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Scattering of Dirac particles from nonlocal separable potentials: The eigenchannel approach

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An application of the new formulation of the eigenchannel method [R. Szmytkowski, Ann. Phys. (N.Y.) **311**, 503 (2004)] to quantum scattering of Dirac particles from nonlocal separable potentials is presented. Eigenchannel vectors, related directly to eigenchannels, are defined as eigenvectors of a certain weighted eigenvalue problem. Moreover, negative cotangents of eigenphase-shifts are introduced as eigenvalues of that spectral problem. Eigenchannel spinor as well as bispinor harmonics are expressed throughout the eigenchannel vectors. Finally, the expressions for the bispinor as well as matrix scattering amplitudes and total cross section are derived in terms of eigenchannels and eigenphase-shifts. An illustrative example is also provided.

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I. INTRODUCTION

Recently, Szmytkowski [1] proposed a new formulation of the eigenchannel method for quantum scattering from Hermitian short-range potentials, different from that presented by Danos and Greiner [2]. Some ideas leading to this method were drawn from works on electromagnetism theory by Garbacz [3] and Harrington and Mautz [4]. This method was further extended to the case of zero-range potentials for Schrödinger particles by Szmytkowski and Gruchowski [5] and then for Dirac particles by Szmytkowski [6] (see also Ref. [7]).

On the other hand, it is a well-known fact that separable potentials, because they provide analytical solutions to the Lippmann-Schwinger equations [8], have found applications in many branches of physics, both in the nonrelativistic and the relativistic cases [9]. (It should be noted that much larger effort has been devoted to the separable potentials in the nonrelativistic regime.) Especially, their utility has been confirmed in nuclear physics by their successful use in describing nucleon-nucleon interactions [10]. Moreover, methods allowing one to approximate an arbitrary nonlocal potential by a separable one are known [11].

In view of what has been said above, it seems interesting to pose the question *how does the new method apply to quantum scattering from nonlocal separable potentials?* Partially, the answer has been given by the author by applying the method to quantum scattering of Schrödinger particles from separable potentials [12]. In the present contribution, we extend considerations from Ref. [12] to the case of Dirac particles.

This article is organized as follows. In Sec. II some facts and notions from the theory of potential scattering of Dirac particles (see Ref. [13]) are provided. In Sec. III we concentrate on the special class of nonlocal potentials, namely, separable potentials. In this context, expressions for the bispinor as well as matrix scattering amplitudes are provided. Section IV contains main ideas and results. Here, I define *eigenchannel vectors*, directly related to eigenchannels, as solutions to a certain weighted eigenproblem. Moreover, I introduce eigenphase-shifts, relating them to eigenvalues of this spectral problem. Within this approach, I also calculate expressions for the scattering amplitude and the average total cross section. In Sec. V, scattering from a rank one δ-like separable potential is discussed as an illustrative example. The article ends with two appendices.

II. QUANTUM SCATTERING OF DIRAC PARTICLES FROM NONLOCAL POTENTIALS

Let us assume that a free Dirac particle of energy E (with $|E| > mc^2$) described by the following monochromatic plane wave

$$\phi_i(\mathbf{r}) \equiv \langle \mathbf{r}|\mathbf{k}_i\chi_i\rangle = U_i(\mathbf{k}_i)e^{i\mathbf{k}_i\cdot\mathbf{r}}, \tag{2.1}$$

where

$$U_i(\mathbf{k}_i) = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{pmatrix}\chi_i\\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_i\,\chi_i\end{pmatrix}, \tag{2.2}$$

$$\varepsilon = \sqrt{\frac{E-mc^2}{E+mc^2}} \tag{2.3}$$

is being scattered from a nonlocal potential given by a kernel $\mathsf{V}(\mathbf{r},\mathbf{r}')$, which in general may be a 4×4 matrix. In the above equation, χ_i stands for a normalized pure spin-$\frac{1}{2}$ state belonging to $\mathbb{C}^2$. Orientation of the spin in $\mathbb{R}^3$ will be denoted by $\boldsymbol{\nu}_i$ and is related to χ_i by $\boldsymbol{\nu}_i = \chi_i^\dagger\boldsymbol{\sigma}\chi_i$, where $\boldsymbol{\sigma}$ is a vector consisting of the standard Pauli matrices, i.e.,

$$\boldsymbol{\sigma} = \left[\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]. \tag{2.4}$$

Moreover, $\mathbf{p}_i = \hbar\mathbf{k}_i$ is a momentum of the incident particle and k denotes the Dirac wave number and is given by

$$k = \mathrm{sgn}(E)\sqrt{\frac{E^2-(mc^2)^2}{c^2\hbar^2}}. \tag{2.5}$$

Thereafter, we shall consider only Hermitian potentials, i.e., those with kernels obeying $\mathsf{V}(\mathbf{r},\mathbf{r}') = \mathsf{V}^\dagger(\mathbf{r}',\mathbf{r})$.

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For this scattering process we may write the Lippmann-Schwinger equation [8] of the form

$$\psi(\mathbf{r}) = \phi_i(\mathbf{r}) - \int_{\mathbb{R}^3} d^3\mathbf{r}' \int_{\mathbb{R}^3} d^3\mathbf{r}'' G(E,\mathbf{r},\mathbf{r}')\mathsf{V}(\mathbf{r}',\mathbf{r}'')\psi(\mathbf{r}''). \tag{2.6}$$

Function $G(E,\mathbf{r},\mathbf{r}')$ appearing above is the relativistic free-particle outgoing Green function given by

$$G(E,\mathbf{r},\mathbf{r}') = \frac{1}{4\pi c^2\hbar^2}(-ic\hbar\boldsymbol{\alpha}\cdot\nabla + \beta mc^2 + E1_4)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.7}$$

and formally is a kernel of the relativistic outgoing Green operator defined as

$$\hat{G}(E) = \lim_{\epsilon\downarrow 0}[\hat{\mathcal{H}}_0 - E - i\epsilon]^{-1}, \tag{2.8}$$

with $\hat{\mathcal{H}}_0 = -ic\hbar\boldsymbol{\alpha}\cdot\nabla + \beta mc^2$ being a Dirac free–particle Hamiltonian. Here

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.9}$$

and $1_4 = 1_2 \otimes 1_2$. It is worth noticing that within the relativistic regime the Green function (2.7) is a 4×4 matrix.

