Anomalous thermodynamics and phase transitions in neutron star matter

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The presence of the long-range Coulomb force in dense stellar matter implies that the total charge cannot be associated with a chemical potential, even if it is a conserved quantity. As a consequence, the analytical properties of the partition sum are modified, changing the order of the phase transitions and affecting the possible occurrence of critical behaviors. The peculiar thermodynamic properties of this system can be understood introducing a model Hamiltonian where each charge is independently neutralized by a uniform background of opposite charge. Some consequences on the characteristics of mixed-phase structures in neutron star crusts and supernova cores are discussed.

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I. INTRODUCTION

Among the major quests in investigating the properties of compact-star matter is the underlying equation of state. In particular, exotic phases (the so-called pasta phases) are predicted in the outer crust of neutron stars as well as in the inner core of exploding supernovae. In this latter physical situation matter attains temperatures of several MeV [1–3], and the possible existence of such dishomogeneous structures is known to have some relevance for hydrodynamic properties of stellar matter and neutrino transport [4,5].

The interplay between the short-range nuclear force and the long-range nonsaturating Coulomb interaction is responsible for the existence of such phases. First calculations of the composition of the stellar crust were done using the liquid drop model or in the Thomas-Fermi approximation [6–10] with an a priori assumption on the different species constituting the matter [6,10] or on the shape of the pasta structures [7–9]: the preferred shape was essentially determined by the balance between surface and Coulomb energies. In these studies, as well as in more recent ones along the same lines [11–14], charge neutrality is imposed only globally, whereas charge fluctuations are allowed on any (small) scale, meaning that the effect of the electron background is neglected. Then the problem reduces [15] to a simple application of Gibbs phase-coexistence conditions in multifluid systems [16]. As a consequence, the transition from the homogeneous outer core to the pasta phase(s) is systematically considered as first order in all these works.

However, the conditions to define a thermodynamic limit in systems involving nonsaturating forces are not trivial and phase transitions may have specific properties analogous to those discussed for finite systems [17]. In this article we show that the suppression of one thermodynamic degree of freedom due to the divergence of the Coulomb energy density for any net charge at the thermodynamic limit implies the disappearance of phase transitions or a modification of their order. Specifically, we show that the crust-core transition is not first order but continuous.

Another limitation of the first seminal works on the phase structure of stellar matter is that most calculations were done in the mean-field approximation, which is known to be especially poor in describing phase transitions. Recent calculations with semiclassical models beyond the mean field [4,5] naturally include thermal fluctuations and allow the rearrangement of the proton distribution due to the Coulomb field under the explicit constraint of charge neutrality. The inclusion of such fluctuations was expected to lead to an increased matter opacity to neutrino scattering with important consequences on the supernova cooling dynamics [5,18]. Such a coherent neutrino-matter scattering is not only expected at low temperature, where the pasta-core transition was supposed to be first order, but even more in the case of the occurrence of a critical point in the postbounce supernova dynamics with the associated phenomenon of critical opalescence [18,19]. Surprisingly, the expected increase in the static form factor is not observed in finite-temperature molecular-dynamics calculations [20], indicating a fluctuation suppression respect to the simplistic scenario of a first-order phase transition.

On the basis of simple scaling arguments, we show in this article that critical behaviors can survive up to the thermodynamic limit only if the particle-density fluctuations associated with the mixed pasta phases do not correspond, close to the critical point, to charge-density fluctuations. In turn, this would imply a strong increase with temperature of free-proton drip or, alternatively, a very strong polarization of the electron field, leading to a complete charge screening of the pasta structures. Because both effects are likely to be unphysical, critical fluctuations and the associated critical opalescence are expected to be quenched.

II. THERMODYNAMIC FEATURES OF (PROTO-)NEUTRON STARS

A. Thermodynamics of charged systems

Let us consider the dense matter in neutron-star crusts and supernova cores formed of electrons (e), neutrons (n), and protons (p). The microscopic Hamiltonian reads

$$\hat{H} = \hat{H}_{np} + \hat{K}_e + \hat{V}_{ee} + \hat{V}_{pp} + \hat{V}_{ep}, \quad (1)$$

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where $\hat{H}_{np}$ is the nuclear strong interaction including the nucleon kinetic energy, $\hat{K}_e$ is the electron kinetic term, and $\hat{V}_{ii'}$ is the Coulomb interaction between different types of particles ($i \in \{e, n, p\}$):

$$\hat{V}_{ii'} = \frac{\alpha q_i q_{i'}}{1+\delta_{ii'}} \int \frac{\rho_i(\vec{r})\rho_{i'}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r} d\vec{r}', \tag{2}$$

where α is the fine-structure constant and $\rho_i(\vec{r})$ is the local density of the particle of type i and charge $q_i = \pm 1$.

