Quark-gluon bags with surface tension

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The temperature and chemical potential dependent surface tension of bags is introduced into the gas of a quark-gluon bags model. This resolves a long standing problem of a unified description of the first- and second-order phase transition with the crossover. Such an approach is necessary to model the complicated properties of quark-gluon plasma and hadronic matter from the first principles of statistical mechanics. The suggested model has an exact analytical solution and allows one to rigorously study the vicinity of the critical endpoint of the deconfinement phase transition. The existence of higher order phase transitions at the critical endpoint is discussed. In addition, we found that at the curve of a zero surface tension coefficient there must exist the surface induced phase transition of the second or higher order, which separates the pure quark-gluon plasma (QGP) from the crossover states, which are the mixed states of hadrons and QGP bags. Thus, the present model predicts that the critical endpoint of quantum chromodynamics is the tricritical endpoint.

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I. INTRODUCTION

Investigation of the strongly interacting matter properties observed in relativistic nuclear collisions has reached the stage where the predictions of the lattice quantum chromodynamics (QCD) can be checked experimentally on the existing data and future mesurements at BNL Relativistic Heavy Ion Collider (RHIC), CERN Super Proton Synchrotron (SPS), and GSI Facility for Antiproton and Ion Research (GSI-FAIR). However, a comparison of the theoretical results with the experimental data is not straightforward because during the collision process the matter can have several phase transformations that are difficult to model. The latter reason stimulated the development of a wide range of phenomenological models of the strongly interacting matter equation of state that are used in dynamical simulations.

One of these models is the famous bag model [1], which treats the hadrons as the bags of the quark-gluon plasma (QGP) confined inside a hadron with the help of bag pressure. The bag model is able to simultaneously describe the hadron mass spectrum, i.e, the hadron masses and their proper volumes, and the properties of the deconfined phase [2,3]. This success led to the development of a statistical model of QGP, the gas of bags model (GBM) [4–6], which itself contains two well-known models of deconfined and confined phases: the bag model of QGP [2] and the hadron gas model [7]. There were hopes [8] that an exact analytical solution of the GBM found in Ref. [4] could be helpful in understanding the properties of strongly interacting matter. However, this solution does not allow one to introduce the critical end point (CEP) of the strongly interacting matter phase diagram. Also, a complicated construction of the line, along which the phase transition order gradually increases, suggested in Ref. [8], does look too artificial. Therefore, the present GBM formulation lacks an important physical input and is interesting only as a toy example that can be solved analytically.

On the other hand, the models, which can correctly reproduce the expectation [9–11] that the endpoint of the first-order phase transition (PT) line to QGP should be the second-order critical point, are indeed necessary for heavy ion phenomenology. In addition, such phenomenological models can provide us with the information about the phase structure and equation of state of strongly interacting matter that is located between the critical endpoint and the region of the color superconductivity because such information is unavailable otherwise. Therefore, the present work is devoted to the extension of the GMB. We think that the GMB can be drastically improved by the inclusion of such a vitally important element as the surface tension of the quark-gluon bags.

The dynamical surface tension of the quark-gluon bags was estimated long ago [12,13], but it was never used in statistical descriptions of the equation of state. Moreover, the estimate of the bag surface tension made in Ref. [13] is negligible for u and d quarks and, hence, can be safely neglected in our treatment. The situation with the surface tension of the quark-gluon bags is somewhat unclear: the early estimates within the MIT Bag Model [14,15] indicate that small hadronic bubbles can exist in the hot QGP, whereas the analysis based on the effective potential of the first-order PT in early Universe [16] does not support the results of Refs. [14] and [15]. Thus, it turns out that the surface energy may play an important role in the properties of hadronic bubbles [14–16] and QGP bags [17]; the surface tension of large bags was not included in a consistent statistical description of QGP. Therefore, the present article is devoted to the investigation and analysis of the critical properties of the model of quark-gluon bags with surface tension (QGBST model hereafter).

In statistical mechanics there are several exactly solvable cluster models with the first-order PT that describe the critical point properties very well. These models are built on the assumptions that the difference of the bulk part (or the volume dependent part) of the free energy of two phases disappears at phase equilibrium and that, in addition, the difference of the surface part (or the surface tension) of the free energy vanishes at the critical point. The most famous of them is the Fisher droplet model (FDM) [18,19], which has been successfully used to analyze the condensation of a gaseous phase (droplets of all sizes) into a liquid. The FDM has been applied to many different systems, including nuclear multifragmentation [20], nucleation of real fluids [21], the compressibility factor of real fluids [22], clusters of the Ising model [23], and percolation clusters [24].

On the basis of the statistical multifragmentation model (SMM) [25] commonly used to study nuclear multifragmentation, there was recently formulated a simplified SMM version [26,27] that was solved analytically both for infinite [28,29] and for finite [30,31] volumes of the system. In the SMM the surface tension temperature dependence differs from that of the FDM, but it had been shown [29] that the value of Fisher exponent $\tau_{\rm SMM} = 1.825 \pm 0.025$, which contradicts the FDM value $\tau_{\rm FDM} \approx 2.16$, is consistent with ISiS Collaboration data [32] and EOS Collaboration data [33]. Our analytical results [29] have been confirmed by numerical studies [34,35].

Such an experimentally obtained range of the τ index is of principal importance because it gives very strong evidence that the SMM, and, thus, the nuclear matter, has a tricritical endpoint rather than a critical endpoint [28,29].

This success of the SMM initiated the studies of the surface partitions of large clusters within the Hills and Dales Model [36,37] and led to the discovery of the origin of the temperature independent surface entropy similar to the FDM. As a consequence, the surface tension coefficient of large clusters consisting of discrete constituents should linearly depend on the temperature of the system [36] and must vanish at the critical endpoint. However, the present formulation of the Hills and Dales Model [36,37], which successfully estimates the upper and lower bounds of the surface deformations of the discrete physical clusters, does not look suitable for quark-gluon bags. Therefore, in this work we assume a certain dependence of the surface tension coefficient on temperature and baryonic chemical potential, and we concentrate on the impact of the surface tension of the quark-gluon bags on the properties of the deconfinement phase diagram and the QCD critical endpoint. A discussion of the origin of the surface tension is a subject of our future work.

Here we show that the existence of a crossover at low values of the baryonic chemical potential along with the first-order deconfinement PT at high baryonic chemical potentials leads to the existence of an additional PT of the second or higher order along the curve where the surface tension coefficient vanishes. Thus, it turns out that the QGBST model predicts the existence of the tricritical rather than the critical endpoint.

This article is organized as follows. Section II contains the formulation of the basic ingredients of the GBM. In Sec. III we formulate the QGBST model and analyze all possible singularities of its isobaric partition for vanishing baryonic densities. This analysis is generalized to nonzero baryonic densities in Sec. IV. Section V is devoted to the analysis of the surface tension induced PT that exists above the deconfinement PT. The conclusions and research perspectives are summarized in Sec. V.