For purposes of further analysis, it is useful to introduce the projector

$$\mathcal{P}(\mathbf{k}) = \frac{c\hbar\boldsymbol{\alpha}\cdot\mathbf{k} + \beta mc^2 + E1_4}{2E}, \tag{2.10}$$

which, as one can immediately infer, may be decomposed in the following way,

$$\mathcal{P}(\mathbf{k}) = \Theta_+(\mathbf{k})\Theta_+^\dagger(\mathbf{k}) + \Theta_-(\mathbf{k})\Theta_-^\dagger(\mathbf{k}) = \frac{1}{1+\varepsilon^2}\begin{pmatrix} 1_2 & \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}} \\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}} & \varepsilon^2 1_2, \end{pmatrix}, \tag{2.11}$$

with $\Theta_\pm(\mathbf{k})$ being defined as

$$\Theta_\pm(\mathbf{k}) = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{pmatrix} \theta_\pm \\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\theta_\pm \end{pmatrix}. \tag{2.12}$$

Spinors $\theta_\pm$ constitute an arbitrary orthonormal basis in $\mathbb{C}^2$, i.e., $\theta_s^\dagger\theta_t = \delta_{st}$ $(s,t=-,+)$ and $\sum_{s=-}^{+}\theta_s\theta_s^\dagger = 1_2$. What is important for further considerations, the matrix (2.10) possesses the obvious property that $\mathcal{P}(\mathbf{k})\Theta_\pm(\mathbf{k}) = \Theta_\pm(\mathbf{k})$ and therefore

$$\mathcal{P}(\mathbf{k}_i)U_i(\mathbf{k}_i) = U_i(\mathbf{k}_i). \tag{2.13}$$

We shall be exploiting this property in later analysis.

Considering scattering processes we usually tend to find expressions for a scattering amplitude and various cross sections. To this aim we need to find an asymptotic behavior of the relativistic outgoing Green function. From Eq. (2.7), using the projector (2.10), we have

$$G(E,\mathbf{r},\mathbf{r}') \overset{r\to\infty}{\sim} \frac{E}{2\pi c^2\hbar^2}\mathcal{P}(\mathbf{k}_f)\frac{e^{ikr}}{r}e^{-i\mathbf{k}_f\cdot\mathbf{r}'}, \tag{2.14}$$

where $\mathbf{k}_f = k\mathbf{r}/r$ is a wave vector of the scattered particle. Notice that, due to the fact that we deal with elastic processes, $|\mathbf{k}_i| = |\mathbf{k}_f| = k$. After application of Eq. (2.14) to Eq. (2.6), we obtain

$$\psi(\mathbf{r}) \overset{r\to\infty}{\sim} \operatorname*{asymp}_{r\to\infty}\ \phi_i(\mathbf{r}) + A_{fi}\frac{e^{ikr}}{r}, \tag{2.15}$$

where A_{fi} is the bispinor scattering amplitude and is defined through the relation

$$A_{fi} = -\frac{E}{2\pi c^2\hbar^2}\mathcal{P}(\mathbf{k}_f)\int_{\mathbb{R}^3} d^3\mathbf{r}' \times \int_{\mathbb{R}^3} d^3\mathbf{r}''\, e^{-i\mathbf{k}_f\cdot\mathbf{r}'}\mathsf{V}(\mathbf{r}',\mathbf{r}'')\psi(\mathbf{r}'') \tag{2.16}$$

and, in general, is of the form

$$A_{fi} = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{pmatrix} \chi_f \\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_f\chi_f \end{pmatrix}. \tag{2.17}$$

Here χ_f is a spinor transformed from the initial spinor χ_i by the scattering process. Vector $\boldsymbol{\nu}_f = (\chi_f^\dagger\boldsymbol{\sigma}\chi_f)/(\chi_f^\dagger\chi_f)$ responds for an orientation of the spin of the scattered particle. Therefore let us assume that there exists a matrix such that $\chi_f = \mathscr{A}_{fi}\chi_i$. Then it is easy to verify that the bispinor scattering amplitude may be written in the form

$$A_{fi} = \mathcal{A}_{fi}U_i(\mathbf{k}_i), \tag{2.18}$$

where the matrix $\mathcal{A}_{fi}$ is related to $\mathscr{A}_{fi}$ by

$$\mathcal{A}_{fi} = \frac{1}{1+\varepsilon^2}\begin{pmatrix} \mathscr{A}_{fi} & \varepsilon\mathscr{A}_{fi}\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_i \\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_f\mathscr{A}_{fi} & \varepsilon^2\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_f\mathscr{A}_{fi}\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_i \end{pmatrix}. \tag{2.19}$$

Henceforth matrices $\mathcal{A}_{fi}$ and $\mathscr{A}_{fi}$ will be called the matrix scattering amplitudes. The differential cross section for scattering from the direction $\mathbf{k}_i$ and the spin arrangement $\boldsymbol{\nu}_i$ onto $\mathbf{k}_f$ and $\boldsymbol{\nu}_f$ is defined as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_f} = \chi_f^\dagger\chi_f = \chi_i^\dagger\mathscr{A}_{fi}^\dagger\mathscr{A}_{fi}\chi_i, \tag{2.20}$$

Subsequently, after integration of the above over all the directions of $\mathbf{k}_f$, we arrive at the total cross section

$$\sigma(\mathbf{k}_i,\boldsymbol{\nu}_i) = \oint_{4\pi} d^2\hat{\mathbf{k}}_f\ \chi_f^\dagger\chi_f. \tag{2.21}$$

Finally, averaging over all directions of incidence $\hat{\mathbf{k}}_i$ and the initial spin orientation $\hat{\boldsymbol{\nu}}_i$, one finds the average total cross section

$$\sigma_t(E) = \frac{1}{(4\pi)^2}\oint_{4\pi} d^2\hat{\mathbf{k}}_i \oint_{4\pi} d^2\hat{\boldsymbol{\nu}}_i \oint_{4\pi} d^2\hat{\mathbf{k}}_f\ \chi_f^\dagger\chi_f. \tag{2.22}$$

Obviously all the mentioned cross sections may be expressed in terms of all the scattering amplitudes A_{fi}, $\mathcal{A}_{fi}$, and $\mathscr{A}_{fi}$.

III. SPECIAL CLASS OF NONLOCAL SEPARABLE POTENTIALS

In this section we employ the above considerations to the special class of nonlocal separable potentials. As previously mentioned, such a class of potentials allows us to find solutions to the Lippmann-Schwinger equations in an analytical way.