Because the Coulomb field is a long-range interaction, the existence of a thermodynamics, i.e., the convergence of a thermodynamic limit and the equivalence between statistical ensembles, is not guaranteed [17]. Let us first consider the canonical ensemble with densities $\rho_i = N_i/\Omega$, where Ω is the volume containing a number N_i of particles $i = n, p$, or e. If the net charge density $\rho_c = \rho_p - \rho_e$ is not strictly zero at the thermodynamic limit, the Coulomb energy per unit volume $\langle V_c \rangle / \Omega \propto \rho_c^2 \Omega^{2/3}$ diverges. The strict charge neutrality demanded by the thermodynamic limit guarantees the additivity of thermodynamic potentials as well as ensemble equivalence. Indeed, the monopole contribution to the Coulomb energy between two separated neutral systems is identically zero. The longer-range multipole interaction is the dipole-dipole term $E \propto \vec{D}_1 \cdot \vec{D}_2 / R^3$, where $\vec{D}$ is the dipole moment in each subsystem. This moment is proportional to $N_c R$, where N_c is the localized charge number and R the maximum size of the actual dipole. To avoid the divergence of the Coulomb-energy density, N_c must scale at maximum proportionally to R^2, indicating the convergence of the dipole part of the intersystem energy density, and the additivity of the two subsystems at the thermodynamic limit. This reasoning can be extended to all multipole-multipole interactions. The Coulomb interaction between neutral systems then behaves like short-range interactions [17], leading to ensemble equivalence according to the standard demonstration of the Van Hove theorem.

B. Thermodynamic consequences of charge neutrality

It is important to remark that the strict neutrality constraint discussed above is very different from the trivial condition of global charge neutrality of the stellar object, which is universally recognized. We have just argued that neutrality is a necessary thermodynamic condition: this means that charge dishomogeneities can appear only at the microscopic level and no locally charged domain can exist at a macroscopic scale, even if this scale is small compared to the size of the star. From a practical point of view, this means that in hydrodynamical calculations the condition $\rho_p(\vec{r}) = \rho_e(\vec{r})$ has to be imposed at every location $\vec{r}$ and that this neutrality constraint must be explicitly applied at the scale of the Wigner Seitz cell in neutron-star crusts. This can have important effects on the phase structure of matter even if the electron background is assumed to be uniform, as we develop in great detail in the next sections.

The important consequence of the charge-neutrality constraint is that the canonical free-energy density f is defined only for $\rho_c = 0$. Hence $f(T, \rho_n, \rho_p, \rho_e) = f(T, \rho_n, \rho)$ with $\rho = (\rho_p + \rho_e)/2$ and the chemical potential associated with ρ_c cannot be defined because the free energy is not differentiable in the total-charge direction. We can notice that this can also be deduced in a grand-canonical treatment. Indeed the divergence of the Coulomb-energy density for $\rho_c \neq 0$ forces the grand-canonical partitions to fulfill the constraint $\rho_p = \rho_e$ exactly for all couples of chemical potentials μ_e and μ_p. If the chemical potentials associated with ρ_c and ρ are introduced in the form $\mu_c = (\mu_p - \mu_e)/2$ and $\mu = \mu_e + \mu_p$, respectively, the grand potential per unit volume g results independent of μ_c, so that $g(T, \mu_n, \mu, \mu_c) = g(T, \mu_n, \mu)$. Indeed, densities are grand-potential derivatives: the condition on the total charge $\rho_c = 0$ then corresponds to a flat grand-potential in the chemical-potential direction μ_c.

This suppression of one degree of freedom arises from the thermodynamic limit and should not be confused with an additional constraint as it is often done in the literature, where the two conditions of charge neutrality (expressed as $n_p = n_e$) and β equilibrium (expressed as $\mu_p = \mu_e$) are treated on the same footing [15] or confused [13]. Additional constraints, such as constant particle fraction or chemical (β) equilibrium, may or may not be realized in the supernova evolution; they are restrictions of the accessible states and do not affect the thermodynamical properties, which are state functions. Conversely, charge neutrality has to be fulfilled for each (macroscopic) physical state. This changes the the number of degrees of freedom of the thermodynamic potentials, which directly affects the thermodynamics. In particular, phase coexistence can occur only between two neutral phases. The practical consequences of this thermodynamic requirement are detailed in the next sections.

C. Quenching of mean-field instabilities in stellar matter

These considerations are especially relevant when phase transitions are concerned. Phase transitions should be analyzed by looking at the derivative and curvature properties of f in the three-dimensional state-variable space (T, ρ_n, ρ). In particular, a first-order phase transition at a given temperature identifies with the linear behavior of the free-energy density f between at least two points $A = (\rho_n^A, \rho^A)$ and $B = (\rho_n^B, \rho^B)$, meaning that all intensive parameters in A and B are equal [16]. However, because the chemical potential associated with the total charge is not a thermodynamic variable, the condition $\mu^A = \mu^B$ does not imply as it is often assumed in the literature [2,12,15,21] that the chemical potentials μ_e and μ_p are both identical in the two phases ($\mu_e^A = \mu_e^B$ and $\mu_p^A = \mu_p^B$). The difference in the chemical potential of charged particles is counterbalanced by the Coulomb force: as some electrons move from one phase to the other driven by the chemical-potential difference, the Coulomb force reacts, forbidding a macroscopic charge to appear. To illustrate this point, we consider a mean-field approximation, where the free energy of the system is defined as the sum of independent baryon f_b and electron f_e free energies,

$$f(T, \rho_n, \rho) = f_b(T, \rho_n, \rho_p = \rho) + f_e(T, \rho_e = \rho). \tag{3}$$