II. BASIC INGREDIENTS OF THE GBM

To review the basic ingredients of the GBM let us consider the Van der Waals gas consisting of n hadronic species, which are called bags in what follows, at zero baryonic chemical potential. Its grand canonical partition (GCP) is given by [4]

$$Z(V,T) = \sum_{\{N_k\}} \left[\prod_{k=1}^{n} \frac{[(V - v_1 N_1 - \cdots - v_n N_n)\,\phi_k(T)]^{N_k}}{N_k!} \right] \times \theta(V - v_1 N_1 - \ldots - v_n N_n), \qquad (1)$$

where the function $\phi_k(T) \equiv g_k \phi(T, m_k)$

$$\phi_k(T) \equiv \frac{g_k}{2\pi^2} \int_0^\infty p^2 dp\, e^{-\frac{(p^2+m_k^2)^{1/2}}{T}} = g_k \frac{m_k^2 T}{2\pi^2} K_2\left(\frac{m_k}{T}\right)$$

is the particle density of bags of mass m_k, eigenvolume v_k, and degeneracy g_k. Using the standard technique of the Laplace transformation [4,28] with respect to volume, one obtains the isobaric partition

$$\hat{Z}(s,T) \equiv \int_0^\infty dV \exp(-sV) Z(V,T) = \frac{1}{[s - F(s,T)]}, \qquad (2)$$

$$\text{with} \quad F(s,T) \equiv \sum_{j=1}^{n} \exp(-v_j s) g_j \phi(T, m_j). \qquad (3)$$

From the definition of pressure in the grand canonical ensemble it follows that, in the thermodynamic limit, the GCP of the system behaves as $Z(V,T) \simeq \exp[pV/T]$. An exponentially increasing $Z(V,T)$ generates the rightmost singularity $s^* = p/T$ of the function $\hat{Z}(s,T)$ in variable s. This is because the integral over V in Eq. (2) diverges at its upper limit for $s < p/T$. Therefore, the rightmost singularity s^* of $\hat{Z}(s,T)$ gives us the system pressure

$$p(T) = T \lim_{V\to\infty} \frac{\ln Z(V,T)}{V} = T s^*(T). \qquad (4)$$

The singularity s^* of $\hat{Z}(s,T)$ (2) can be calculated from the transcendental equation [4,28]

$$s^*(T) = F(s^*, T). \qquad (5)$$

As long as the number of bags, n, is finite, the only possible singularities of $\hat{Z}(s,T)$ (2) are simple poles. For example, for the ideal gas [$n = 1; v_1 = 0$ in Eq. (5)] $s^* = g_1\phi(T, m_1)$ and thus from Eq. (4) one gets $p = T g_1 \phi(T, m_1)$, which corresponds to the grand canonical ensemble ideal gas equation of state for the particles of mass m_1 and degeneracy g_1.

However, in the case of an infinite number of sorts of bags an essential singularity of $\hat{Z}(s,T)$ may appear. This property is used in the GBM: to the finite sum over different bag states in Eq. (2) the integral $\int_{M_0}^\infty dm dv \ldots \rho(m,v)$ is added with the bag mass-volume spectrum, $\rho(m,v)$, which defines the number of bag states in the mass-volume region $[m, v; m+dm, v+dv]$. In this case the function $F(s,T)$ in Eqs. (2) and (5) should be replaced by

$$F(s,T) \equiv F_H(s,T) + F_Q(s,T) = \sum_{j=1}^{n} g_j e^{-v_j s} \phi(T, m_j) + \int_{V_o}^{\infty} dv \int_{M_o+Bv}^{\infty} dm \rho(m,v) \exp(-sv) \phi(T,m). \quad (6)$$

The first term of Eq. (6), F_H, represents the contribution of a finite number of low-lying hadron states. This function has no s singularities at any temperature T and can generate a simple pole of the isobaric partition, whereas the mass-volume spectrum of the bags $F_Q(s,T)$ can be chosen to generate an essential singularity $s_Q(T) \equiv p_Q(T)/T$ that defines the QGP pressure $p_Q(T)$ at zero baryonic densities [4,38,39].

The mass-volume spectrum is the generalization of the exponential mass spectrum introduced by Hagedorn [40,41]. The usage of the grand canonical description for the exponential mass spectrum was recently strongly criticized [42–44] because of the thermostatic properties of this spectrum. Fortunately, the Van der Waals repulsion compensates the growing part of the mass-volume spectrum and, hence, the criticism of Refs. [42–44] is irrelevant to the present model.

There are several possibilities to parametrize the mass-volume spectrum $\rho(m,v)$. Thus, in the simplest case one can assume that for heavy resonances their mass and eigenvolume are proportional, i.e., the spectrum $\rho(m,v)$ contains the function $\delta(m - v\,\mathrm{Const})$. An alternative choice was suggested in Ref. [38], but in either case the resulting expression for the continuum spectrum of the GBM $F_Q(s,T)$ can be cast as

$$F_Q(s,T) = u(T) \int_{V_0}^{\infty} dv \frac{\exp[-v(s - s_Q(T))]}{v^{\tau}}, \quad (7)$$

where $u(T)$ and $\tau > 0$ are the model parameters. The QGP pressure $p(T) = T s_Q(T)$ can be parameterized in many ways. For instance, the MIT bag model equation of state [1] corresponds [38] to $s_Q(T) \equiv \frac{1}{3}\sigma_Q T^3 - \frac{B}{T}$ and $u(T) = C\pi^{-1}\sigma_Q^{\delta+1/2} T^{4+4\delta}(\sigma_Q T^4 + B)^{3/2}$. Here B denotes the bag constant, $\sigma_Q = \frac{\pi^2}{30}\frac{95}{2}$ is the Stefan-Boltzmann constant counting gluons (spin, color) and (anti-)quarks (spin, color, and u, d, and s flavors) degrees of freedom; and the constants $C, \delta < 0, V_o \approx 1$ fm^3, and $M_o \approx 2$ GeV are the parameters of the mass-volume spectrum. A recent attempt to derive the mass-volume spectrum that accounts for additional constraints can be found in Ref. [39].

III. THE ROLE OF SURFACE TENSION

At the moment the particular choice of function $F_Q(s,T)$ (7) is not important. The key point for our study is that it should have the form of Eq. (7), which has a singularity at $s = s_Q$ because for $s < s_Q$ the integral over dv diverges at its upper limit. Note that the exponential in Eq. (7) is nothing else, but a difference of the bulk free energy of a bag of volume v, i.e., $-Tsv$, which is under external pressure Ts, and the bulk free energy of the same bag filled with QGP, i.e., $-Ts_Q v$. At phase equilibrium this difference of the bulk free energies vanishes. Despite all positive features, Eq. (7) lacks the surface part of the free energy of bags, which will be called a surface energy hereafter. In addition to the difference of the bulk free energies the realistic statistical models that demonstrated their validity, the FDM [18] and the SMM [25], have the contribution of the surface energy that plays an important role in defining the phase diagram structure [28,31]. Therefore, we modify Eq. (7) by introducing the surface energy of the bags in a general fashion [29]:

$$F_Q = u(T) \int_{V_0}^{\infty} dv \frac{\exp[(s_Q(T) - s)v - \sigma(T)v^{\varkappa}]}{v^{\tau}}, \quad (8)$$

where the ratio of the temperature dependent surface tension coefficient to T (the reduced surface tension coefficient hereafter) has the form $\sigma(T) = \frac{\sigma_o}{T} \cdot [\frac{T_{\rm cep} - T}{T_{\rm cep}}]^{2k+1}$ $(k = 0, 1, 2, \ldots)$. Here $\sigma_o > 0$ can be a smooth function of the temperature, but for simplicity we fix it to be a constant. For $k = 0$ the two terms in the surface (free) energy of a v-volume bag have a simple interpretation [18]: thus, the surface energy of such a bag is $\sigma_0 v^{\varkappa}$, whereas the free energy, which comes from the surface entropy $\sigma_o T_{\rm cep}^{-1} v^{\varkappa}$, is $-T\sigma_o T_{\rm cep}^{-1} v^{\varkappa}$. Note that the surface entropy of a v-volume bag counts its degeneracy factor or the number of ways to make such a bag with all possible surfaces. This interpretation can be extended to $k > 0$ on the basis of the Hills and Dales Model [36,37].

