + \Pi + \pi$, $\mathcal{P}_\perp = p + \Pi - \pi/2$, $u^\mu = (\gamma, \gamma v_r, 0, 0)$, and $m^\mu = (\gamma v_r, \gamma, 0, 0)$. The heat four-current and the pressure tensor can be written, respectively, as

$$q^\mu = q m^\mu, \tag{A5}$$
$$P^\mu_\nu = -\mathcal{P}_{\rm eff}\Delta^\mu_\nu + (\mathcal{P}_r - \mathcal{P}_\perp)\big[m^\mu m_\nu + \tfrac{1}{3}\Delta^\mu_\nu\big]. \tag{A6}$$

The local rest frame net charge four-current and the energy-momentum stress tensor are obtained by boosting back with $u^\mu(t,r,\phi,\theta) = \widehat{u}^\mu(t,r,\phi,\theta) = (1,0,0,0)$ and $m^\mu(t,r,\phi,\theta) = \widehat{m}^\mu(t,r,\phi,\theta) = (0,1,0,0)$. The nonvanishing components of the net charge and the energy-momentum tensor are

$$N^0 = n\gamma, \quad N^r = n\gamma v_r, \tag{A7}$$

$$T^0_0 = \mathcal{W}\gamma^2\mathcal{P}_r + 2q\gamma^2 v_r = T^{00}, \tag{A8}$$

$$-T^0_r = \mathcal{W}\gamma^2 v_r + q\gamma^2\big(1 + v_r^2\big) = T^r_0 = T^{0r}, \tag{A9}$$

$$-T^r_r = \mathcal{W}\gamma^2 v_r^2 + \mathcal{P}_r + 2q\gamma^2 v_r = T^{rr}, \tag{A10}$$

$$-T^\phi_\phi = -T^\theta_\theta = \mathcal{P}_\perp, \tag{A11}$$

with $\mathcal{W} \equiv \varepsilon + \mathcal{P}_r$. The local velocity, energy density, and net charge density are obtained from

$$N^0 = \gamma n, \quad \varepsilon = T^{00} - (T^{0r} + q)v_r, \quad T^{0r} = (T^{00} + \mathcal{P}_r)v_r + q, \tag{A12}$$

that is,

$$v_r = \frac{T^{0r} - q}{T^{00} + \mathcal{P}_r}, \tag{A13}$$

$$\varepsilon = T^{00} - \frac{T^{0r\,2} - q^2}{T^{00} + \mathcal{P}_r}, \tag{A14}$$

$$n = \big(1 - v_r^2\big)^{1/2} N^0. \tag{A15}$$

The net charge conservation $\partial_\mu N^\mu = 0$ and the energy-momentum conservation $\partial_\mu T^\mu_\nu = 0$ can be written as

$$\partial_\mu N^\mu \equiv 0 \Longrightarrow \partial_t N^0 + \frac{1}{r^\alpha}\partial_r\{r^\alpha N^0 v_r\} = 0, \tag{A16}$$

$$\partial_\mu T^\mu_0 \equiv 0 \Longrightarrow \partial_t T^{00} + \frac{1}{r^\alpha}\partial_r\{r^\alpha T^{r0}\} = 0, \tag{A17}$$

$$\partial_\mu T^\mu_r \equiv 0 \Longrightarrow \partial_t T^{0r} + \frac{1}{r^\alpha}\partial_r\{r^\alpha T^{rr}\} - \frac{\alpha}{r}T^\phi_\phi = 0, \tag{A18}$$

where $\alpha = 2$ for spherical geometry. For numerical purposes, we recast the above equations as follows:

$$\partial_t N^0 + \partial_r\{N^0 v_r\} = \frac{\alpha}{r}N^0 v_r, \tag{A19}$$

$$\partial_t T^{00} + \partial_r\{(T^{00} + \mathcal{P}_r)v_r + q\} = -\alpha\frac{v_r}{r}(T^{00} + \mathcal{P}_r) - \alpha\frac{q}{r}, \tag{A20}$$

$$\partial_t T^{0r} + \partial_r\{(T^{0r} + q)v_r + \mathcal{P}_r\} = -\alpha\frac{v_r}{r}(T^{0r} + q) - \frac{\alpha}{r}(\mathcal{P}_r - \mathcal{P}_\perp). \tag{A21}$$

The structure of these equations is similar to the one-dimensional case with additional geometrical terms.