To spot a phase transition we can study the convexity anomalies of f looking at the curvature matrix

$$C = \begin{pmatrix} \partial\mu_n/\partial\rho_n & \partial\mu_n/\partial\rho_p \\ \partial\mu_p/\partial\rho_n & \partial\mu_p/\partial\rho_p + \partial\mu_e/\partial\rho_e \end{pmatrix}, \quad (4)$$

where we have introduced the chemical potentials $\mu_n = \partial f_b/\partial\rho_n$, $\mu_p = \partial f_b/\partial\rho_p$, and $\mu_e = \partial f_e/\partial\rho_e$. The additional term $\chi_e^{-1} = \partial\mu_e/\partial\rho_e$ in the matrix modifies the stability conditions with respect to the nuclear-matter part, i.e., to the curvature of f_b. In general, we can expect a quenching of the instability: because the electron susceptibility χ_e^{-1} is always positive, the instability conditions corresponding to Det $C \leqslant 0$ or tr $C \leqslant 0$ are more difficult to fulfill. Moreover, due to their very small mass, electrons are almost always a relativistic degenerate Fermi gas:

$$\chi_e^{-1} = \hbar c \left(\frac{\pi^2}{9\rho_e^2}\right)^{1/3}. \quad (5)$$

Because χ_e^{-1} is large at subsaturation densities, the quenching is expected to be strong. A quantitative application is shown in Fig. 1, which displays the minimal eigenvalue of the free-energy curvature matrix Eq. (4) as a function of the isospin asymmetry $\rho_3/\rho_b = (\rho_n - \rho_p)/(\rho_n + \rho_p)$, evaluated with the effective Sly230a Skyrme interaction [22,23].

At the thermodynamic limit, the nuclear free-energy density appearing in Eq. (3) is defined for each temperature and each point $(\rho_n, \rho_p = \rho)$ in the density plane by a Legendre transform of the Gibbs free-energy density

$$f_b = g + \sum_{q=n,p} \mu_q \rho_q. \quad (6)$$

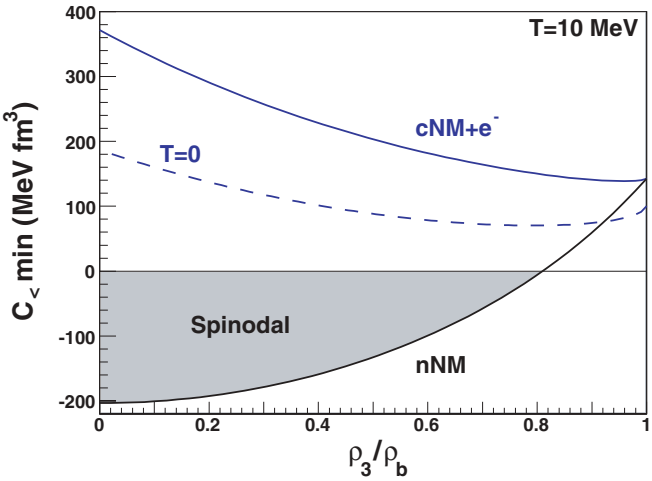

FIG. 1. (Color online) Minimal free-energy curvature of homogeneous matter at the fixed finite temperature $T = 10$ MeV in the mean-field approximation, obtained with the Sly230a interaction. Without Coulomb interaction : nNM (neutral nuclear matter). Switching on Coulomb interaction: cNM+e (charged nuclear matter neutralized by an electron gas). (Dotted line) Nuclear matter with electrons at $T = 0$.

In the mean-field approximation, this latter can be approximated using independent-particle averages only [24,25]:

$$g \approx -T\ln z_0 - \frac{1}{\Omega}(\langle\hat{W}\rangle - \langle\hat{H}\rangle). \quad (7)$$

In this expression, z_0 is the grand-canonical partition sum of a noninteracting Fermi gas of free protons and neutrons with effective mass m_q^* and chemical potentials μ_q^* shifted of the value of the mean field $\mu_q^* = \mu_q - U_q$. $\langle\hat{W}\rangle = \sum_q \langle\hat{p}^2/2m_q^* + U_q\rangle$ is the average single particle energy, $\langle\hat{H}\rangle = \Omega\mathcal{H}$ is the average energy with the chosen Skyrme functional, and the local mean field and effective mass are defined as partial derivatives of the energy density $\mathcal{H}$ with respect to the particle densities ρ_q and the momentum densities $\tau_q = \langle p^2\rangle/\hbar^2$:

$$U_q = \frac{\partial\mathcal{H}}{\partial\rho_q}; \quad \frac{\hbar^2}{2m_q^*} = \frac{\partial\mathcal{H}}{\partial\tau_q}. \quad (8)$$

All averages are thermal averages over the modified single-particle Fermi distribution

$$n_q(p) = \frac{1}{1 + \exp[\beta(p^2/2m_q^* - \mu_q^*)]}. \quad (9)$$

From Fig. 1 we can see that ordinary neutral nuclear matter (nNM) shows a wide unstable (spinodal) region characterized by a direction of negative convexity of the free energy. In this region, any density fluctuation of homogeneous matter is spontaneously amplified, and leads to phase separation. This result is drastically modified considering charged nuclear matter neutralized by an electron gas (cNM+e) : the free energy is everywhere convex, and this stays true at zero temperature (dashed line). Because the extension of the spinodal region monotonically decreases with increasing temperatures, this latter result implies that stellar matter does not present any thermodynamic instability in this model. This quenching is due to the fact that matter dishomogeneities cause charge dishomogeneities, which cannot be fully screened by the electron gas because of its high incompressibility and are thus forbidden at any (macroscopic) scale. As a consequence, the different structures in the stellar crust (free nucleons and nuclei, bubbles, or pasta) have to be considered as intrinsically microscopic objects and cannot be treated as different phases in coexistence, obeying standard phase equilibrium rules, as it is often done in the literature [10,11]. We will see in Sec. II F that such microscopic charge fluctuations naturally appear beyond the mean field, and they indeed dominate the density region between the liquid core and the gazeous outer crust. Such structures are, however, associated to an important energetic cost, which, as we will demonstrate, leads to an expansion of the dishomogeneous phase.