Consider the following class of potential kernels:

$$\mathsf{V}(\mathbf{r},\mathbf{r}') = \sum_{\mu} \omega_\mu \mathsf{u}_\mu(\mathbf{r})\mathsf{u}_\mu^\dagger(\mathbf{r}'), \tag{3.1}$$

where it is assumed that in general μ may denote the arbitrary finite set of indices, i.e., $\mu = \{\mu_1, \ldots, \mu_k\}$ and all the coefficients ω_μ different from zero. Functions $\mathsf{u}_\mu(\mathbf{r})$ are assumed to be four-element columns.

Substitution of Eq. (3.1) to Eq. (2.6) leads us to the Lippmann-Schwinger equation for the separable potentials

$$\psi(\mathbf{r}) = \phi_i(\mathbf{r}) - \sum_\mu \omega_\mu \int_{\mathbb{R}^3} d^3\mathbf{r}'\, G(E,\mathbf{r},\mathbf{r}')\mathsf{u}_\mu(\mathbf{r}') \times \int_{\mathbb{R}^3} d^3\mathbf{r}''\, \mathsf{u}_\mu^\dagger(\mathbf{r}'')\psi(\mathbf{r}''), \tag{3.2}$$

which may be equivalently rewritten as a set of linear algebraic equations. Indeed, using the Dirac notation one finds

$$\sum_\mu [\delta_{\nu\mu} + \langle \mathsf{u}_\nu|\hat{G}(E)|\mathsf{u}_\mu\rangle\omega_\mu]\langle \mathsf{u}_\mu|\psi\rangle = \langle \mathsf{u}_\nu|\phi_i\rangle. \tag{3.3}$$

For further convenience we introduce the notations

$$\langle \mathsf{u}|\varphi\rangle = \begin{pmatrix} \langle \mathsf{u}_1|\varphi\rangle \\ \langle \mathsf{u}_2|\varphi\rangle \\ \vdots \end{pmatrix}, \qquad \langle\varphi|\mathsf{u}\rangle = \langle \mathsf{u}|\varphi\rangle^\dagger = (\langle\varphi|\mathsf{u}_1\rangle\langle\varphi|\mathsf{u}_2\rangle\ldots). \tag{3.4}$$

Consequently, we may rewrite Eq. (3.3) as a matrix equation $(1 + \mathsf{G}\Omega)\langle \mathsf{u}|\psi\rangle = \langle \mathsf{u}|\phi_i\rangle$ or equivalently as

$$\langle \mathsf{u}|\psi\rangle = (1 + \mathsf{G}\Omega)^{-1}\langle \mathsf{u}|\phi_i\rangle, \tag{3.5}$$

with G being a matrix composed of the elements $\langle \mathsf{u}_\nu|\hat{G}(E)|\mathsf{u}_\mu\rangle$ and $\Omega = \mathrm{diag}[\omega_\mu]$. Similarly, substituting Eq. (3.1) to Eq. (2.16) and again using Eq. (2.10), we arrive at the bispinor scattering amplitude for the separable potentials in the form

$$A_{fi} = \frac{-E}{2\pi c^2\hbar^2}\mathcal{P}(\mathbf{k}_f)\sum_\mu \omega_\mu \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{-i\mathbf{k}_f\cdot\mathbf{r}}\mathsf{u}_\mu(\mathbf{r}) \times \int_{\mathbb{R}^3} d^3\mathbf{r}'\, \mathsf{u}_\mu^\dagger(\mathbf{r}')\psi(\mathbf{r}'), \tag{3.6}$$

which, by virtue of Eqs. (2.11) and (3.5), reduces to

$$A_{fi} = \frac{-E}{2\pi c^2\hbar^2}\sum_{s=-}^{+}\Theta_s(\mathbf{k}_f)\langle\mathbf{k}_f\theta_s|\mathsf{u}\rangle\Omega\left(1+\mathsf{G}\Omega\right)^{-1}\langle \mathsf{u}|\phi_i\rangle \tag{3.7}$$

and, utilizing the fact that for all invertible matrices X and Y the relation $(\mathsf{XY})^{-1} = \mathsf{Y}^{-1}\mathsf{X}^{-1}$ is satisfied, finally to

$$A_{fi} = \frac{-E}{2\pi c^2\hbar^2}\sum_{s=-}^{+}\Theta_s(\mathbf{k}_f)\langle\mathbf{k}_f\theta_s|\mathsf{u}\rangle\left(\Omega^{-1}+\mathsf{G}\right)^{-1}\langle \mathsf{u}|\phi_i\rangle. \tag{3.8}$$

Subsequently, using Eq. (2.13), we obtain the bispinor scattering amplitude in the following form

$$A_{fi} = \frac{-E}{2\pi c^2\hbar^2}\sum_{s,t=-}^{+}\Theta_s(\mathbf{k}_f)\langle\mathbf{k}_f\theta_s|\mathsf{u}\rangle \times (\Omega^{-1}+\mathsf{G})^{-1}\langle\mathsf{u}|\mathbf{k}_i\theta_t\rangle\Theta_t^\dagger(\mathbf{k}_i)U_i(\mathbf{k}_i), \tag{3.9}$$

which, after comparison with Eq. (2.18), gives the formula for the 4×4 matrix scattering amplitude:

$$\mathcal{A}_{fi} = \frac{-E}{2\pi c^2\hbar^2}\sum_{s,t=-}^{+}\Theta_s(\mathbf{k}_f)\langle\mathbf{k}_f\theta_s|\mathsf{u}\rangle \times (\Omega^{-1}+\mathsf{G})^{-1}\langle\mathsf{u}|\mathbf{k}_i\theta_t\rangle\Theta_t^\dagger(\mathbf{k}_i), \tag{3.10}$$

and finally, after straightforward movements, for the 2×2 matrix scattering amplitude as

$$\mathscr{A}_{fi} = \frac{-E}{2\pi c^2\hbar^2}\sum_{s,t=-}^{+}\theta_s\langle\mathbf{k}_f\theta_s|\mathsf{u}\rangle\left(\Omega^{-1}+\mathsf{G}\right)^{-1}\langle\mathsf{u}|\mathbf{k}_i\theta_t\rangle\theta_t^\dagger. \tag{3.11}$$

IV. THE EIGENCHANNEL METHOD

Now we are in position to apply the eigenchannel method proposed recently by Szmytkowski [1] to scattering of the Dirac particles from potentials of the form of Eq. (3.1). As we see below, such a class of potentials allows us to formulate this method in a simplified algebraic form.

