Two-flavor QCD phases and condensates at finite isospin chemical potential

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We study the phase structure and condensates of two-flavor QCD at finite isospin chemical potential in the framework of a confining, Dyson-Schwinger equation model. We find that the pion superfluidity phase is favored at high enough isospin chemical potential. A new gauge-invariant mixed quark-gluon condensate induced by isospin chemical potential is proposed based on operator product expansion. We investigate the sign and magnitude of this new condensate and show that it is an important condensate in QCD sum rules at finite isospin density.

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I. INTRODUCTION

The phase structure of QCD at nonzero temperature and baryon chemical potential has been intensively investigated throughout the past decade. In reality, dense baryonic matter obeys isospin asymmetry; that is, in the case of two light flavors, the densities of u and d quarks are different. For QCD to adequately describe isotopically asymmetric matter (such as in a compact star), isospin asymmetric nucleon matter, and heavy-ion collisions, the isospin chemical potential $\mu_I = (\mu_u - \mu_d)$ is usually introduced in the theory [1,2]. Different approaches—such as lattice QCD [3,4], chiral perturbation theory [1,2,5,6], ladder QCD [7], a Nambu-Jona-Lasinio-type model [8–12], and a random matrix model [13]—have been used to explore the QCD phase structure at finite isospin density. It has been widely confirmed that there is a phase transition from the normal phase to the pion superfluidity phase that is characterized by a pion condensate $\langle \bar{d}\gamma_5 u + \text{H.c.}\rangle$ at high enough isospin chemical potential. It is also found that the kaon superfluidity phase characterized by kaon condensate $\langle \bar{s}\gamma_5 u + \text{H.c.}\rangle$ appears at high isospin and strangeness chemical potential in the three light-flavors case [2,11].

The previous studies on the effects of finite isospin chemical potential and strangeness are mostly focused on two types of condensates, $\langle \bar{u}\gamma_5 d + \text{H.c.}\rangle$ and $\langle \bar{s}\gamma_5 d + \text{H.c.}\rangle$, which are order parameters for the corresponding superfluidity phase transitions. It is generally believed that the vacuum of QCD has complicated structure and it is expected that all gauge-invariant Lorentz singlet local operators built of quarks and/or gluons have nonvanishing vacuum expectation values according to QCD sum rules [14,15]. For example, the well-known low-dimensional condensates, such as quark condensate $\langle \bar{q}q\rangle$, gluon condensate $g^2\langle GG\rangle$, mixed quark gluon condensate $g\langle \bar{q}\sigma Gq\rangle$, and four-quark condensate $\langle \bar{q}\Gamma_1 q\bar{q}\Gamma_2 q\rangle$, play significant roles in the hadronic studies based on QCD sum rules.

Because of the presence of the flavor mixed condensates $\langle \bar{u}\gamma_5 d + \text{H.c.}\rangle$ and $\langle \bar{s}\gamma_5 d + \text{H.c.}\rangle$ at finite isospin and strangeness chemical potential, it is natural to expect that there should exist other new types of flavor-mixed condensates induced by isospin chemical potential and strangeness chemical potential according to operator product expansion (OPE). Besides the pion condensate and kaon condensate, the possible low-dimensional flavor-mixed condensates are mixed quark-gluon condensate $g\langle \bar{d}\gamma_5\sigma G u + \text{H.c.}\rangle$ induced by isospin density and $g\langle \bar{s}\gamma_5\sigma G u + \text{H.c.}\rangle$ induced by strangeness density. (For convenience, we call the former pion mixed quark-gluon condensate and the later kaon mixed quark-gluon condensate.) In addition, new forms of four-quark condensates, such as $\langle \bar{q}\gamma_5\tau_i\Gamma_1 q\bar{q}\Gamma_2 q\rangle$, may also appear in OPE. We expect that these induced low-dimensional condensates also play important roles on the hadronic physical observables in the framework of QCD sum rules.