In choosing such a simple surface energy parametrization we follow the original Fisher idea [18], which allows one to account for the surface energy by considering some mean bag of volume v and surface $v^{\varkappa}$. The consideration of the general mass-volume-surface bag spectrum we leave for the future investigation. The power $\varkappa < 1$ that describes the bag's effective surface is a constant, which, in principle, can differ from the typical FDM and SMM value $\frac{2}{3}$. This is so because near the deconfinement PT region QGP has low density and, hence, like in the low density nuclear matter [45], the nonspherical bags (spaghetti-like or lasagna-like [45]) can be favorable (see also Refs. [14] and [15] for the bubbles of complicated shapes). A similar idea of "polymerization" of gluonic quasiparticles was introduced recently [46].

The second essential difference with the FDM and SMM surface tension parametrization is that we do not require the vanishing of $\sigma(T)$ above the CEP. As will be shown later, this is the most important assumption that, in contrast to the GBM, allows one to naturally describe the crossover from hadron gas to QGP. Note that a negative value of the reduced surface tension coefficient $\sigma(T)$ above the CEP does not mean anything wrong. As we discussed previously, the surface tension coefficient consists of energy and entropy parts that have opposite signs [18,36,37]. Therefore, $\sigma(T) < 0$ does not mean that the surface energy changes the sign, but it rather means that the surface entropy, i.e., the logarithm of the degeneracy of bags of a fixed volume, simply exceeds the surface energy. In other words, the number of nonspherical bags of a fixed volume becomes so large that the Boltzmann exponent, which accounts for the energy "costs" of these bags, cannot suppress them anymore.

Finally, the third essential difference with the FDM and the SMM is that we assume that the surface tension in the QGBST model happens at some line in the $\mu_B - T$ plane, i.e., $T_{\rm cep} = T_{\rm cep}(\mu_B)$. However, in subsequent sections we consider $T_{\rm cep} =$ Const for simplicity, and in Sec. V we discuss the necessary modifications of the model with $T_{\rm cep} = T_{\rm cep}(\mu_B)$.

The surface energy should, in principle, be introduced into a discrete part of the mass-volume spectrum F_H, but a successful fitting of the particle yield ratios [7] with the experimentally determined hadronic spectrum F_H does not indicate such a necessity.

In principle, besides the bulk and surface parts of free energy, the spectrum (8) could include the curvature part as well, which may be important for small hadronic bubbles [14,15] or for cosmological PT [16]. We stress, however, that the critical properties of the present model are defined by the infinite bag; therefore the inclusion in Eq. (8) of a curvature term of any sign could affect the thermodynamic quantities of this model at $s = s_Q(T)$ and $\sigma(T) = 0$, which is possible at the (tri)critical endpoint only (see below). If, the curvature term was really important for cluster models like the present one, then it should have been seen also at the (tri)critical points of the FDM, the SMM, and many systems described by the FDM [20–24], but this is not the case. Indeed, recently the Complement method [47] was applied to the analysis of the largest, but still mesoscopic, drop of a radius $R_{\rm dr}$ representing the liquid in equilibrium with its vapor. The method allows one to find out the concentrations of the vapor clusters in a finite system under a whole range of temperatures and to determine the free energy difference of two phases with high precision. The latter enables us to extract not only the critical temperature, the surface tension coefficient, and even the value of Fisher index τ of the infinite system but also such a delicate effect as the Gibbs-Thomson correction [48] to the free energy of a liquid drop. Note that the Gibbs-Thomson correction behaves as $R_{\rm dr}^{-1}$, but the Complement method [47] allows one to find it, whereas the curvature part of the free energy, which is proportional to $R_{\rm dr}$, is not seen for either a drop or for smaller clusters. Such a result is directly related to the QGP bags because QCD is expected to be in the same universality class [9] as the three-dimensional Ising model whose clusters were analyzed in Ref. [47]. Therefore, admitting that for finite QGP bags the curvature effects may be essential, we leave them out because the critical behavior of the present model is defined by the properties of the infinite bag. On the other hand, similar to the FDM, the SMM, and FDM-like systems [20–24,47], we assume that the curvature part of the free energy of the infinite QGP bag is not important and leave for future analysis the question of why this is so.

According to the general theorem [4] the analysis of PT existence of the GCP is now reduced to the analysis of the rightmost singularity of the isobaric partition (2). Depending on the sign of the reduced surface tension coefficient, there are three possibilities.

(**I**) The first possibility corresponds to $\sigma(T) > 0$. Its treatment is very similar to the GBM choice (7) with $\tau > 2$ [4]. In this case at low temperatures the QGP pressure $T s_Q(T)$ is

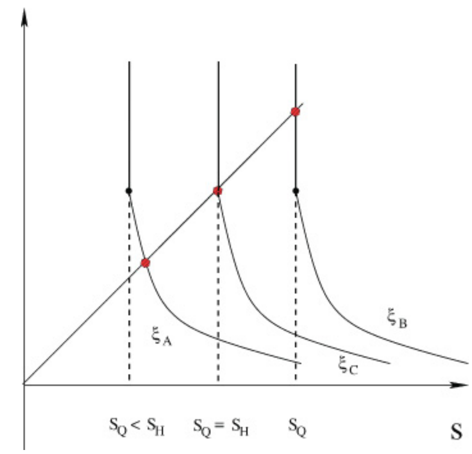

FIG. 1. (Color online) Graphical solution of Eq. (5) that corresponds to a PT. The solution of Eq. (5) is shown by a filled hexagon. The function $F(s,\xi)$ is shown by a solid curve for a few values of the parameter ξ. The function $F(s,\xi)$ diverges for $s < s_Q(\xi)$ (shown by dashed lines), but is finite at $s = s_Q(\xi)$ (shown by black circle). At low values of the parameter $\xi = \xi_A$, which can be either T or μ_B, the simple pole s_H is the rightmost singularity and it corresponds to the hadronic phase. For $\xi = \xi_B \gg \xi_A$ the rightmost singularity is an essential singularity $s = s_Q(\xi_B)$, which describes QGP. At intermediate value $\xi = \xi_C$ both singularities coincide, $s_H(\xi_C) = s_Q(\xi_C)$, and this condition is a Gibbs criterion.