We start from the decomposition of the matrix $\Omega^{-1} + \mathsf{G}_\mathrm{D}$ into its Hermitian and non-Hermitian parts, i.e.,

$$\Omega^{-1} + \mathsf{G} = \mathsf{A} + i\mathsf{B}, \tag{4.1}$$

where matrices A and B are defined through relations

$$\mathsf{A} = \Omega^{-1} + \frac{1}{2}\left(\mathsf{G}+\mathsf{G}^\dagger\right), \quad \mathsf{B} = \frac{1}{2i}\left(\mathsf{G}-\mathsf{G}^\dagger\right). \tag{4.2}$$

It is evident from these definitions that both matrices A and B are Hermitian. Moreover, utilizing the fact that

$$\nabla\frac{e^{ik|\boldsymbol{\varrho}|}}{|\boldsymbol{\varrho}|} = \left(\frac{ike^{ik|\boldsymbol{\varrho}|}}{|\boldsymbol{\varrho}|} - \frac{e^{ik|\boldsymbol{\varrho}|}}{|\boldsymbol{\varrho}|^2}\right), \tag{4.3}$$

where $\boldsymbol{\varrho} = \mathbf{r} - \mathbf{r}'$, the straightforward calculations lead us to their matrix elements of the form

$$\mathsf{A}_{\nu\mu} = \omega_\nu^{-1}\delta_{\nu\mu} - \frac{k}{4\pi c^2\hbar^2}\int_{\mathbb{R}^3} d^3\mathbf{r}\int_{\mathbb{R}^3} d^3\mathbf{r}'\mathsf{u}_\nu^\dagger(\mathbf{r}) \times \left[ic\hbar k\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\varrho}}{|\boldsymbol{\varrho}|}y_1(k|\boldsymbol{\varrho}|) + (\beta mc^2 + E)y_0(k|\boldsymbol{\varrho}|)\right]\mathsf{u}_\mu(\mathbf{r}') \tag{4.4}$$

and

$$\mathsf{B}_{\nu\mu} = \frac{k}{4\pi c^2\hbar^2}\int_{\mathbb{R}^3} d^3\mathbf{r}\int_{\mathbb{R}^3} d^3\mathbf{r}'\mathsf{u}_\nu^\dagger(\mathbf{r}) \times \left[ic\hbar k\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\varrho}}{|\boldsymbol{\varrho}|}j_1(k|\boldsymbol{\varrho}|) + (\beta mc^2 + E)j_0(k|\boldsymbol{\varrho}|)\right]\mathsf{u}_\mu(\mathbf{r}'), \tag{4.5}$$

where $j_0(z)$, $j_1(z)$, $y_0(z)$, and $y_1(z)$ are, respectively, the Bessel and Neumann spherical functions [14]. Recall that in general $j_0(z) = (\sin z)/z$, $y_0(z) = (\cos z)/z$ and

$$j_1(z) = -\frac{\cos z}{z} + \frac{\sin z}{z^2}, \quad y_1(z) = -\frac{\sin z}{z} - \frac{\cos z}{z^2}. \tag{4.6}$$

The main idea of the present article, adopted from Ref. [1], is to construct the following weighted spectral problem:

$$\mathsf{A}X_\gamma(E) = \lambda_\gamma(E)\mathsf{B}X_\gamma(E), \tag{4.7}$$

where $X_\gamma(E)$ and $\lambda_\gamma(E)$ are, respectively, an eigenvector and an eigenvalue. Hereafter the eigenvectors $\{X_\gamma(E)\}$ are called *eigenchannel vectors*. They are directly related to the eigenchannels defined in Ref. [1] as state vectors. In fact, they constitute a projection of eigenchannels onto subspace spanned by $\mathsf{u}_\mu(\mathbf{r})$.

Using the facts that matrices A and B are Hermitian and, as it is proven in Appendix A, that the matrix B is positive semidefinite, one finds that the eigenvalues $\{\lambda_\gamma(E)\}$ are real, i.e., $\lambda_\gamma^*(E) = \lambda_\gamma(E)$. Moreover, the eigenchannels associated with different eigenvalues obey the orthogonality relation

$$X_{\gamma'}^\dagger(E)\mathsf{B}X_\gamma(E) = 0 \qquad (\lambda_{\gamma'}(E) \neq \lambda_\gamma(E)). \tag{4.8}$$

In case of degeneration of some eigenvalues one may always choose the corresponding eigenvectors to be orthogonal according to the above relation. Then, imposing the normalization $X_\gamma^\dagger(E)\mathsf{B}X_\gamma(E) = 1$, one obtains the following orthonormality relation:

$$X_{\gamma'}^\dagger(E)\mathsf{B}X_\gamma(E) = \delta_{\gamma'\gamma}. \tag{4.9}$$

From Eqs. (4.7) and (4.9) one infers that the eigenvalues $\{\lambda_\gamma(E)\}$ may be related to the matrix A as

$$\lambda_\gamma(E) = X_\gamma^\dagger(E)\mathsf{A}X_\gamma(E). \tag{4.10}$$

Similar reasoning may be carried out employing the matrices A and $\Omega^{-1} + \mathsf{G}$. Indeed, after algebraic manipulations we arrive at

$$\begin{aligned} X_{\gamma'}^\dagger(E)\mathsf{A}X_\gamma(E) &= \lambda_\gamma(E)\delta_{\gamma'\gamma}, \\ X_{\gamma'}^\dagger(E)(\Omega^{-1} + \mathsf{G})X_\gamma(E) &= [i + \lambda_\gamma(E)]\delta_{\gamma'\gamma}, \end{aligned} \tag{4.11}$$

and $\lambda_\gamma(E) = X_\gamma^\dagger(E)(\Omega^{-1} + \mathsf{G})X_\gamma(E) - i$. Because the eigenchannels $\{X_\gamma(E)\}$ are the solutions of the Hermitian eigenvalue problem, they may satisfy the following closure relations,

$$\begin{aligned} \sum_\gamma X_\gamma(E)X_\gamma^\dagger(E)\mathsf{B} &= 1, \\ \sum_\gamma \lambda_\gamma^{-1}(E)X_\gamma(E)X_\gamma^\dagger(E)\mathsf{A} &= 1, \end{aligned} \tag{4.12}$$

and

$$\sum_\gamma \frac{1}{i + \lambda_\gamma(E)} X_\gamma(E)X_\gamma^\dagger(E)(\Omega^{-1} + \mathsf{G}) = 1, \tag{4.13}$$

where 1 is an identity matrix, which dimension depends on the dimension of the matrix G. For purposes of further analyzes the above closure relations are assumed to hold. Below, we employ the above reasoning to the derivation of the scattering amplitudes. From Eq. (4.13) one deduces that

$$(\Omega^{-1} + \mathsf{G})^{-1} = \sum_\gamma \frac{1}{i + \lambda_\gamma(E)} X_\gamma(E)X_\gamma^\dagger(E). \tag{4.14}$$