If the qualitative effect of electron quenching of the instability is independent of the model, the net quantitative effect on the spinodal zone depends on the parameters of the nuclear interaction: the different Skyrme forces we have analyzed [22] all lead to a complete instability suppression, whereas a small spinodal region still subsists at zero temperature in NL3 RMF stellar matter calculations [26].

D. Suppression of first-order transitions in stellar matter

To expose how the charge-neutrality constraint affects the thermodynamics of star matter, we now study the core-crust transition in two different approaches. One corresponds to the physical situation, for which electroneutrality has to be ensured by the correlation between proton and electron density distributions. For the other approach, we consider a model system where each charged particle species is independently neutralized by a uniform background of opposite charge.

The modified Hamiltonian reads

$$\hat{H}' = \hat{H}_{np} + \hat{K}_e + \hat{V}'_{ee} + \hat{V}'_{pp} + \hat{V}'_{ep}, \tag{10}$$

where the interaction with the opposite background charge has been subtracted to the various Coulomb terms

$$\hat{V}'_{ii'} = \frac{\alpha q_i q_{i'}}{1+\delta_{ii'}} \int \frac{\delta\rho_i(\vec{r})\delta\rho_{i'}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r} d\vec{r}', \tag{11}$$

with $\delta\rho_i(\vec{r}) = \rho_i(\vec{r}) - \langle\rho_i\rangle$. This system presents no divergence, so the free energy $f'(T, \rho_n, \rho_p, \rho_e)$ can always be defined. When $\rho_p = \rho_e$ the introduced backgrounds cancel out exactly. In this case, the canonical ensemble of our initial stellar problem is equivalent to the canonical ensemble of the modified Coulomb system along the $\rho_p = \rho_e$ surface:

$$f(T, \rho_n, \rho) = f'(T, \rho_n, \rho_p = \rho, \rho_e = \rho). \tag{12}$$

The two pressures are connected by the relation

$$g(T, \mu_n, \mu) = g'(T, \mu'_n, \mu'_p, \mu'_e), \tag{13}$$

where the chemical potentials $\mu'_i = \partial f'/\partial\rho_i$ of the different species $i = n, p, e$, are linked to the physical chemical potentials by the relations $\mu_n = \mu'_n, \mu = \mu'_p + \mu'_e$. Let us, however, note that this equivalence breaks down in phase coexistence, because a phase mixture in the neutralized model belongs to the physical model only if $\rho_c = 0$ in each phase separately.

In the neutralized Coulomb system, all chemical potentials and densities are allowed. This model may present phase transitions according to the usual phenomenology of multifluid systems [16]. In particular, first-order phase transitions are characterized by the equality of all intensive parameters between two points A' and B' of the particle-density space: $(\rho_n^{A'}, \rho_p^{A'}, \rho_e^{A'})$ and $(\rho_n^{B'}, \rho_p^{B'}, \rho_e^{B'})$. The linear behavior of $f'(T, \rho_n, \rho_p, \rho_e)$ between A' and B' corresponds to a single point in the intensive-parameter space $(T, \mu'_n, \mu'_p, \mu'_e)$ with different densities (i.e., pressure derivatives) on both sides.

An illustration of this approach is given by Figs. 2 and 3, presenting mean-field calculations of such neutralized model, where nuclear and electron contributions are treated independently. The strong Hamiltonian is parametrized by the Skyrme Sly230a interaction [23] and the nuclear-matter thermodynamics is evaluated within the mean-field approximation, as in the previous section [22,25,27]. Until a critical temperature, nuclear matter (nNM) presents a first-order liquid-gas phase transition. Electrons, conversely, are treated as a single homogeneous Fermi gas. This results in a violation of the relation $\rho_p = \rho_e$ in each phase, because we have $\rho_p > (<)\rho_e$ in the dense (dilute) phase. In other words,

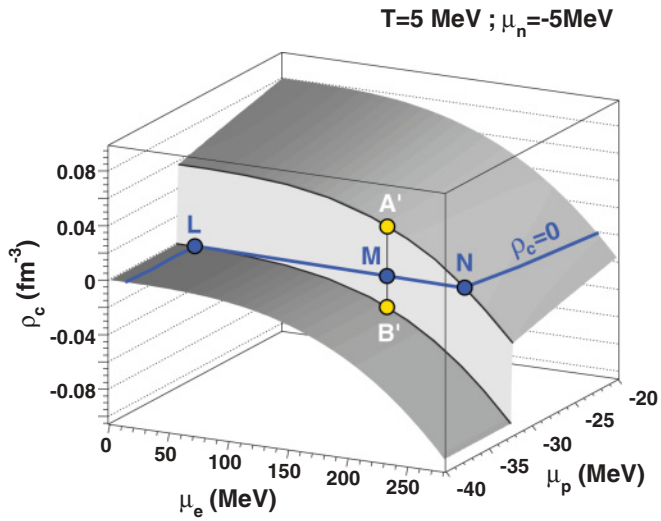

FIG. 2. (Color online) Net charge density $\rho_c = -\partial g'/\partial\mu'_c$ as a function of the proton and electron chemical potential for a neutralized Coulomb system at fixed temperature and neutron chemical potential, with the Sly230a effective interaction. The solid line gives the physical constraint $\rho_c = \text{cte} = 0$. The neutralized model presenting a first-order phase transition with a ρ_c discontinuity, the physical constraint forces the system to follow the coexistence line, leading to a continuous transition for the physical system.