For the transport equations, recall that there is only one independent component of the heat flux and of the shear stress tensor. For the heat flux, this translates into the equation for q, the heat flow. For the shear stress tensor, we choose either the $\phi\phi$ or the $\theta\theta$ components. This is just for convenience since these choices make the calculations more tractable. The simplicity comes from the property that these transverse components of the shear stress tensor do not change under the boost and hence they do not pick up the velocity or the Lorentz γ factors which would otherwise render the solution more difficult. Thus we see that in (1+1)-dimensional spherical symmetric flows, all the dissipative fluxes can be represented by scalar quantities. In the case in which there are no viscous/heat couplings (α_i's and l_A's are zero), the weak coupling limit of the causal transport equations in the cylindrical and fireball geometry takes the form

$$\tau_\Pi\dot{\Pi} + \Pi = -\zeta\Theta - \frac{1}{2}\Pi\left(\tau_\Pi\Theta + \zeta T\left(\frac{\tau_\Pi}{\zeta T}\right)^{\cdot}\right), \tag{A22}$$

$$\tau_q\dot{q} + q = -\kappa T\left(\frac{T'}{T} + a\right) - \frac{1}{2}q\left(\tau_q\Theta - \kappa T^2\left(\frac{\tau_q}{\kappa T^2}\right)^{\cdot}\right), \tag{A23}$$

$$\tau_\pi\dot{\pi} + \pi = -2\cdot 2\eta\sigma - \frac{1}{2}\pi\left(\tau_\pi\Theta + \eta T\left(\frac{\tau_\pi}{\eta T}\right)^{\cdot}\right), \tag{A24}$$

where

$$\sigma = \left[-\frac{\gamma v_r}{r} + \frac{1}{3}\Theta\right], \tag{A25}$$

$$\dot{f} \equiv \left[\gamma\frac{\partial}{\partial t} + \gamma v_r\frac{\partial}{\partial r}\right] f, \tag{A26}$$

$$f' \equiv \gamma\left[\gamma v_r\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial r}\right] f, \tag{A27}$$

$$\Theta \equiv \theta + \alpha \frac{\gamma v_r}{r}, \tag{A28}$$

$$\theta \equiv \frac{\partial}{\partial t}\gamma + \frac{\partial}{\partial r}\gamma v_r. \tag{A29}$$

The relationship between the calculational frame and the local rest frame components of the shear tensor can be read from Eqs. (A4) and (A2). In particular,

$$q^r = \gamma \mathcal{Q}^r = \gamma q, \tag{A30}$$

$$\pi^{rr} = \gamma^2 \tau^{rr} = \gamma^2 \pi, \tag{A31}$$

$$\pi^\theta_\theta = \tau^{\theta\theta} = -\frac{\pi}{2}, \tag{A32}$$

$$\pi^\phi_\phi = \tau^{\phi\phi} = -\frac{\pi}{2}. \tag{A33}$$

Including the bulk equation, we only have three transport equations. Again these equations have the same structure as in the one-dimensional case, except for geometrical terms.

From the energy-momentum conservation, $\partial_\mu T^\mu_\nu = 0$, in the calculational frame we have