After substitution of Eq. (4.14) to Eq. (3.10) and rearranging terms, we have

$$\begin{aligned} \mathcal{A}_{fi} = \frac{-E}{2\pi c^2\hbar^2} \sum_\gamma \frac{1}{i + \lambda_\gamma(E)} \sum_{s=-}^{+} \Theta_s(\mathbf{k}_f)\langle \mathbf{k}_f\theta_s|\mathsf{u}\rangle X_\gamma(E) \\ \times \sum_{t=-}^{+} X_\gamma^\dagger(E)\langle \mathsf{u}|\mathbf{k}_i\theta_t\rangle\Theta_t^\dagger(\mathbf{k}_i). \end{aligned} \tag{4.15}$$

Let us define the following angular functions,

$$\mathcal{Y}_\gamma(\mathbf{k}) = \sqrt{\frac{Ek}{8\pi^2c^2\hbar^2}} \sum_{s=-}^{+} \Theta_s(\mathbf{k})\langle \mathbf{k}\theta_s|\mathsf{u}\rangle X_\gamma(E), \tag{4.16}$$

hereafter termed the *eigenchannel bispinor harmonics*. The functions $\{\mathcal{Y}_\gamma(\mathbf{k})\}$ are orthonormal on the unit sphere (for proof, see Appendix B), i.e.,

$$\oint_{4\pi} d^2\hat{\mathbf{k}}\, \mathcal{Y}_{\gamma'}^\dagger(\mathbf{k})\mathcal{Y}_\gamma(\mathbf{k}) = \delta_{\gamma'\gamma}. \tag{4.17}$$

Application of Eq. (4.16) to Eq. (4.15) yields

$$\mathcal{A}_{fi} = \frac{4\pi}{k} \sum_\gamma e^{i\delta_\gamma(E)} \sin\delta_\gamma(E)\mathcal{Y}_\gamma(\mathbf{k}_f)\mathcal{Y}_\gamma^\dagger(\mathbf{k}_i), \tag{4.18}$$

where $\{\delta_\gamma(E)\}$ are called *eigenphase-shifts* and are related to $\{\lambda_\gamma(E)\}$ according to

$$\lambda_\gamma(E) = -\cot\delta_\gamma(E). \tag{4.19}$$

Similar considerations may be carried out for the 2×2 matrix scattering amplitude $\mathscr{A}_{fi}$. Indeed, in virtue of Eq. (2.19) we may rewrite Eq. (3.11) in the form

$$\mathscr{A}_{fi} = \frac{4\pi}{k} \sum_\gamma e^{i\delta_\gamma(E)} \sin\delta_\gamma(E)\Upsilon_\gamma(\mathbf{k}_f)\Upsilon_\gamma^\dagger(\mathbf{k}_i), \tag{4.20}$$

where the angular functions $\{\Upsilon_\gamma(\mathbf{k})\}$, hereafter called *eigenchannel spinor harmonics*, are defined as

$$\Upsilon_\gamma(\mathbf{k}) = \sqrt{\frac{Ek}{8\pi^2c^2\hbar^2}} \sum_{s=-}^{+} \theta_s\langle \mathbf{k}\theta_s|\mathsf{u}\rangle X_\gamma(E). \tag{4.21}$$

Moreover, they are orthogonal on the unit sphere (for proof, see Appendix B)

$$\oint_{4\pi} d^2\hat{\mathbf{k}}\, \Upsilon_{\gamma'}^\dagger(\mathbf{k})\Upsilon_\gamma(\mathbf{k}) = \delta_{\gamma'\gamma}, \tag{4.22}$$

and, as one can verify, are related to the eigenchannel bispinor harmonics $\{\mathcal{Y}_\gamma(\mathbf{k})\}$ via the relation

$$\mathcal{Y}_\gamma(\mathbf{k}) = \frac{1}{\sqrt{1+\varepsilon^2}} \begin{pmatrix} \Upsilon_\gamma(\mathbf{k}) \\ \varepsilon\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\Upsilon_\gamma(\mathbf{k}) \end{pmatrix}. \tag{4.23}$$

Now we are in position to compute scattering cross sections. Substitution of Eq. (4.20) to Eq. (2.20) and integration over all directions of scattering $\hat{\mathbf{k}}_f$, by virtue of Relation (4.22), yields

$$\sigma(\mathbf{k}_i, \boldsymbol{\nu}_i) = \frac{16\pi^2}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E) |\chi_i^\dagger \Upsilon_\gamma(\mathbf{k}_i)|^2. \tag{4.24}$$

To compute the total cross section averaged over all arrangements of spin of the incident particle, we have to notice that the projector onto the pure state χ_i may be written as $\chi_i \chi_i^\dagger = (1/2)[1_2 + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}]$, with $|\boldsymbol{\nu}_i| = 1$. Therefore, substituting of the above to Eq. (4.24) and averaging over all directions of $\boldsymbol{\nu}_i$, we arrive at

$$\sigma(\mathbf{k}_i) = \frac{8\pi^2}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E) \Upsilon_\gamma^\dagger(\mathbf{k}_i) \Upsilon_\gamma(\mathbf{k}_i). \tag{4.25}$$

Finally, averaging the above scattering cross section over all directions of incidence $\hat{\mathbf{k}}_i$, again by virtue of Eq. (4.22), we get the total cross section in the form

$$\sigma_t(E) = \frac{2\pi}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E). \tag{4.26}$$

It should be emphasized that all the above considerations respecting scattering cross sections may be repeated using the eigenchannel bispinor harmonics $\{\mathcal{Y}_\gamma(\mathbf{k})\}$ instead of the eigenchannel spinor harmonics $\{\Upsilon_\gamma(\mathbf{k})\}$. The significant difference is that then the integrals over $\hat{\mathbf{k}}_f$ and $\hat{\mathbf{k}}_i$ need to be calculated using Relation (4.17) instead of Eq. (4.22).