$\rho_c = \rho_p - \rho_e$ is an order parameter of the phase coexistence. This is permitted within the neutralized model where proton and electron charge are independently canceled in each phase according to Eq. (11), suppressing the Coulomb contribution.

Figure 2 emphasizes the role of ρ_c as an order parameter for a phase coexistence occurring in the neutralized model. In the neutron-star phenomenology, this neutralized system corresponds to nuclear-matter calculations [12] that do not explicitly impose the neutrality constraint on the scale of the Wigner-Seitz cell. The high-density phase represents the homogeneous liquid core, whereas the low-density one is the unbound vapor. Let us consider a system such that $\rho_c = 0$. The corresponding path is represented on Fig. 3 as a projection in the chemical-potential plane (μ_e, μ_p), for different values of the neutron chemical potential μ_n. Starting from a high-density homogeneous system, and decreasing the density, the liquid-gas phase coexistence is reached at point L. Afterwards, the global requirement $\rho_c = 0$ has to be fulfilled by phase coexistence between two phases for which $\rho_c \neq 0$. An exemple is given by points A' and B', which obey the phase-equilibrium condition of equal intensive parameters. This is a first-order phase transition. Similar features are found for any fixed undercritical temperature.

However, this does not correspond to the physical situation, where electroneutrality imposes $\rho_c = 0$ at any macroscopic scale. Let us consider the line $\rho_c = 0$ represented on Fig. 2. In the neutralized model, this path crosses a coexistence region (N-L portion). Turning to the physical system, the states of this region cannot be described in terms of phase coexistence, as discussed previously: they instead consist in microscopic fluctuations constituting a single dishomogeneous phase. In this restrained $\rho_c = 0$ subspace, the grand-potential

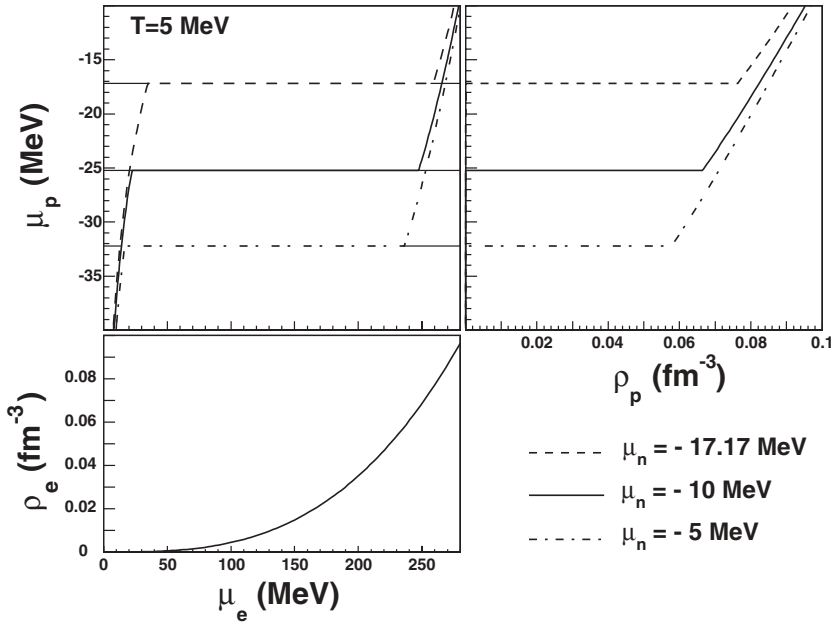

FIG. 3. (Upper left) Representation of the constraint $\rho_p = \rho_e$ as paths in the chemical-potential plane (μ_e, μ_p). (Upper righ) For each given neutron chemical potential, proton density is determined by μ_p. (Lower part) Electron density is uniquely determined by μ_e, according to Eq. (5). $\mu_n = -17.17$ MeV corresponds to the coexistence zone of symmetric matter, higher μ_n gives a matter increasingly neutron rich.

derivatives remains continuous (e.g., proton density evolves continuously): the physical free energy does not present a linear behavior, which means that the phase transition is no longer of first order. However, the path $\rho_c = 0$ is nonanalytical when it enters the region of ρ_c discontinuity (points L and N on Fig. 2). The pressure is nonanalytical at such points, which trace continuous transitions in the physical system.

To summarize, as the neutralized-model system enters the coexistence region and undergoes a first-order phase transition, the physical system restrained to the subspace $\rho_c = 0$ turns into a dishomogeneous phase *via* a continuous transition. The only way to keep a first-order phase transition in the physical system would be to have no ρ_c discontinuity at the phase transition in the neutralized system, which could be realized only if the electron distribution followed closely the proton distribution leading to a complete screening.