negative and, therefore, the rightmost singularity is a simple pole of the isobaric partition $s^* = s_H(T) = F(s_H(T), T) > s_Q(T)$, which is mainly defined by a discrete part of the mass-volume spectrum $F_H(s,T)$. The last inequality provides the convergence of the volume integral in Eq. (8) (see Fig. 1). On the other hand at very high T the QGP pressure dominates and, hence, the rightmost singularity is the essential singularity of the isobaric partition $s^* = s_Q(T)$. The phase transition occurs when the singularities coincide:

$$s_H(T_c) \equiv \frac{p_H(T_c)}{T_c} = s_Q(T_c) \equiv \frac{p_Q(T_c)}{T_c}, \tag{9}$$

which is nothing else but the Gibbs criterion. The graphical solution of Eq. (5) for all these possibilities is shown in Fig. 1. Like in the GBM [4,8], the necessary condition for the PT existence is the finiteness of $F_Q(s_Q(T), T)$ at $s = s_Q(T)$. It can be shown that the necessary conditions are the following inequalities: $F_Q(s_Q(T), T) > s_Q(T)$ for low temperatures and $F(s_Q(T), T) < s_Q(T)$ for $T \to \infty$. These conditions provide that at low T the rightmost singularity of the isobaric partition is a simple pole, whereas for high T the essential singularity $s_Q(T)$ becomes its rightmost one (see Fig. 1 and a detailed analysis of case $\mu_B \neq 0$).

The PT order can be found from the T derivatives of $s_H(T)$. Thus, differentiating Eq. (5) one finds

$$s'_H = \frac{G + u\,\mathcal{K}_{\tau-1}(\Delta, -\sigma)\cdot s'_Q}{1 + u\,\mathcal{K}_{\tau-1}(\Delta, -\sigma)}, \tag{10}$$

where the functions G and $\mathcal{K}_{\tau-a}(\Delta, -\sigma)$ are defined as

$$G \equiv F'_H + \frac{u'}{u} F_Q + \frac{(T_{\rm cep} - 2kT)\sigma(T)}{(T_{\rm cep} - T)T} \times u\mathcal{K}_{\tau-\varkappa}(\Delta, -\sigma), \quad (11)$$

$$\mathcal{K}_{\tau-a}(\Delta, -\sigma) \equiv \int_{V_o}^{\infty} dv \frac{\exp[-\Delta v - \sigma(T)v^{\varkappa}]}{v^{\tau-a}}, \quad (12)$$

where $\Delta \equiv s_H - s_Q$.

Now it is easy to see that the transition is of the first order, i.e., $s'_Q(T_c) > s'_H(T_c)$, provided $\sigma(T) > 0$ for any τ. The second or higher order phase transition takes place provided $s'_Q(T_c) = s'_H(T_c)$ at $T = T_c$. The latter condition is satisfied when $\mathcal{K}_{\tau-1}$ diverges to infinity at $T \to (T_c - 0)$, i.e., for T approaching T_c from below. Like for the GBM choice (7), such a situation can exist for $\sigma(T_c) = 0$ and $\frac{3}{2} < \tau \leqslant 2$. Studying the higher T derivatives of $s_H(T)$ at T_c, one can show that for $\sigma(T) \equiv 0$ and for $(n+1)/n \leqslant \tau < n/(n-1)$ ($n = 3, 4, 5, \ldots$), there is a n^{th} order phase transition

$$s_H(T_c) = s_Q(T_c), \qquad s'_H(T_c) = s'_Q(T_c), \cdots$$
$$s_H^{(n-1)}(T_c) = s_Q^{(n-1)}(T_c), \qquad s_H^{(n)}(T_c) \neq s_Q^{(n)}(T_c), \quad (13)$$

with $s_H^{(n)}(T_c) = \infty$ for $(n+1)/n < \tau < n/(n-1)$ and with a finite value of $s_H^{(n)}(T_c)$ for $\tau = (n+1)/n$.

(II) The second possibility, $\sigma(T) \equiv 0$, described in the preceding paragraph, does not give anything new compared to the GBM [4,8]. If the PT exists, then the graphical picture of singularities is basically similar to Fig. 1. The only difference is that, depending on the PT order, the derivatives of $F(s, T)$ function with respect to s should diverge at $s = s_Q(T_c)$.

(III) A principally new possibility exists for $T > T_{\rm cep}$, where $\sigma(T) < 0$. In this case there exists a crossover, if for $T \leqslant T_{\rm cep}$ the rightmost singularity is $s_H(T)$, which corresponds to the leftmost curve in Fig. 1. Under the latter, its existence can be shown as follows. Let us solve the equation for singularities (5) graphically (see Fig. 2). For $\sigma(T) < 0$ the function $F_Q(s, T)$ diverges at $s = s_Q(T)$. On the other hand, the partial derivatives $\frac{\partial F_H(s,T)}{\partial s} < 0$ and $\frac{\partial F_Q(s,T)}{\partial s} < 0$ are always negative. Therefore, the function $F(s, T) \equiv F_H(s, T) + F_Q(s, T)$ is a monotonically decreasing function of s, which vanishes at $s \to \infty$. Since the left-hand side of Eq. (5) is a monotonically increasing function of s, then there can exist a single intersection s^* of s and $F(s, T)$ functions. Moreover, for finite $s_Q(T)$ values this intersection can occur on the right-hand side of the point $s = s_Q(T)$, i.e., $s^* > s_Q(T)$ (see Fig. 2). Thus, in this case the essential singularity $s = s_Q(T)$ can become the rightmost one for infinite temperature only. In other words, the pressure of the pure QGP can be reached at infinite T, whereas for finite T the hadronic mass spectrum gives a nonzero contribution into all thermodynamic functions. Note that such a behavior is typical for the lattice QCD data at zero baryonic chemical potential [49].

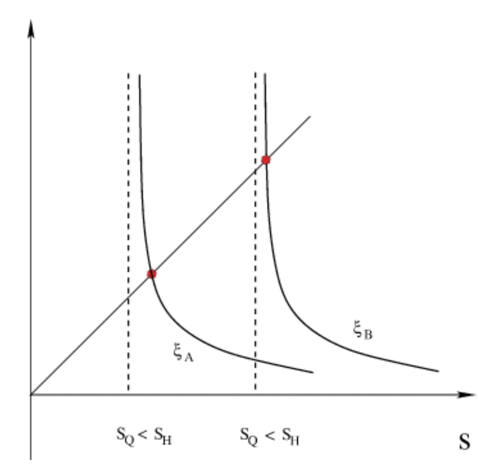

FIG. 2. (Color online) Graphical solution of Eq. (5) that corresponds to a crossover. The notations are the same as those in Fig. 1. Now the function $F(s, \xi)$ diverges at $s = s_Q(\xi)$ (shown by dashed lines). In this case the simple pole s_H is the rightmost singularity for any value of ξ.