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + \mathcal{P}_{\rm eff})\Theta + \partial_\mu q^\mu = (\mathcal{P}_r - \mathcal{P}_\perp)\left[m_\mu m_\nu + \frac{1}{3}\Delta_{\mu\nu}\right]\sigma^{\mu\nu} + q a^\mu m_\mu, \tag{A34}$$

$$(\varepsilon + \mathcal{P}_{\rm eff})a^\mu + \Delta^{\mu\nu}\left(m_\nu u^\alpha \partial_\alpha q + q u^\alpha \partial_\alpha m_\nu - \partial_\nu \mathcal{P}_{\rm eff} + \partial_\beta\left\{(\mathcal{P}_r - \mathcal{P}_\perp)\left[m^\beta m_\nu + \frac{1}{3}\Delta^\beta_\nu\right]\right\}\right) + q m_\nu \sigma^{\mu\nu} + \frac{4}{3}\Theta q m^\mu = 0, \tag{A35}$$

which are the local energy density conservation equation, Eq. (A34), and the local rest frame equations of motion, Eqs. (A35). Contracting the second equation with m_μ, we get

$$m^\mu \partial_\mu \mathcal{P}_r + (\mathcal{P}_r - \mathcal{P}_\perp)\partial_\mu m^\mu - (\varepsilon + \mathcal{P}_\perp)a_\mu m^\mu + \frac{4}{3}\Theta q + u^\mu \partial_\mu q - q m^\mu m^\nu \sigma_{\mu\nu} = 0, \tag{A36}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}\sigma\left(m^\mu m^\nu + \frac{1}{3}\Delta^{\mu\nu}\right), \tag{A37}$$

$$\sigma = -\left[2\theta - 2\frac{\gamma v_r}{r}\right] \Longrightarrow \frac{1}{2}\sigma + \Theta = 3\frac{\gamma v_r}{r}. \tag{A38}$$

Explicitly, the net charge density, the energy, and the radial pressure equations are

$$\dot{n} + n\Theta = 0, \tag{A39}$$

$$\dot{\varepsilon} + (\varepsilon + \mathcal{P}_{\rm eff})\Theta + \frac{1}{\gamma}q' = -\frac{1}{3}(\mathcal{P}_r - \mathcal{P}_\perp)\sigma - 2q\frac{a}{\gamma}, \tag{A40}$$

$$\mathcal{P}_r' + \gamma \dot{q} = -(\varepsilon + \mathcal{P}_\perp)a - (\mathcal{P}_r - \mathcal{P}_\perp)\gamma \mathcal{A} - \frac{2}{3}[2\Theta + \sigma]\gamma q, \tag{A41}$$

where

$$a = \frac{\gamma}{v_r}\left[\partial_t \gamma + v_r^2 \partial_r \gamma v_r\right], \tag{A42}$$

$$\mathcal{A} = \frac{1}{\gamma}\left(a + 2\frac{\gamma}{r}\right), \tag{A43}$$

$$\dot{f} \equiv \gamma[\partial_t + v_r \partial_r]f, \tag{A44}$$

$$f' \equiv \gamma^2[v_r \partial_t + \partial_r]f. \tag{A45}$$

APPENDIX B: PHYSICAL COMPONENTS AND TRANSFORMATION MATRICES

Although the covariant and contravariant components of the vectors (e.g, four-velocity) and tensors (energy-momentum tensor) in curved space coordinates (e.g., cylindrical and spherical coordinates) are useful for calculations, the normalizations brought in by the metric make them inconvenient for physical interpretations. More convenient are components on orthonormal tetrads carried by the fluid elements. The line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu\, dx^\nu. \tag{B1}$$

Since the metric is diagonal, we will denote the nonvanishing components by $g^{(\nu)(\nu)}$ and, respectively, $g_{(\nu)(\nu)}$ for its inverse. For a vector A^ν, with the abstract contravariant component $A^{(\nu)}$ we must associate the *physical component*

$$A(\nu) = |g_{(\nu)(\nu)}|^{1/2} A^\nu, \tag{B2}$$

with no summation implied. For covariant components, the physical components are related to covariant components by the expression

$$A(\nu) = |g^{(\nu)(\nu)}|^{1/2} A_\nu, \tag{B3}$$

with no summation implied. To compute the physical components of a tensor, the *physical components* of $T^{\mu\nu}$ in terms of its covariant components are

$$T(\mu,\nu) = |g^{(\mu)(\mu)}|^{1/2} |g^{(\nu)(\nu)}|^{1/2} T_{\mu\nu}. \tag{B4}$$