V. EXAMPLE

We conclude our considerations by providing here an illustrative example concerning the scattering from a spherical shell of radius R, centered at the origin of the coordinate system. Because of the assumption of nonlocality of potentials under consideration, we shall simulate this process by using a potential of the form

$$\mathsf{V}(\mathbf{r}, \mathbf{r}') = \omega v(\mathbf{r}) v(\mathbf{r}') 1_4, \quad v(\mathbf{r}) = \frac{1}{\sqrt{4\pi}} \frac{\delta(r - R)}{R^2}, \tag{5.1}$$

where $\omega \neq 0$. Notice that the potential defined above is the special case of that proposed recently by de Prunelé [15] (see also Ref. [16]). Scattering of the Dirac particles from δ-like potentials was also studied, e.g., in Refs. [17,18]. However, in these articles the authors considered only local potentials and not nonlocal ones.

At the very beginning, we need to bring the Potential (5.1) to the previously postulated form (3.1). To this aim, let e_1 and e_2 constitute a standard basis in $\mathbb{C}^2$, i.e., $\mathsf{e}_1 = (1\ 0)^T$ and $\mathsf{e}_2 = (0\ 1)^T$. Moreover, let $\mathsf{e}_{ij} = \mathsf{e}_i \otimes \mathsf{e}_j$ and then by virtue of the fact that $1_4 = \sum_{i,j=1}^2 \mathsf{e}_{ij} \mathsf{e}_{ij}^\dagger$, we may rewrite Eq. (5.1) as

$$\mathsf{V}(\mathbf{r}, \mathbf{r}') = \omega \sum_{i,j=1}^{2} \mathsf{u}_{ij}(\mathbf{r}) \mathsf{u}_{ij}^\dagger(\mathbf{r}), \qquad \mathsf{u}_{ij}(\mathbf{r}) = v(\mathbf{r}) \mathsf{e}_{ij}. \tag{5.2}$$

Now, we are in position to compute the matrix G. Using Eqs. (2.7) and (5.1), after straightforward integrations we have

$$\mathsf{G} = ikj_0(kR) h_0^{(+)}(kR) \begin{pmatrix} \eta_+ 1_2 & 0 \\ 0 & \eta_- 1_2 \end{pmatrix}, \tag{5.3}$$

where $\eta_\pm = (E \pm mc^2)/c^2\hbar^2$ and $h_0^{(+)}(z) = j_0(z) + iy_0(z)$ is the spherical Hankel function of the first kind. Hence, by the definitions given in Eq. (4.2), we find that the explicit forms of matrices A and B are

$$\mathsf{A} = \begin{pmatrix} [\omega^{-1} - kj_0(kR)y_0(kR)\eta_+]1_2 & 0 \\ 0 & [\omega^{-1} - kj_0(kR)y_0(kR)\eta_-]1_2 \end{pmatrix} \tag{5.4}$$

and

$$\mathsf{B} = kj_0^2(kR) \begin{pmatrix} \eta_+ 1_2 & 0 \\ 0 & \eta_- 1_2 \end{pmatrix}. \tag{5.5}$$

According to the method formulated in Sec. IV, we may construct the following spectral problem

$$\mathsf{A} X_\gamma(E) = \lambda_\gamma(E) \mathsf{B} X_\gamma(E) \qquad (\gamma = 1, 2, 3, 4), \tag{5.6}$$

which, as one can easily verify, has two different eigenvalues

$$\lambda_\pm(E) = \frac{\omega^{-1} - kj_0(kR)y_0(kR)\eta_\pm}{kj_0^2(kR)\eta_\pm} \tag{5.7}$$

and respective eigenvectors

$$\begin{aligned} X_+^{(1(2))}(E) &= \frac{1}{\sqrt{k\eta_+} j_0(kR)} \mathsf{e}_1 \otimes \mathsf{e}_{1(2)}, \\ X_-^{(1(2))}(E) &= \frac{1}{\sqrt{k\eta_-} j_0(kR)} \mathsf{e}_2 \otimes \mathsf{e}_{1(2)}. \end{aligned} \tag{5.8}$$

Then, using Eq. (4.16) and by virtue of the fact that

$$\begin{aligned} \langle \mathbf{k}\chi | \mathsf{u} \rangle &= \sqrt{\frac{4\pi}{1 + \varepsilon^2}} j_0(kR) \\ &\quad \times (\chi^\dagger \mathsf{e}_1 \ \ \chi^\dagger \mathsf{e}_2 \ \ \varepsilon \chi^\dagger \boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\, \mathsf{e}_1 \ \ \varepsilon \chi^\dagger \boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\, \mathsf{e}_2), \end{aligned} \tag{5.9}$$

we arrive at the four eigenchannel bispinor harmonics $\{\mathcal{Y}_\gamma(\mathbf{k})\}$ in the form

$$\mathcal{Y}_+^{(1(2))}(\mathbf{k}) = \frac{1}{\sqrt{4\pi(1 + \varepsilon^2)}} \begin{pmatrix} \mathsf{e}_{1(2)} \\ \varepsilon \boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\, \mathsf{e}_{1(2)} \end{pmatrix} \tag{5.10}$$

and

$$\mathcal{Y}_-^{(1(2))}(\mathbf{k}) = \frac{1}{\sqrt{4\pi(1 + \varepsilon^2)}} \begin{pmatrix} \varepsilon \boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\, \mathsf{e}_{1(2)} \\ \mathsf{e}_{1(2)} \end{pmatrix}. \tag{5.11}$$

Then, by virtue of Eq. (4.21), one obtains the eigenchannel spinor harmonics $\{\Upsilon_\gamma(\mathbf{k})\}$ in the form

$$\Upsilon_+^{(1(2))}(\mathbf{k}) = \frac{1}{\sqrt{4\pi}} \mathsf{e}_{1(2)}, \quad \Upsilon_-^{(1(2))}(\mathbf{k}) = \frac{1}{\sqrt{4\pi}} \boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\, \mathsf{e}_{1(2)}. \tag{5.12}$$

The latter may be equivalently obtained by combining Eqs. (4.23) and (5.12). Moreover, as one may easily verify, functions given by Eqs. (5.10) and (5.12) are orthonormal, respectively, in the sense of Eqs. (4.17) and (4.22).

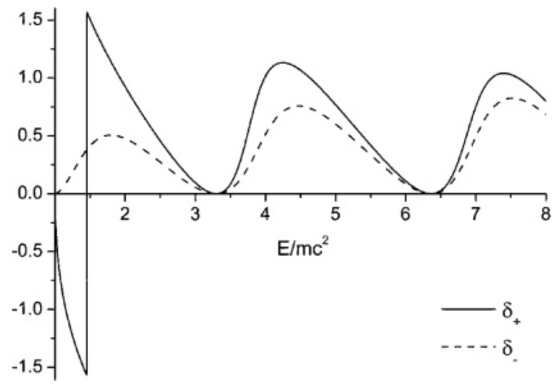

FIG. 1. Behavior of eigenphase-shifts $\delta_+(E)$ (solid curve) and $\delta_-(E)$ (dashed curve) as functions of energy E (in units of mc^2) for $\omega = -\hbar^3/m^2c$ and $R = \hbar/mc$. The eigenphase-shift $\delta_+(E)$ has been constrained to the range $[-\pi/2, \pi/2]$.