It is clear that in terms of the present model a crossover existence means a fast transition of energy or entropy density in a narrow T region from a dominance of the discrete mass-volume spectrum of light hadrons to a dominance of the continuous spectrum of heavy QGP bags. This is exactly the case for $\sigma(T) < 0$ because in the right vicinity of the point $s = s_Q(T)$ the function $F(s, T)$ decreases very fast and then it gradually decreases as a function of the s variable. Because $F_Q(s, T)$ changes quickly from $F(s, T) \sim F_Q(s, T) \sim s_Q(T)$ to $F(s, T) \sim F_H(s, T) \sim s_H(T)$, their s derivatives should change quickly as well. Now, recalling that the change from $F(s, T) \sim F_Q(s, T)$ behavior to $F(s, T) \sim F_H(s, T)$ in the s variable corresponds to the cooling of the system (see Fig. 2), we conclude that there exists a narrow region of temperatures, where the T derivative of system pressure, i.e., the entropy density, drops down from $\frac{\partial p}{\partial T} \sim s_Q(T) + T\frac{ds_Q(T)}{dT}$ to $\frac{\partial p}{\partial T} \sim s_H(T) + T\frac{ds_H(T)}{dT}$, very fast compared to other regions of T, if the system cools. If, however, in the vicinity of $T = T_{\rm cep} - 0$ the rightmost singularity is $s_Q(T)$, then for $T > T_{\rm cep}$ the situation is different and the crossover does not exist. A detailed analysis of this situation is given in Sec. V.

Note also that all these nice properties would vanish if the reduced surface tension coefficient were zero or positive above $T_{\rm cep}$. This is one of the crucial points of the present model that puts forward certain doubts about the vanishing of the reduced surface tension coefficient in the FDM [18] and the SMM [25]. These doubts are also supported by the first principle results obtained by the Hills and Dales Model [36,37], because the surface entropy simply counts the degeneracy of a cluster of a fixed volume and it does not physically affect the surface energy of this cluster.

IV. GENERALIZATION TO NONZERO BARYONIC DENSITIES

The possibilities **(I)**–**(III)** discussed in the preceding section remain unchanged for nonzero baryonic numbers. The latter should be taken into consideration to make our model more realistic. To keep the presentation simple, we do not account for strangeness. The inclusion of the baryonic charge of the quark-gluon bags does not change the two types of singularities of the isobaric partition (2) and the corresponding equation for them (5), but it leads to the following modifications of the F_H and F_Q functions:

$$F_H(s,T,\mu_B) = \sum_{j=1}^{n} g_j e^{\frac{b_j \mu_B}{T} - v_j s} \phi(T, m_j), \tag{14}$$

$$F_Q(s,T,\mu_B) = u(T,\mu_B) \int_{V_0}^{\infty} dv \times \frac{\exp[(s_Q(T,\mu_B) - s)v - \sigma(T) v^{\varkappa}]}{v^{\tau}}. \tag{15}$$

Here the baryonic chemical potential is denoted as μ_B, and the baryonic charge of the jth hadron in the discrete part of the spectrum is b_j. The continuous part of the spectrum, F_Q, can be obtained from some spectrum $\rho(m, v, b)$ in the spirit of Refs. [38] and [39], but this will lead us away from the main subject.

The QGP pressure $p_Q = T s_Q(T, \mu_B)$ can also be chosen in several ways. Here we use the bag model pressure

$$p_Q = \frac{\pi^2}{90} T^4 \left[\frac{95}{2} + \frac{10}{\pi^2} \left(\frac{\mu_B}{T} \right)^2 + \frac{5}{9\pi^4} \left(\frac{\mu_B}{T} \right)^4 \right] - B, \tag{16}$$

but the more complicated model pressures, even with the PT of other kinds like the transition between the color superconducting QGP and the usual QGP, can be, in principle, used.

The conditions necessary for a PT existence are

$$F((s_Q(T,\mu_B = 0) + 0), T, \mu_B = 0) > s_Q(T, \mu_B = 0), \tag{17}$$

$$F((s_Q(T,\mu_B) + 0), T, \mu_B) < s_Q(T,\mu_B), \forall \mu_B > \mu_A. \tag{18}$$

Condition (17) provides that the simple pole singularity $s^* = s_H(T, \mu_B = 0)$ is the rightmost one at vanishing $\mu_B = 0$ and given T, whereas condition (18) ensures that $s^* = s_Q(T, \mu_B)$ is the rightmost singularity of the isobaric partition for all values of the baryonic chemical potential above some positive constant μ_A. This can be seen in Fig. 1 for μ_B being a variable. Because $F(s, T, \mu_B)$, where it exists, is a continuous function of its parameters, one concludes that, if conditions (17) and (18), are fulfilled, then at some chemical potential $\mu_B^c(T)$, both singularities should be equal. Thus, one arrives at the Gibbs criterion (9), but for two variables,

$$s_H(T, \mu_B^c(T)) = s_Q(T, \mu_B^c(T)). \tag{19}$$

It is easy to see that the inequalities (17) and (18) are the sufficient conditions of a PT existence that can be used for more complicated functional dependencies of $F_H(s, T, \mu_B)$ and $F_Q(s, T, \mu_B)$ than the ones used here.

For our choice, Eqs. (14), (15), and (16), of $F_H(s, T, \mu_B)$ and $F_Q(s, T, \mu_B)$ functions, the PT exists at $T < T_{\rm cep}$, because the sufficient conditions (17) and (18) can be easily fulfilled by a proper choice of the bag constant B and the function $u(T, \mu_B) > 0$ for the interval $T \leqslant T_{\rm up}$ with the constant $T_{\rm up} > T_{\rm cep}$. Clearly, this is the first-order PT, because the surface tension is finite and it provides the convergence of the integrals (11) and (12) in the expression (10), where the usual T derivatives should be now understood as the partial ones for $\mu_B = $ const.

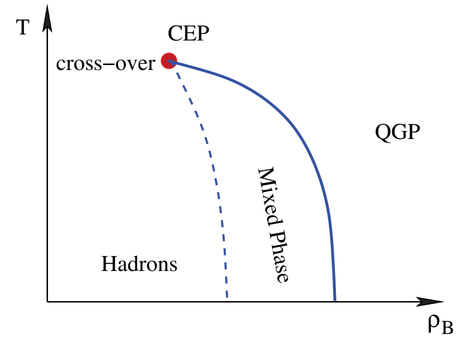

FIG. 3. (Color online) A schematic picture of the deconfinement phase transition diagram in the plane of baryonic density ρ_B and T for the second-order PT at the critical endpoint (CEP), i.e., for $\frac{3}{2} < \tau \leqslant 2$. For the third-order (or higher) PT, the boundary of the mixed and hadronic phases (dashed curve) should have the same slope as the boundary of the mixed phase and QGP (solid curve) at the CEP.

Assuming that the conditions (17) and (18) are fulfilled by the correct choice of the model parameters B and $u(T, \mu_B) > 0$, one can see now that at $T = T_{\rm cep}$ there exists a PT as well, but its order is defined by the value of τ. As was discussed in the preceding section for $\frac{3}{2} < \tau \leqslant 2$, there exists the second-order PT. For $1 < \tau \leqslant \frac{3}{2}$ there exists a PT of higher order, defined by the conditions formulated in Eq. (13). This is a new possibility, which, to our best knowledge, does not contradict any general physical principle (see Fig. 3).