Similarly for contravariant components, we have

$$T(\mu,\nu) = |g_{(\mu)(\mu)}|^{1/2} |g_{(\nu)(\nu)}|^{1/2} T^{\mu\nu}. \tag{B5}$$

The parenthetic index has no tensorial significance being merely a label. The physical components do not of course transform as tensors, but their transformation law can be easily deduced. One notices that the diagonal elements of a mixed second rank tensor are the same as the physical components. For mixed tensors, the physical components are related to the abstract components by

$$T(\mu,\nu) = \left|\frac{g_{(\mu)(\mu)}}{g_{(\nu)(\nu)}}\right|^{1/2} T^\mu_\nu = \left|\frac{g_{(\nu)(\nu)}}{g_{(\mu)(\mu)}}\right|^{1/2} T^\nu_\mu \tag{B6}$$

or

$$T(\mu,\nu) = \left|\frac{g_{(\mu)(\mu)}}{g_{(\nu)(\nu)}}\right|^{1/2} g^{\mu\alpha} T_{\alpha\nu} = \left|\frac{g_{(\mu)(\mu)}}{g_{(\nu)(\nu)}}\right|^{1/2} g_{\nu\alpha} T^{\alpha\mu}. \tag{B7}$$

(i) *Spherical symmetry*
The metric and the line element in spherical coordinates read

$$g^{\mu\nu} = \text{diag}(1, -1, -r^{-2}, -r^{-2}\sin^{-2}\phi),$$
$$g_{\mu\nu} = \text{diag}(1, -1, -r^{2}, -r^{2}\sin^{2}\phi), \tag{B8}$$
$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - r^2\sin^2\phi d\theta^2. \tag{B9}$$

and the determinant $g = -r^4 \sin\phi$, while the transformation matrix for the derivatives is

$$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ 0 & \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ 0 & \cos\phi & -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta} \end{pmatrix}. \tag{B10}$$

The relations between the abstract and physical components of the four-velocity in spherical coordinates read (respectively, contravariant and covariant)

$$u^{(0)} = u_t, \quad u^{(1)} = u_r, \quad u^{(2)} = u_\phi/r,$$
$$u^{(3)} = u_\theta/(r\sin\phi), \tag{B11}$$
$$u_0 = u_t, \quad u_1 = u_r, \quad u_2 = r u_\phi,$$
$$u_3 = r\sin\phi u_\theta. \tag{B12}$$

For a symmetric tensor in spherical coordinates we have, for contravariant components, the relations between physical and abstract components

$$T^{00} = T_{tt}, \quad T^{11} = T_{rr}, \quad T^{12} = T_{r\phi}/r,$$
$$T^{13} = T_{r\theta}/(r\sin\phi), \tag{B13}$$

$$T^{22} = T_{\phi\phi}/r^2, \quad T^{23} = T_{\phi\theta}/(r^2\sin\phi),$$
$$T^{33} = T_{\theta\theta}/(r\sin\phi)^2, \tag{B14}$$

with analogous expressions for covariant components.

(ii) *Cylindrical symmetry* The metric and the line element in cylindrical coordinates read

$$g^{\mu\nu} = \text{diag}(1, -1, -r^{-2}, -1),$$
$$g_{\mu\nu} = \text{diag}(1, -1, -r^{2}, -1), \tag{B15}$$
$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2. \tag{B16}$$

and the determinant $g = r^2$, while the transformation of the derivatives is

$$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}. \tag{B17}$$

The relations between the abstract and physical components of the four-velocity in cylindrical coordinates are (respectively, for contravariant and covariant components)

$$u^{(0)} = u_t, \quad u^{(1)} = u_r, \quad u^{(2)} = u_\phi/r,$$
$$u^{(3)} = u_z, \tag{B18}$$
$$u_0 = u_t, \quad u_1 = u_r, \quad u_2 = r u_\phi, \quad u_3 = u_z. \tag{B19}$$

For a symmetric tensor in cylindrical coordinates, we have (for contravariant components)

$$T^{00} = T_{tt}, \quad T^{11} = T_{rr}, \quad T^{12} = T_{r\phi}/r,$$
$$T^{13} = T_{rz}, \tag{B20}$$
$$T^{22} = T_{\phi\phi}/r^2, \quad T^{23} = T_{\phi z}/r, \quad T^{33} = T_{zz}, \tag{B21}$$

with analogous expressions for covariant components.