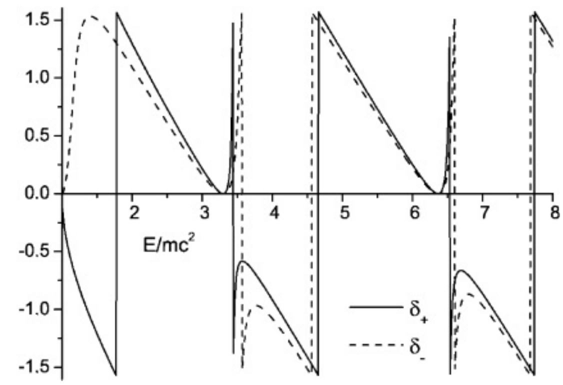

FIG. 2. Behavior of eigenphase-shifts $\delta_+(E)$ (solid curve) and $\delta_-(E)$ (dashed curve) as functions of energy E (in units of mc^2) for $\omega = -5\hbar^3/m^2c$ and $R = \hbar/mc$. Both eigenphase-shifts have been constrained to the range $[-\pi/2, \pi/2]$.

Before we find an expression for total cross section, we compute the scattering amplitude. Because, as shown in Sec. II, the bispinor and both matrix scattering amplitudes are mutually related, we restrict our considerations to the 2×2 scattering amplitude. Thus, combining Eqs. (4.20), (5.5), and (5.12) we obtain

$$\mathscr{A}_{fi} = -j_0^2(kR)\left[\frac{1_2}{ikj_0(kR)h_0^{(+)}(kR) + (\omega\eta_+)^{-1}} + \frac{(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_f)(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_i)}{ikj_0(kR)h_0^{(+)}(kR) + (\omega\eta_-)^{-1}}\right]. \tag{5.13}$$

Finally, substitution of Eqs. (5.8) and (5.12) to Eq. (4.24) with the aid of Eq. (4.19) yields

$$\sigma(\mathbf{k}_i, \boldsymbol{\nu}_i) = \frac{4\pi}{k^2} j_0^4(kR) \times \left\{\frac{1}{[(k\omega\eta_+)^{-1} - j_0(kR)y_0(kR)]^2 + j_0^4(kR)} + \frac{1}{[(k\omega\eta_-)^{-1} - j_0(kR)y_0(kR)]^2 + j_0^4(kR)}\right\}. \tag{5.14}$$

Here it is evident that $\sigma(\mathbf{k}_i, \boldsymbol{\nu}_i) = \sigma(\mathbf{k}_i) = \sigma_t(E)$.

To illustrate the obtained results, the eigenphase-shifts for two different values of ω, derived from Eqs. (4.19) and (5.7), are plotted in Figs. 1 and 2. Figures 3 and 4 present partial $\sigma_\pm(E)$ as well as total $\sigma_t(E)$ cross sections.

It seems interesting to investigate the behavior of both eigenvalues $\lambda_\pm(E)$ in the nonrelativistic limit, i.e., when $c \to \infty$. From Eq. (2.5) one concludes that

$$\eta_+ \xrightarrow{c\to\infty} \frac{2m}{\hbar^2}, \qquad \eta_- \xrightarrow{c\to\infty} 0, \tag{5.15}$$

and therefore

$$\lambda_+(E) \xrightarrow{c\to\infty} \frac{(\hbar^2/2m\omega) - kj_0(kR)y_0(kR)}{kj_0^2(kR)} \tag{5.16}$$

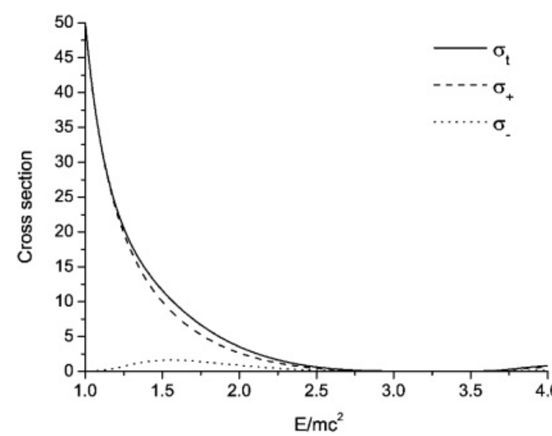

FIG. 3. Partial $\sigma_+(E)$ (dashed curve), $\sigma_-(E)$ (dotted curve), and total $\sigma_t(E)$ (solid curve) cross sections (all in units of R^2) as functions of energy E (in units of mc^2) for $\omega = -\hbar^3/m^2c$ and $R = \hbar/mc$.

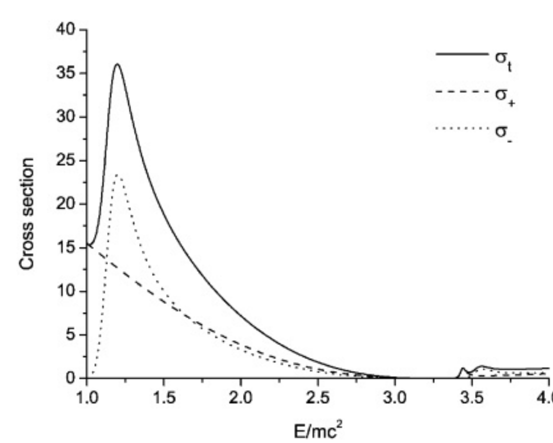

FIG. 4. Partial $\sigma_+(E)$ (dashed curve), $\sigma_-(E)$ (dotted curve), and total $\sigma_t(E)$ (solid curve) cross sections (all in units of R^2) as functions of energy E (in units of mc^2) for $\omega = -5\hbar^3/m^2c$ and $R = \hbar/mc$.

and

$$\lambda_-(E) \xrightarrow{c\to\infty} \mathrm{sgn}(\omega)\infty. \tag{5.17}$$

This means that $\delta_-(E) \to n\pi$ $(n \in \mathbb{Z})$ in the limit of $c \to \infty$. Therefore the cross section $\sigma_-(E)$ vanishes in the nonrelativistic limit and in this sense it has a purely relativistic character leading to the fact that the resonance appearing in Fig. 4 at about 1.25 mc^2 is purely relativistic effect.