It is well known that, in the chiral limit, both chiral condensate and mixed quark-gluon condensate are ideal order parameters for the chiral phase transition of QCD. Similarly, pion mixed quark-gluon condensate and kaon mixed quark-gluon condensate can play the roles of order parameters for the pion superfluidity phase transition and the kaon superfluidity phase transition at finite isospin chemical potential and strangeness chemical potential, respectively. Though both pion condensate and pion mixed quark-gluon condensate can be used as order parameters to describe the pion superfluidity phase transition, they reflect different aspects of the nonperturbative structure of the ground state: The former reflects the correlation between different flavors with color-singlet component, whereas the later reflects the correlation between different flavors with color-octet components. Therefore, pion mixed quark-gluon condensate will give new and important information on the pion superfluidity phase transition. The same thing is true for kaon condensate and kaon mixed quark-gluon condensate.

Therefore, it is interesting to investigate the thermal and dense properties of these new types of low-dimensional condensates and their effects on the physical hadronic observables. Since the global color model (GCM) [16–19] is an effective quark and gluon field theory and has been successfully used to investigate the property of the traditional mixed quark-gluon condensate [20,21] and other QCD condensates [22], we will adopt this model to explore the thermal and dense properties of these induced mixed condensates. In this paper, we only consider the possible pion superfluidity phase transition and pion mixed quark-gluon condensate.

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II. MEAN-FIELD THEORY OF GCM AT FINITE ISOSPIN CHEMICAL POTENTIAL

In the Euclidean metric, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma_\mu^+ = \gamma_\mu$, the generating functional of GCM with quark and gluon degrees is

$$Z[J, \bar\eta, \eta] = \int D\bar q Dq DA \exp\big(-S_{\rm GCM}[\bar q, q, A_\mu^a] + \bar\eta q + \bar q \eta + J_\mu^a A_\mu^a\big) \tag{1}$$

with the action

$$S_{\rm GCM}[\bar q, q, A_\mu^a] = \int \left[\bar q \left(\gamma\cdot\partial + M - igA_\mu^a \frac{\lambda^a}{2}\gamma_\mu\right) q + \frac{1}{2} A_\mu^a D_{\mu\nu}^{-1}(i\partial) A_\nu^a\right]. \tag{2}$$

The essence of GCM is that it models the QCD local gluonic action $\int F_{\mu\nu}^a F_{\mu\nu}^a$, which has local color symmetry, by a highly nonlocal action that has a global color symmetry. The main aspects of GCM have been reviewed in Refs. [17–19].

By integrating over the gluon degrees, the partition function of the GCM with two quark flavors at finite baryon and isospin chemical potential has the form

$$Z(\mu, \mu_I) = \int D\bar q(x) Dq(x) \exp\left[-\int_x \bar q(x)(\gamma_\mu \partial_\mu + M - \mu\gamma_4 - \delta\mu\tau_3\gamma_4) q(x) - \frac{1}{2}\int_x\int_y j_\mu^a(x) g^2 D_{\mu\nu}(x-y;\mu,\mu_I) j_\nu^a(y)\right], \tag{3}$$

where $M = {\rm diag}(m_u, m_d)$, $\int_x = \int d^4x$, and $j_\mu^a(x) = \bar q(x)\gamma_\mu \frac{\lambda_C^a}{2} q(x)$. (Note that we only consider the case with temperature $T = 0$ in this paper.) In Eq. (3), $\tau_i (i = 1, 2, 3)$ are the Pauli matrices in flavor space, $\mu \equiv \mu_B/3$ is the chemical potential associated with baryon number, and the quantity $\delta\mu$ is half of the isospin chemical potential (i.e. $\delta\mu = \mu_I/2$). (In this paper, we only consider $\mu_I > 0$.) The effective gluon propagator $g^2 D_{\mu\nu}(x-y;\mu,\mu_I)$ is generally a (μ,μ_I)-dependent function, which is parametrized to model the low-energy dynamics of QCD.

In this study, we will take $m_u = m_d = m$. Evidently, the Lagrangian here is invariant under the baryon $U_B(1)$ symmetry and the parity symmetry transformation P. In the case with $\mu_I \neq 0$, the traditional isospin $SU_I(2)$ symmetry is reduced to $U_{I_3}(1)$ symmetry. Usually, the quark condensate $\langle\bar q q\rangle$ is responsible for the chiral symmetry breaking of the ground state and does not spoil the parity and isotopical symmetry, whereas the nonzero pion condensate $\langle\bar q\gamma_5\tau_1 q\rangle$ breaks both the parity and isotopical symmetry of the ground state.