APPENDIX C: NONIDEAL FLUID DYNAMICS IN THE NONRELATIVISTIC LIMIT

In Cartesian coordinates (t, x, y, z), the governing equations can be written compactly as the single equation

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0, \tag{C1}$$

where

$$U \equiv \begin{bmatrix} \rho \\ \mathcal{M}^x \\ \mathcal{M}^y \\ \mathcal{M}^z \\ E \end{bmatrix},$$

$$F \equiv \begin{bmatrix} \rho v_x \\ \mathcal{M}^x v_x + \mathcal{P}_x \\ \mathcal{M}^y v_x + \pi^{xy} \\ \mathcal{M}^z v_x + \pi^{xz} \\ (E + \mathcal{P}_x)v_x + \pi^{yx}v_y + \pi^{zx}v_z + \mathcal{Q}^x \end{bmatrix},$$

$$G \equiv \begin{bmatrix} \rho v_y \\ \mathcal{M}^x v_y + \pi^{yx} \\ \mathcal{M}^y v_y + \mathcal{P}_y \\ \mathcal{M}^z v_y + \pi^{yz} \\ (E + \mathcal{P}_y)v_y + \pi^{xy}v_x + \pi^{zy}v_z + \mathcal{Q}^y \end{bmatrix}, \tag{C2}$$

$$H \equiv \begin{bmatrix} \rho v_z \\ \mathcal{M}^x v_z + \pi^{zx} \\ \mathcal{M}^y v_z + \pi^{zy} \\ \mathcal{M}^z v_z + \mathcal{P}_z \\ (E + \mathcal{P}_z)v_z + \pi^{xz}v_x + \pi^{yz}v_y + \mathcal{Q}^z \end{bmatrix},$$

in which $\mathcal{M}^i \equiv \rho v_i$ and $\mathcal{P}_i \equiv p + \Pi + \pi^{ij}\delta_{ij}$. The dissipative fluxes are governed by the following relaxational transport equation

$$\frac{\partial U}{\partial t} + v_x\frac{\partial U}{\partial x} + v_y\frac{\partial U}{\partial y} + v_z\frac{\partial U}{\partial z} = -\frac{1}{\tau_U}[U + F], \tag{C3}$$

where $\tau_U = \{\tau_\Pi, \tau_q, \tau_\pi\}$ and

$$U \equiv \begin{bmatrix} \Pi \\ Q^x \\ Q^y \\ Q^z \\ \pi^{xx} \\ \pi^{yy} \\ \pi^{zz} \\ \pi^{xy} = \pi^{yx} \\ \pi^{yz} = \pi^{zy} \\ \pi^{zx} = \pi^{xz} \end{bmatrix}, \quad F \equiv \begin{bmatrix} -\zeta\Theta \\ -\kappa\frac{\partial T}{\partial x} \\ -\kappa\frac{\partial T}{\partial y} \\ -\kappa\frac{\partial T}{\partial z} \\ -2\eta\left(\frac{\partial v_x}{\partial x} - \frac{1}{3}\Theta\right) \\ -2\eta\left(\frac{\partial v_y}{\partial y} - \frac{1}{3}\Theta\right) \\ -2\eta\left(\frac{\partial v_z}{\partial z} - \frac{1}{3}\Theta\right) \\ -\eta\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right] \\ -\eta\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right] \\ -\eta\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right] \end{bmatrix}, \tag{C4}$$

in which

$$\Theta = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{C5}$$

For one-dimensional Cartesian coordinates (t, z), pure cylindrical coordinates (t, r, ϕ, z) with $v_\phi = v_z = 0$ and pure spherical coordinates (t, r, ϕ, θ) with $v_\phi = v_\theta = 0$, the governing equations can be written in one single equation