The case $\tau > 2$ can be ruled out because there must exist the first-order PT for $T \geqslant T_{\rm cep}$, whereas for $T < T_{\rm cep}$ there exists the crossover. Thus, the critical endpoint in the $T - \mu_B$ plane will correspond to the critical interval in the temperature-baryonic density plane. because such a structure of the phase diagram in the variable's temperature-density has, to our knowledge, never been observed, we conclude that the case $\tau > 2$ is unrealistic (see Fig. 4). Note that a similar phase diagram exists in the FDM with the only difference that the boundary of the mixed and liquid phases (the latter in

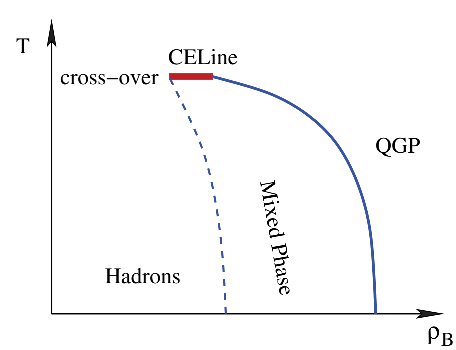

FIG. 4. (Color online) A schematic picture of the deconfinement phase transition diagram in the plane of baryonic density ρ_B and T for $\tau > 2$. The critical endpoint in the $\mu_B - T$ plane generates the critical end line (CELine) in the $\rho_B - T$ plane shown by the thick horizontal line. This occurs because of the discontinuity of the partial derivatives of s_H and s_Q functions with respect to μ_B and T.

the QGBST model corresponds to QGP) is moved to infinite particle density.

V. SURFACE TENSION INDUCED PHASE TRANSITION

Using our results for Case **III** of the preceding section, we conclude that above $T_{\rm cep}$ there is a crossover, i.e., the QGP and hadrons coexist together up to the infinite values of T and/or μ_B. Now, however, it is necessary to answer the question: How can the two different sets of singularities that exist on two sides of the line $T = T_{\rm cep}$ provide the continuity of the solution of Eq. (5)?

It is easy to answer this question for $\mu_B < \mu_B^c(T_{\rm cep})$ because in this case all partial T derivatives of $s_H(T, \mu_B)$, which is the rightmost singularity, exist and are finite at any point of the line $T = T_{\rm cep}$. This can be seen from the fact that for the considered region of parameters $s_H(T, \mu_B)$ is the rightmost singularity and, consequently, $s_H(T, \mu_B) > s_Q(T, \mu_B)$. The latter inequality provides the existence and finiteness of the volume integral in $F_Q(s, T, \mu_B)$. In combination with the power T dependence of the reduced surface tension coefficient $\sigma(T)$ the same inequality provides the existence and finiteness of all its partial T derivatives of $F_Q(s, T, \mu_B)$, regardless of the sign of $\sigma(T)$. Thus, using the Taylor expansion in powers of $(T - T_{\rm cep})$ at any point of the interval $T = T_{\rm cep}$ and $\mu_B < \mu_B^c(T_{\rm cep})$, one can calculate $s_H(T, \mu_B)$ for the values of $T > T_{\rm cep}$ that are inside the convergency radius of the Taylor expansion.

The other situation is for $\mu_B \geqslant \mu_B^c(T_{\rm cep})$ and $T > T_{\rm cep}$, namely, in this case above the deconfinement PT there must exist a weaker PT induced by the disappearance of the reduced surface tension coefficient. To demonstrate this we have solve Eq. (5) in the limit, when T approaches the curve $T = T_{\rm cep}$ from above, i.e., for $T \to T_{\rm cep} + 0$, and study the behavior of T derivatives of the solution of Eq. (5) s^* for fixed values of μ_B. For this purpose we have to evaluate the integrals $\mathcal{K}_\tau(\Delta, \gamma^2)$ introduced in Eq. (12). Here the notations $\Delta \equiv s^* - s_Q(T, \mu_B)$ and $\gamma^2 \equiv -\sigma(T) > 0$ are introduced for convenience.

To avoid the unpleasant behavior for $\tau \leqslant 2$ it is convenient to transform Eq. (12) further on by integrating by parts:

$$\mathcal{K}_\tau(\Delta, \gamma^2) \equiv g_\tau(V_0) - \frac{\Delta}{(\tau-1)}\mathcal{K}_{\tau-1}(\Delta, \gamma^2) + \frac{\varkappa\gamma^2}{(\tau-1)}\mathcal{K}_{\tau-\varkappa}(\Delta, \gamma^2), \quad (20)$$

where the regular function $g_\tau(V_0)$ is defined as

$$g_\tau(V_0) \equiv \frac{1}{(\tau-1)V_0^{\tau-1}} \exp\left[-\Delta V_0 + \gamma^2 V_0^{\varkappa}\right]. \quad (21)$$

For $\tau - a > 1$ one can change the variable of integration $v \to z/\Delta$ and rewrite $\mathcal{K}_{\tau-a}(\Delta, \gamma^2)$ as

$$\mathcal{K}_{\tau-a}(\Delta, \gamma^2) = \Delta^{\tau-a-1}\int_{V_0\Delta}^{\infty} dz \frac{\exp\left[-z + \frac{\gamma^2}{\Delta^{\varkappa}}z^{\varkappa}\right]}{z^{\tau-a}} \equiv \Delta^{\tau-a-1}\mathcal{K}_{\tau-a}(1, \gamma^2\Delta^{-\varkappa}). \quad (22)$$

This result shows that in the limit $\gamma \to 0$, when the rightmost singularity must approach $s_Q(T, \mu_B)$ from above, i.e., $\Delta \to 0^+$, the function (22) behaves as $\mathcal{K}_{\tau-a}(\Delta, \gamma^2) \sim \Delta^{\tau-a-1} + O(\Delta^{\tau-a})$. This is so because for $\gamma \to 0$ the ratio $\gamma^2\Delta^{-\varkappa}$ cannot go to infinity; otherwise the function $\mathcal{K}_{\tau-1}\left(1, \gamma^2\Delta^{-\varkappa}\right)$, which enters into the right-hand side of Eq. (20), would diverge exponentially and this makes impossible the existence of the solution of Eq. (5) for $T = T_{\rm cep}$. The analysis shows that for $\gamma \to 0$ there exist two possibilities: either $\nu \equiv \gamma^2\Delta^{-\varkappa} \to$ Const or $\nu \equiv \gamma^2\Delta^{-\varkappa} \to 0$. The most straightforward way to analyze these possibilities for $\gamma \to 0$ is to assume the following behavior,

$$\Delta = A\gamma^\alpha + O(\gamma^{\alpha+1}), \quad (23)$$

$$\frac{\partial\Delta}{\partial T} = \frac{\partial\gamma}{\partial T}[A\,\alpha\gamma^{\alpha-1} + O(\gamma^\alpha)] \sim \frac{(2k+1)A\,\alpha\gamma^\alpha}{2(T-T_{\rm cep})}, \quad (24)$$

and find out the α value by equating Eq. (24) with the T derivative (10).