In the neutron-star phenomenology, the mixed phase consists in a Coulomb lattice of neutron-rich nuclei at zero temperature, or the pasta phase(s) at finite temperature. Most calculations presented in literature treat the electron dynamics at the mean-field level either by considering electrons as a uniform background [4,12,28] or allowing for a slight polarization of the electron gas, leading to a screening length of some tens of fm [5]. An explicit calculation of the electron screening in the RMF coupled to the electric field [29] confirms this assumption. In this condition, the clusters (nuclei or pasta) show a net positive charge at their (microscopic) scale, meaning that the total charge is an order parameter for the transition in the model neutralized system. As a result, the physical crust matter cannot be described as a phase coexistence: the core-crust transition occurs between uniform matter and a mixed phase presenting microscopic dishomogeneities. This transition is necessarily continuous, contrary to what is almost universally assumed in the literature [10,12–14,30]. Of course, for such dishomogeneous structures to emerge, calculations beyond the mean field are necessary, as will be discussed in the next section. In this case the phase diagram can be more complicated and present other bifurcations if the different pasta structures, here represented by a single coexistence region, correspond to different phases as suggested by molecular-dynamics calculations [30]. The limited increase of the static form factor to neutrino scattering observed in the numerical calculations of Ref. [20] is consistent with the thermodynamic arguments developed in this section.

E. Quenching of critical behavior in stellar matter

Turning now to the mixed-phase phenomenology of high-temperature supernova cores, it is interesting to remark that long-range Coulomb interactions also strongly affect critical behaviors. Indeed, the Coulomb-energy density can be expressed as a function of charge fluctuations as

$$\frac{\langle \hat{V}_c' \rangle}{\Omega} = \frac{\alpha}{2\Omega} \int \frac{\sigma_c(\vec{r}, \vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r} d\vec{r}' = 2\pi\alpha \int \sigma_c(r) r dr, \quad (14)$$

where $\sigma(\vec{r}, \vec{r}') = \langle \delta\rho_c(\vec{r}) \delta\rho_c(\vec{r}') \rangle$ is the charge-density fluctuation, and translational and rotational invariance imposes $\sigma_c(\vec{r}, \vec{r}') = \sigma_c(|\vec{r} - \vec{r}'|)$. If the electron screening is a small effect even at high temperature, i.e., if ρ_c keeps being an order parameter up to the critical point, then $\sigma_c(r)$ is expected to scale as

$$\sigma_c(r) \propto r^{2-d-\eta}, \quad (15)$$

where d is the space dimensionality and η is a critical exponent which turns out to be close to zero in most physical systems ($\eta = 0.017$ for the liquid-gas universality class). A critical point would then correspond to a divergent Coulomb

energy. Hence, only three behaviors are allowed. Either the critical temperature goes to infinity, or the order parameter becomes independent of the total charge or the phase transition in the neutralized system ends at a first-order point. The corresponding second-order point in the physical system cannot then be a critical point, and critical opalescence is suppressed. Concerning supernova cores, the presence of critical opalescence may have important consequences on the opacity to neutrino scattering [18], and neutrino transport is crucial for simulation studies of supernova explosions [4,5]. According to this qualitative argument, we do not expect a strong increase of matter opacity to neutrino scattering. To have some quantitative estimation, it is clear that calculations of stellar phase structure going beyond the mean-field approximation are necessary.

Our recent finite-size-scaling calculations within a lattice gas model confirm the suppression of critical behavior in stellar matter, as we now show [31]. In this schematic but exactly solvable model, each site of a three-dimensional lattice of $\Omega = L^3$ sites is characterized by an occupation number $n_i = 0, 1$ and an effective charge $q_i = n_i - \Sigma_{j=1}^{\Omega} n_j/\Omega$. The effective charge represents the proton distribution screened by a uniform electron background.

The schematic Hamiltonian $H = H_N + H_C$ with

$$H_N = -\frac{\epsilon}{2}\sum_{\langle ij\rangle} n_i n_j \tag{16}$$

$$H_C = \frac{\kappa}{2}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}} = \frac{\kappa}{2}\sum_{i\neq j} n_i n_j C_{ij} \tag{17}$$

is introduced to study the interplay of nuclear-like (H_N) and Coulomb-like forces (H_C). $\Sigma_{\langle ij\rangle}$ is a sum extended over closest neighbors, and r_{ij} is the distance between sites i and j. The short-range and long-range interactions are characterized by the coupling constants ϵ and $\kappa = \alpha\rho_0^{1/3}x^2$, respectively, where ρ_0 is the nuclear saturation density and x is the proton fraction.

To accelerate thermodynamic convergence, the finite lattice is repeated in all three directions of space a large number N_R of times. The Coulomb interaction with the different replicas of each site is analytically calculated and shown to be equivalent to a renormalization of the long-range couplings C_{ij} [31]. The phase diagram of the model is evaluated with standard Metropolis techniques [31] and shown to contain for all simulated lattice sizes L a coexistence region terminating at a limiting temperature $\tilde{T}_{\rm lim}(L)$. The quenching of criticality can be formally verified in terms of critical exponents. If the asymptotic value of the limiting temperature $T_{\rm lim} = lim_{L\to\infty}\tilde{T}_{\rm lim}(L)$ did correspond to a critical point, any generic thermodynamic observable Y characterized by the critical exponent Φ should fulfill close to $T_{\rm lim}$:

$$Y(L,t) = f(L/\xi(t))|t|^{\Phi}, \tag{18}$$

with $t = T/T_{\rm lim} - 1$. Scale independence close to the critical point imposes for the scaling function $f(s) \propto s^{-\Phi/\nu}$, where ν rules the divergence of the correlation length, $\xi \propto t^{-\nu}$. Specifying Eq. (18) to the behavior of the limiting temperature gives