$$\frac{\partial U}{\partial t} + \frac{1}{r^{\alpha-1}}\frac{\partial}{\partial r}\{r^{\alpha-1}Uv\} + \frac{1}{r^{\alpha-1}}\frac{\partial}{\partial r}\{r^{\alpha-1}F\} + \frac{\partial G}{\partial r} + \frac{(\alpha-1)}{r}H = 0, \tag{C6}$$

where $\alpha = 1, 2, 3$ for Cartesian, cylindrical, and spherical coordinates, respectively, and in particular for cylindrical coordinates,

$$U \equiv \begin{bmatrix} \rho \\ \mathcal{M}^r \\ E \end{bmatrix}, \quad F \equiv \begin{bmatrix} 0 \\ \pi^{rr} \\ \mathcal{P}v_r + Q^r \end{bmatrix},$$
$$G \equiv \begin{bmatrix} 0 \\ \mathcal{P} \\ 0 \end{bmatrix}, \quad H \equiv \begin{bmatrix} 0 \\ -\pi^{\phi\phi} \\ 0 \end{bmatrix}. \tag{C7}$$

The governing equations for the dissipative fluxes can be written in the single compact equation

$$\frac{\partial U}{\partial t} + v_r\frac{\partial U}{\partial r} = -\frac{1}{\tau_U}[U + F(U)], \tag{C8}$$

where in cylindrical coordinates,

$$U \equiv \begin{bmatrix} \Pi \\ Q^r \\ \pi^{rr} \\ \pi^{\phi\phi} \\ \pi^{zz} \end{bmatrix}, \quad F \equiv \begin{bmatrix} -\zeta\Theta \\ -\kappa\frac{\partial T}{\partial r} \\ -2\eta\left(\frac{\partial v_r}{\partial r} - \frac{1}{3}\Theta\right) \\ -2\eta\left(\frac{v_r}{r} - \frac{1}{3}\Theta\right) \\ \frac{2}{3}\eta\Theta \end{bmatrix}, \tag{C9}$$

and in spherical coordinates,

$$U \equiv \begin{bmatrix} \Pi \\ Q^r \\ \pi^{rr} \\ \pi^{\phi\phi} \\ \pi^{\theta\theta} \end{bmatrix}, \quad F \equiv \begin{bmatrix} -\zeta\Theta \\ -\kappa\frac{\partial T}{\partial r} \\ -2\eta\left(\frac{\partial v_r}{\partial r} - \frac{1}{3}\Theta\right) \\ -2\eta\left(\frac{v_r}{r} - \frac{1}{3}\Theta\right) \\ -2\eta\left(\frac{v_r}{r} - \frac{1}{3}\Theta\right) \end{bmatrix}, \tag{C10}$$

while in Cartesian coordinates,

$$U \equiv \begin{bmatrix} \Pi \\ Q^z \\ \pi^{zz} \\ \pi^{yy} \\ \pi^{xx} \end{bmatrix}, \quad F \equiv \begin{bmatrix} -\zeta\Theta \\ -\kappa\frac{\partial T}{\partial z} \\ -2\eta\left(\frac{\partial v_z}{\partial z} - \frac{1}{3}\Theta\right) \\ \frac{2}{3}\eta\Theta \\ \frac{2}{3}\eta\Theta \end{bmatrix}, \tag{C11}$$

in which

$$\Theta = \frac{1}{r^{\alpha-1}}\frac{\partial}{\partial r}(r^{\alpha-1}v_r). \tag{C12}$$

Note that in the Cartesian coordinates ($\alpha = 1$), the subscript/superscript r is replaced by z and the differentiation with respect to r becomes differentiation with respect to z. Also, the transverse components of $\pi^{\mu\nu}$, namely, π^{xx} and π^{yy}, do not vanish; and they are equal to $-\pi^{zz}/2$. In the spherical ($\alpha = 3$) case, the transverse components are the $\pi^{\phi\phi}$ and the $\pi^{\theta\theta}$ and each equals $-\pi^{rr}/2$. In the cylindrical coordinates, $\alpha = 2$, the transverse components of $\pi^{\mu\nu}$ are $\pi^{\phi\phi}$ and π^{zz}.

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