Indeed, using Eqs. (10), (11), and (12), one can write

$$\frac{\partial\Delta}{\partial T} = \frac{G_2 + u\mathcal{K}_{\tau-\varkappa}(\Delta, \gamma^2)\,2\gamma\gamma'}{1 + u\mathcal{K}_{\tau-1}(\Delta, \gamma^2)} \approx \frac{\Delta^{2-\tau}G_2}{u\mathcal{K}_{\tau-1}(1, \nu)} + \frac{2\gamma\gamma'\Delta^{1-\varkappa}[\nu\varkappa\mathcal{K}_{\tau-2\varkappa}(1, \nu) - \mathcal{K}_{\tau-1-\varkappa}(1, \nu)]}{(\tau-1-\varkappa)\mathcal{K}_{\tau-1}(1, \nu)}, \quad (25)$$

where the prime denotes the partial T derivative. Note that the function $G_2 \equiv F' + u'\mathcal{K}_\tau(\Delta, \gamma^2) - s_Q'$ can vanish for a few values of μ_B only. In the last step of deriving Eq. (25) we used the identities (20) and (22) and dropped the nonsingular terms. As we discussed above, in the limit $\gamma \to 0$ the function ν either remains a constant or vanishes, then the term $\nu\varkappa\mathcal{K}_{\tau-2\varkappa}(1, \nu)$ in Eq. (25) either is of the same order as the constant $\mathcal{K}_{\tau-1-\varkappa}(1, \nu)$ or vanishes. Thus, to reveal the behavior of (25) for $\gamma \to 0$ it is sufficient to find a leading term out of $\Delta^{2-\tau}$ and $\gamma\gamma'\Delta^{1-\varkappa}$ and compare it with the assumption (23).

The analysis shows that for $\Delta^{2-\tau} \leqslant \gamma\gamma'\Delta^{1-\varkappa}$ the last term in the right-hand side of Eq. (25) is the leading one. Consequently, equating the powers of γ of the leading terms in Eqs. (24) and (25), one finds

$$\gamma^{\alpha-2} \sim \Delta^{1-\varkappa} \Rightarrow \alpha\varkappa = 2 \quad \text{for} \quad \tau \leqslant 1 + \frac{\varkappa}{2k+1}, \quad (26)$$

where the last inequality follows from the fact that the term $\gamma\gamma'\Delta^{1-\varkappa}$ in Eq. (25) is the dominant one.

Similarly, for $\Delta^{2-\tau} \geqslant \gamma\gamma'\Delta^{1-\varkappa}$ one obtains $\gamma^{\alpha-1}\gamma' \sim \Delta^{2-\tau}$ and, consequently,

$$\alpha = \frac{2}{(\tau-1)(2k+1)} \quad \text{for} \quad \tau \geqslant 1 + \frac{\varkappa}{2k+1}. \quad (27)$$

Summarizing our results for $\gamma \to 0$ as

$$\frac{\partial\Delta}{\partial T} \sim \frac{T_{\rm cep}\gamma^\alpha}{T - T_{\rm cep}} = \begin{cases} \left[\frac{T-T_{\rm cep}}{T_{\rm cep}}\right]^{\frac{2k+1}{\varkappa}-1}, & \tau \leqslant 1 + \frac{\varkappa}{2k+1}, \\ \left[\frac{T-T_{\rm cep}}{T_{\rm cep}}\right]^{\frac{2-\tau}{\tau-1}}, & \tau \geqslant 1 + \frac{\varkappa}{2k+1}, \end{cases} \quad (28)$$

we can also write the expression for the second derivative of Δ as

$$\frac{\partial^2 \Delta}{\partial T^2} \sim \begin{cases} \left[\frac{T-T_{\rm cep}}{T_{\rm cep}}\right]^{\frac{2k+1}{\varkappa}-2}, & \tau \leqslant 1+\frac{\varkappa}{2k+1}, \\ \left[\frac{T-T_{\rm cep}}{T_{\rm cep}}\right]^{\frac{3-2\tau}{\tau-1}}, & \tau \geqslant 1+\frac{\varkappa}{2k+1}. \end{cases} \quad (29)$$

The last result shows us that, depending on $\varkappa$ and k values, the second derivatives of $s*$ and $s_Q(T, \mu_B)$ can differ from each other for $\frac{3}{2} < \tau < 2$ or can be equal for $1 < \tau \leqslant \frac{3}{2}$. In other words, we found that at the line $T = T_{\rm cep}$ there exists the second-order PT for $\frac{3}{2} < \tau < 2$ and the higher order PT for $1 < \tau \leqslant \frac{3}{2}$, which separates the pure QGP phase from the region of a crossover, i.e., the mixed states of hadronic and QGP bags. Because it exists at the line of a zero surface tension, this PT will be called the *surface induced PT*. For instance, from Eq. (29) it follows that for $k = 0$ and $\varkappa > \frac{1}{2}$ there is the second-order PT, whereas for $k = 0$ and $\varkappa = \frac{1}{2}$ or for $k > 0$ and $\varkappa < 1$ there is the third-order PT, and so on.

Because the analysis performed in the present section did not include any μ_B derivatives of Δ, it remains valid for the μ_B dependence of the reduced surface tension coefficient, i.e., for $T_{\rm cep}(\mu_B)$. However, it is necessary to make a few comments on a possible location of the *surface tension null line* $T_{\rm cep}(\mu_B)$. In principle, such a null line can be located anywhere, if its location does not contradict the necessary conditions (17) and (18) of the first deconfinement PT existence. Thus, the surface tension null line must cross the deconfinement line in the $\mu_B - T$ plane at a single point, which is the tricritical endpoint $(\mu_B^{\rm cep}; T_{\rm cep}(\mu_B^{\rm cep}))$, whereas for $\mu_B > \mu_B^{\rm cep}$ the null line should have a temperature for the same μ_B higher than that of the deconfinement one, i.e., $T_{\rm cep}(\mu_B) > T_c(\mu_B)$ (see Fig 5). Clearly, there exist two distinct cases for the surface tension null line: either it is endless or it ends at zero temperature. But recalling that at low temperatures and high values of the baryonic chemical potential there may exist the Color-Flavor-Locked phase [50], it is possible that the null line may also cross the boundary of the Color-Flavor-Locked phase and, perhaps, it may create another special point at this intersection. From the present lattice QCD data, case C in Fig. 5 is the least possible.

One may wonder why this surface induced PT has not been observed so far. The main reason is that the lattice QCD calculations at nonzero μ_B are very difficult, and because of this the identification of the precise location of the critical endpoint is a highly nontrivial task [9–11]. Therefore, the identification of the second or higher order PT, which might be located in the vicinity of the deconfinement PT, can be a real challenge. In addition, for all $\mu_B > \mu_B^{\rm cep}$ the surface induced PT may lie so close to the deconfinement PT line that it would be extremely difficult to observe it at the present lattices.