$$|\tilde{T}_{\rm lim}(L) - T_{\rm lim}| \propto L^{-1/\nu}. \tag{19}$$

The scaling of the order parameter $\Delta\rho = \langle n\rangle_L - \langle n\rangle_G$ and of the susceptibility $\chi = \sum_{ij}^{\Omega}\langle \delta n_i \delta n_j\rangle/T$ results

$$L^{\beta/\nu}\Delta\rho \propto (L^{1/\nu}|t|)^{\beta} \quad T < \tilde{T}_{\rm lim}(L), \quad L \gg \xi \tag{20}$$

$$L^{-\gamma/\nu}\chi \propto (L^{1/\nu}|t|)^{-\gamma} \quad T \geqslant \tilde{T}_{\rm lim}(L), \quad L \gg \xi, \tag{21}$$

whereas both expressions should become independent of L and T as soon as the critical point is approached for each given finite size, $L \approx \xi$. For $T < \tilde{T}_{\rm lim}(L)$, χ contains also jumps between the low-density solution $\langle n\rangle_G$ and the high-density one $\langle n\rangle_L$ and should obey first-order scaling as $t^{-2\beta}\chi \propto L^d$. Introducing the hyperscaling relation $d = (\gamma + 2\beta)/\nu$, we get

$$L^{-\gamma/\nu}\chi \propto (L^{1/\nu}|t|)^{2\beta} \quad T < \tilde{T}_{\rm lim}(L), \quad L \gg \xi. \tag{22}$$

Figure 4 illustrates that finite-size scaling is violated for the neutralized long-range Ising model at the approach of the limiting temperature: it is not possible to find a

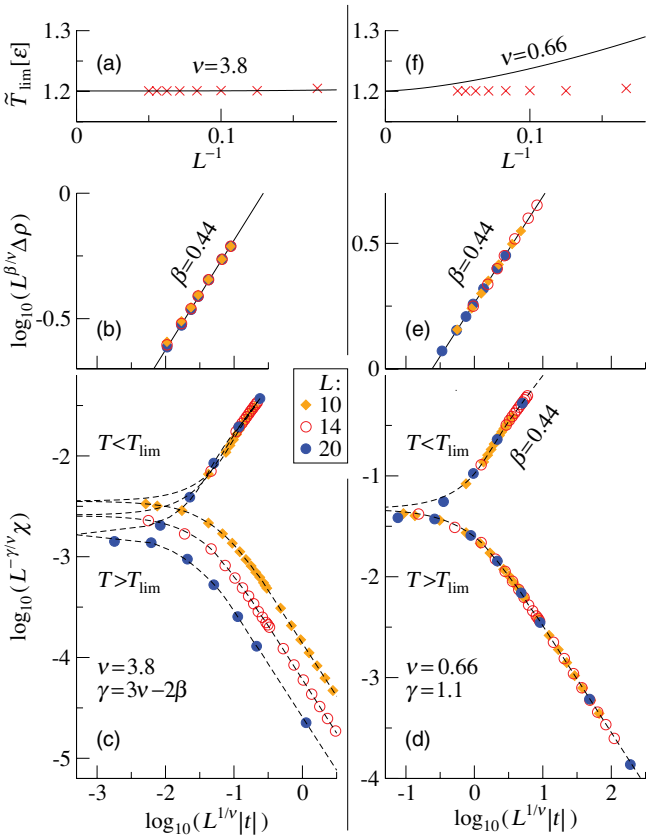

FIG. 4. (Color online) Finite-size scaling for the neutralized Ising model with long-range interactions. (Upper part) Limiting temperature as a function of the lattice linear size giving the exponent ν. (Medium part) Scaling of the order parameter giving the consistence between β and ν. (Lower part) Scaling of the susceptibility giving the consistence between γ and ν. The right and left parts of the figure give two different ways of fixing the critical exponents. None of them gives a coherent scaling of all the quantities.

thermodynamically consistent set of critical exponents fulfilling at the same time Eqs. (19)–(22). If we fix β and ν from Eqs. (19) and (20) [panels (a) and (b) in Fig. 4], the value of γ is fixed from the hyperscaling relation. The calculations with different lattice sizes do not collapse on a single curve in the representation of Eqs. (21) [Fig. 4(c)], even if the first-order scaling Eqs. (22) is nicely respected up to $\tilde{T}_{\rm lim}(L)$. If conversely we fix γ and ν such as to fulfill Eq. (21) [Fig. 4(d)], this does not affect the behavior of the order parameter Eq. (20) [Fig. 4(e)], but the scaling of the limiting temperature Eq. (19) is violated [Fig. 4(f)].

The loss of critical behavior has been already observed in Ising models with long-range frustrating interactions, where the coexistence region was seen to end at a first-order point [32]. This effect was absent in the mean-field approximation and was attributed to fluctuations.

The fact that no enhancement is observed in the static form factor of clusterized matter in the molecular-dynamics simulations by Horowitz and coworkers [20] may also be an indication of this fluctuation quenching.