One sees that in the nonrelativistic limit the cross section (5.14) reduces to

$$\sigma_t(E) \xrightarrow{c\to\infty} \frac{4\pi}{k^2} j_0^4(kR) \times \left\{ \frac{1}{[(\hbar^2/2mk\omega) - j_0(kR)y_0(kR)]^2 + j_0^4(kR)} \right\}. \tag{5.18}$$

The above cross section may also be obtained using nonrelativistic formulation of the present method given in Ref. [12].

VI. CONCLUSIONS

In this work, an application of the recently proposed eigenchannel method [1] to the scattering of Dirac particles from nonlocal separable potentials has been presented. Application of such a particular case of the nonlocal potentials reduces naturally the general weighted eigenvalue problem stated in Ref. [1] to its matrix counterpart given by Eq. (4.7) leading to the definition of eigenchannel vectors. Using the notion of the eigenchannel vectors the definitions of eigenchannel spinor and bispinor harmonics have been given. The latter provide us with the formulas for scattering amplitudes similar to those well-known for central potentials, generalizing them at the same time to the case of nonlocal separable potentials.

The general considerations have been extended with an illustrative example in which the Dirac particles are scattered from a nonlocal, δ-like potential. In this particular case, the general eigenvalue problem (4.7) becomes just a 4×4 matrix equation and therefore is easily solvable (notice that in the case of nonrelativistic scattering it would be just a one-dimensional problem). The eigenvalues of this problem are twofold degenerated and therefore give two different eigenphase-shifts from which one has a purely relativistic character in the sense that it tends to $n\pi$ $(n \in \mathbb{Z})$ whenever $c \to \infty$, giving no contribution to total cross sections in the nonrelativistic limit. One sees also that even such a simple example of nonlocal potentials may lead to some resonances (see Fig. 4).

The next step in our considerations will be to investigate the applicability of the new formulation of the eigenchannel method in the case of inelastic scattering from separable potentials. Moreover it seems also interesting to investigate the applicability of the method to the other, more complicated examples of separable potentials.

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APPENDIX A: POSITIVE SEMIDEFINITENESS OF THE MATRIX B

The proof follows the suggestions of Szmytkowski [19]. Positive semidefiniteness of the matrix B means that the inequality

$$X_\gamma^\dagger(E)\mathsf{B}X_\gamma(E) \geqslant 0 \tag{A1}$$

is satisfied. To prove the above statement let us notice that

$$\oint_{4\pi} d^2\hat{\mathbf{k}}\, e^{i\mathbf{k}\cdot\varrho}(c\hbar\boldsymbol{\alpha}\cdot\mathbf{k} + \beta mc^2 + E1_4) = 4\pi \times \left[ic\hbar k j_1(k|\varrho|)\boldsymbol{\alpha}\cdot\frac{\varrho}{|\varrho|} + (\beta mc^2 + E1_4) j_0(k|\varrho|) \right], \tag{A2}$$

where $\varrho = \mathbf{r} - \mathbf{r}'$. Then using Eq. (2.10), we may rewrite Eq. (4.5) in the form

$$\mathsf{B}_{\nu\mu} = \frac{Ek}{8\pi^2c^2\hbar^2} \oint_{4\pi} d^2\hat{\mathbf{k}} \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} \mathsf{u}_\nu^\dagger(\mathbf{r})\mathcal{P}(\mathbf{k}) \times \int_{\mathbb{R}^3} d^3\mathbf{r}'\, e^{-i\mathbf{k}\cdot\mathbf{r}'}\mathsf{u}_\mu(\mathbf{r}'), \tag{A3}$$

which after application to Eq. (A1) yields

$$X_\gamma^\dagger(E)\mathsf{B}X_\gamma(E) = \frac{mk}{8\pi^2\hbar^2}\oint_{4\pi} d^2\hat{\mathbf{k}} \times \left\| \sum_\nu X_{\gamma\nu}^*(E) \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}\mathsf{u}_\nu^\dagger(\mathbf{r})\mathcal{P}(\mathbf{k}) \right\|^2 \geqslant 0, \tag{A4}$$

finishing obviously the proof. Here $X_{\gamma\nu}(E)$ denotes the νth element of the eigenchannel vector $X_\gamma(E)$ and $||w|| = \sqrt{w^\dagger w}$ for an arbitrary vector w.

APPENDIX B: ORTHONORMALITY OF THE ANGULAR FUNCTIONS $\mathcal{Y}_\gamma(\mathbf{k})$ AND $\Upsilon_\gamma(\mathbf{k})$

We begin with the proof for the functions $\mathcal{Y}_\gamma(\mathbf{k})$. By application of Eq. (4.16) to Eq. (4.17) with the aid of Eq. (2.11) and the fact that $\mathcal{P}(\mathbf{k})$ is a projector, we can deduce that

$$\oint_{4\pi} d^2\hat{\mathbf{k}}\, \mathcal{Y}_{\gamma'^\dagger}(\mathbf{k})\mathcal{Y}_\gamma(\mathbf{k}) = \frac{Ek}{8\pi^2c^2\hbar^2}\sum_{\nu\mu} X_{\gamma'\nu}^*(E)\left[\int_{\mathbb{R}^3} d^3\mathbf{r}\int_{\mathbb{R}^3} d^3\,\mathbf{r}'\mathsf{u}_\nu^\dagger(\mathbf{r}) \times \oint_{4\pi} d^2\hat{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathcal{P}(\mathbf{k})\mathsf{u}_\mu(\mathbf{r}')\right] X_{\gamma\mu}(E), \tag{B1}$$

Comparison with Eq. (A3) shows that the square brackets in the above equation contain the expression proportional to certain elements of the matrix $\mathcal{B}$. Therefore, we may rewrite Eq. (B1) as

$$\oint_{4\pi} d^2\hat{\mathbf{k}}\, \mathcal{Y}_{\gamma'^\dagger}(\mathbf{k})\mathcal{Y}_\gamma(\mathbf{k}) = X_{\gamma'}^\dagger(E)\mathsf{B}X_\gamma(E). \tag{B2}$$

Finally, substitution of Eq. (4.9) to Eq. (B2) leads us directly to Eq. (4.17), finishing the proof. To prove the orthonormality relation for the functions $\{\Upsilon_\gamma(\mathbf{k})\}$, it suffices to combine Eq. (4.23) with Eq. (B2).

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