To understand the meaning of the surface induced PT it is instructive to quantify the difference between phases by looking into the mean size of the bag:

$$\langle v \rangle \equiv -\frac{\partial \ln F(s, T, \mu_B)}{\partial s}\bigg|_{s=s^*-0}. \quad (30)$$

As was shown in hadronic phase phase $\Delta > 0$ and, hence, it consists of the bags of finite mean volumes, whereas, by construction, the QGP phase is a single infinite bag. For the crossover states $\Delta > 0$ and, therefore, they are the bags of finite mean volumes, which gradually increase, if the rightmost singularity approaches $s_Q(T, \mu_B)$, i.e., at very large values of T and/or μ_B. Such a classification is useful to distinguish QCD phases of present model: it shows that hadronic and crossover states are separated from the QGP phase by the first-order deconfinement PT and by the second or higher order PT, respectively.

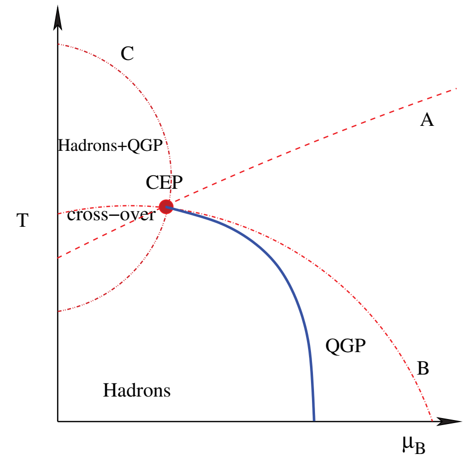

FIG. 5. (Color online) A schematic picture of the deconfinement phase transition diagram (full curve) in the plane of baryonic chemical potential μ_B and T for the second-order PT at the tricritical endpoint (CEP). The model predicts an existence of the surface induced PT of the second or higher order (depending on the model parameters). This PT starts at the CEP and goes to higher values of T and/or μ_B. Here it is shown by the dashed curve CEP-A, if the phase diagram is endless, by the dashed-dot curve CEP-B, if the phase diagram ends at $T = 0$, or by the dashed-double-dot curve CEP-C, if the phase diagram ends at $\mu_B = 0$. Below (above) each of A or B curves the reduced surface tension coefficient is positive (negative). For the curve C the surface tension coefficient is positive outside of it.

VI. CONCLUSIONS AND PERSPECTIVES

Here we suggest an analytically solvable statistical model that simultaneously describes the first- and second-order PTs with a crossover. The approach is general and can be used for more complicated parameterizations of the hadronic mass-volume spectrum, if in the vicinity of the deconfinement PT region the discrete and continuous parts of this spectrum can be expressed in the form of Eqs. (14) and (15), respectively. Also the actual parametrization of the QGP pressure $p = T s_Q(T, \mu_B)$ has not been used so far, which means that our result can be extended to more complicated functions, which can contain other phase transformations (chiral PT, or the PT to color superconducting phase) provided that the necessary conditions (17) and (18) for the deconfinement PT existence are satisfied.

In this model the desired properties of the deconfinement phase diagram are achieved by accounting for the temperature dependent surface tension of the quark-gluon bags. As we show, it is crucial for the crossover existence that at $T = T_{\rm cep}$

the reduced surface tension coefficient vanishes and remains negative for temperatures above $T_{\rm cep}$. Then the deconfinement $\mu_B - T$ phase diagram has the first-order PT at $\mu_B > \mu_B^c(T_{\rm cep})$ for $\frac{3}{2} < \tau < 2$, which degenerates into the second-order PT (or higher order PT for $\frac{3}{2} \geqslant \tau > 1$) at $\mu_B = \mu_B^c(T_{\rm cep})$, and a crossover for $0 \leqslant \mu_B < \mu_B^c(T_{\rm cep})$. These two ingredients drastically change the critical properties of the GBM [4] and resolve the long-standing problem of a unified description of the first- and second-order PTs and a crossover, which, despite all claims, was not resolved in Ref. [8]. In addition, we found that at the null line of the surface tension there must exist a surface induced PT of the second or higher order that separates the pure QGP from the mixed states of hadrons and QGP bags, which coexist above the crossover region (see Fig. 5). Thus, the QGBST model predicts that the QCD critical endpoint is the tricritical endpoint. It would be interesting to verify this prediction with the help of the lattice QCD analysis. For this, one will need to study the behavior of the bulk and surface contributions to the free energy of the QGP bags and/or the string connecting the static quark-antiquark pair.

In contrast to popular mean-field models, the PT mechanism in the present model is clear: it happens because of the competition of the rightmost singularities of the isobaric partition function. Because the GCP function of the QGBST model does not depend on any (baryonic, entropy, or energy) density, but depends exclusively on T, μ_B, and V, its phase diagram does not contain any back bending and/or spinodal instabilities [51] that are typical for the mean-field (=classical) models. The found exact analytical solution does not require a complicated and artificial procedure of conjugating the two parts of the equation of state in the vicinity of the critical endpoint like is done by hands in Refs. [52] and [53] because all this is automatically included in the statistical description.

Also in the QGBST model the pressure of the deconfined phase is generated by the infinite bag, whereas the discrete part of the mass-volume spectrum plays an auxiliary role even above the crossover region. Therefore, there is no reason to believe that any quantitative changes of the properties of low-lying hadronic states generated by the surrounding media (like the mass shift of the ω and ρ mesons [54]) would be the robust signals of the deconfinement PT. On the other hand, the QGP bags created in the experiments have finite mass and volume and, hence, the strong discontinuities that are typical for the first-order PT should be smeared out, which would make them hardly distinguishable from the crossover. Thus, to seriously discuss the signals of the first-order deconfinement PT and/or the tricritical endpoint, one needs to solve the finite volume version of the QGBST model like it was done for the SMM [30] and the GBM [31]. This, however, is not sufficient because, to make any reliable prediction for experiments, the finite volume equation of state must be used in hydrodynamic equations, which, unfortunately, are not suited for such a purpose. Thus, we are facing a necessity to return to the foundations of heavy ion phenomenology and to modify them according to the requirements of the experiments. The present model can be considered the next step in this direction.

Although the present model has a great advantage compared to other models because, in principle, it can be formulated on the basis of the experimental data on the degeneracies, masses, and eigenvolumes of hadronic resonances in the spirit of Ref. [38], a lot of additional work is necessary to properly study the issues addressed in Ref. [55]. Thus, above the surface tension null line the hadrons can coexist with QGP at high temperatures. Consequently, the nonrelativistic consideration of hard core repulsion in the present model should be modified to its relativistic treatment for light hadrons as is suggested in Refs. [56] and [57]. This can lead to some new effects discussed recently in Ref. [57]. Also, the realistic equation of state requires the inclusion of the temperature and mass dependent width of heavy resonances into a continuous part of the mass-volume spectrum, which may essentially modify our understanding of the crossover mechanism [58].

Finally, a precise temperature dependence of the surface tension coefficient along with the role of the curvature part of free energy of the bags should be investigated and their relation to the interquark string tension should be studied in detail. For this it will be necessary to modify the Hills and Dales Model [36,37] to include the surface deformations with the base of arbitrary size, because its present formulation is suited for discrete clusters and, hence, for discrete bases of surface deformations.

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