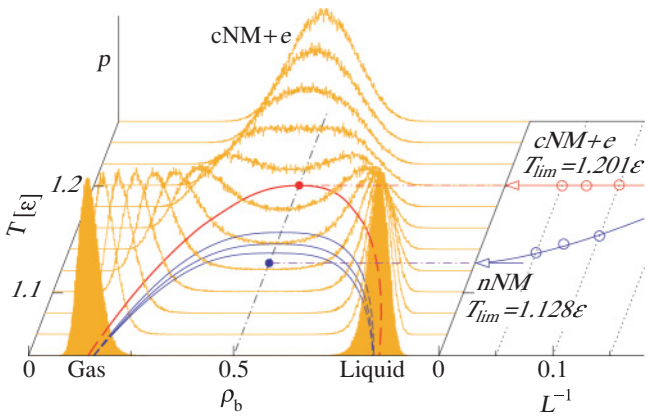

FIG. 5. (Color online) Metropolis calculations of the density distributions of the neutralized Ising model with long range interactions at different temperatures, for a cubic lattice of linear size $L = 10$ and with a proton fraction $\rho_p/\rho_b = 0.3$ (cNM). The extension of the coexistence zone, corrected for finite size effects through finite size scaling, is also shown. The lines labeled nNM give the coexistence zone of neutral matter for different lattice sizes and the extrapolation to the thermodynamic limit.

F. Extension of the pasta phases

Another remarkable effect that should manifest beyond the mean-field approximation is an extension of the mixed-phase pasta region in stellar matter with respect to standard uncharged nuclear matter. Even if at the ending point of the coexistence region the correlation length may not diverge as discussed in the previous section, we can still expect that it will be characterized by large correlated structures. As we infer from Eq. (11) and Eq. (14), in the neutralized system the Coulomb energy will be minimized in homogeneous partitions corresponding to pure phases, and it will be maximal in clusterized partitions. This increase of the energy difference between the pure phases and the mixed phase results in an extension of the coexistence region, i.e., an extension of the dishomogeneous phase corresponding to the mixed partitions situated in between the pure phases for each temperature. Consequently, the limiting temperature for the existence of such mixed phase is increased with respect to the uncharged system. This effect has been recently observed in the neutralized long-range lattice-gas model presented in the previous section [31], as shown in Fig. 5. The phase diagram of this model (cNM line in Fig. 5) is compared to the one obtained when the Coulomb interaction is neglected, $\kappa = 0$ in Eq. (17) (nNM lines in Fig. 5). For both models the phase diagram is extracted from the Metropolis simulated density distribution in the grand-canonical ensemble, also shown in the figure for different chosen temperatures.

As we have argued in Sec. II D, at the mean-field level the two models only differ for the presence of the electron pressure in cNM, which is not added in Fig. 5 and would transform the coexistence curve into a second-order transition line. For a given value of ρ_b, the charged-particle density $\rho_p = \rho_e = x\rho_b$ is fixed and so is the electron pressure [see Eq. (5)], meaning that the microstates explored by the calculation with (cNM) or without (nNM) Coulomb in the canonical ensemble are the same at the level of mean field. Figure 5 shows that this is not true once the phase diagram is numerically calculated from the exact Hamiltonian without any mean-field approximation. The limiting temperature decreases with the lattice size for nNM fulfilling finite-size scaling with standard 3d-Ising exponents [31], whereas it is almost completely independent of L when Coulomb is accounted for, reflecting the finite correlation lenght at the limiting point and criticality quenching discussed in the previous section. More important, the limiting temperature is increased for increasing strenght of the Coulomb field (about 6% for the proton fraction $x = 1/3$ considered in Fig. 5). This expansion of the coexistence region is due to the energy cost of fragmented inhomogeneous configurations at $\rho/\rho_0 \approx 0.5$ respect to the the largely uniform proton-charge distribution in pure phases, whose Coulomb energy is almost exactly compensated by the electron background. This effect is entirely due to the screening effect of the electrons, under the assumption that they can be approximated by a fixed background.

As such, the coexistence-region expansion is specific of the stellar problem, whereas other physical systems subject to Coulomb frustration exhibit the opposite behavior. This is notably the case of frustrated Ising ferromagnets, as well as of finite atomic nuclei. In such cases the Coulomb repulsion is known to reduce the limiting temperature [32–36]. This reduction is also a usual expectation in the astrophysical context [2,7,9,12–14,16]. An amplification of the mixed-phase region at zero temperature has already been reported in Ref. [29]. In this work a consistent treatment of the proton-density rearrangement effect due to the Coulomb field under the constraint of strict charge neutrality in the Wigner Seitz cell is seen to defavor, at each given total density, the formation of large charged structures (rods respect to droplets, bubbles respect to tubes). The net effect is a density amplification of the mixed phase. Our results imply that this effect should persist at finite temperature. An increase of the maximum temperature for the coexistence region might be expected,

and the mixed-phase phenomenology may be relevant for the proto-neutron-star structure in a wider temperature range than usually expected [16].

III. CONCLUSION

In this article we have shown that long-range Coulomb forces in dense stellar matter require a nontrivial discussion on the definition of the thermodynamic limit. When the constraint of total charge is not strictly zero, the divergence of the Coulomb energy implies that the chemical potential associated with the total charge loses its thermodynamic meaning. Hence, the grand potential, i.e., the system pressure, and its derivatives, i.e., the particle densities, result independent of the charge chemical potential. The suppression of a thermodynamic degree of freedom strongly affects the thermodynamics of the charged system and the phase-transition phenomenology. In general, the Coulomb field modifies the nonanalytical properties of the partition sums and changes the order of the transitions: the first-order core-crust transition obtained when the coulomb energy is disregarded, turns into a continuous transition from the homogeneous core phase to a mixed phase (the so-called pasta phases), when the coulomb energy is accounted for. A critical point can be found at finite temperature if and only if the net charge does not present fluctuations. Only in this case, critical opalescence in stellar matter can be compatible with the long-range Coulomb force.

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