Within the GCM formalism, the ground state of QCD is defined by the saddle point of the action and the quark gap equation at the mean-field level is determined by the rainbow truncated quark Dyson-Schwinger equation (DSE)

$$\Sigma(p) = \frac{4}{3}\int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q)\gamma_\mu S(q)\gamma_\nu. \tag{4}$$

(The application of the DSE model to finite temperature and density is reviewed in Ref. [23].) At finite (μ, μ_I) with u and d quarks, the inverse of the quark propagator can be written in the form

$$S^{-1}(p,\mu,\mu_I) = S_0^{-1}(p,\mu,\mu_I) + \begin{pmatrix} \Sigma_{uu}(p,\mu,\mu_I) & \Sigma_{ud}(p,\mu,\mu_I) \\ \Sigma_{du}(p,\mu,\mu_I) & \Sigma_{dd}(p,\mu,\mu_I)\end{pmatrix}, \tag{5}$$

where

$$S_0^{-1}(p,\mu,\mu_I) = \begin{pmatrix} i\vec\gamma\cdot\vec p + i\gamma_4 w_u + m & \\ & i\vec\gamma\cdot\vec p + i\gamma_4 w_d + m\end{pmatrix}, \tag{6}$$

with

$$\begin{aligned} w_u &= (p_4 + i\mu + i\delta\mu), \quad w_d = (p_4 + i\mu - i\delta\mu),\\ \Sigma_{aa}(p,\mu,\mu_I) &= i\vec\gamma\cdot\vec p A_a(\vec p, w_u, w_d) + i\gamma_4 w_a B_a(\vec p, w_u, w_d) + C_a(\vec p, w_u, w_d),\end{aligned} \tag{7}$$

$$\Sigma_{ud}(p,\mu,\mu_I) = \Sigma_{du}(p,\mu,\mu_I) = i\gamma_5 D(\vec p, w_u, w_d), \tag{8}$$

and where A_a, B_a, C_a, and D are momentum-dependent scalar functions. Nonzero C_a and D are responsible for the dynamical chiral symmetry breaking and isotopical symmetry breaking, respectively.

Note that the possible diquark condensation is not considered here and only single Lorentz structure is concerned in Σ_{ud} and Σ_{du}. The four matrix elements of the momentum-dependent quark propagator

$$S(p,\mu,\mu_I) = \begin{pmatrix} S_{uu}(p,\mu,\mu_I) & S_{ud}(p,\mu,\mu_I) \\ S_{du}(p,\mu,\mu_I) & S_{dd}(p,\mu,\mu_I)\end{pmatrix} \tag{9}$$

take the form

$$\begin{aligned} S_{uu} &= [(X_d C_u + D^2 C_d) - i\vec\gamma\cdot\vec p(X_d A_u + D^2 A_d) - i\gamma_4(X_d w_u B_u + D^2 w_d B_d)]/H,\\ S_{dd} &= [(X_u C_d + D^2 C_u) - i\vec\gamma\cdot\vec p(X_u A_d + D^2 A_u) - i\gamma_4(X_u w_d B_d + D^2 w_u B_u)]/H,\\ S_{ud} &= -iD\big[\gamma_5 Y - i\gamma_5\vec\gamma\cdot\vec p S_{ud}^{\gamma_5\vec\gamma} - i\gamma_5\gamma_4 S_{ud}^{\gamma_5\gamma_4} + \gamma_5\vec\gamma\cdot\vec p\gamma_4 S_{ud}^{\gamma_5\vec\gamma\gamma_4}\big]/H,\\ S_{du} &= -iD\big[\gamma_5 Y - i\gamma_5\vec\gamma\cdot\vec p S_{du}^{\gamma_5\vec\gamma} - i\gamma_5\gamma_4 S_{du}^{\gamma_5\gamma_4} - \gamma_5\vec\gamma\cdot\vec p\gamma_4 S_{du}^{\gamma_5\vec\gamma\gamma_4}\big]/H,\end{aligned} \tag{10}$$

with

$$\begin{aligned} S_{ud}^{\gamma_5\vec\gamma} &= -S_{du}^{\gamma_5\vec\gamma} = A_u C_d - C_u A_d,\\ S_{ud}^{\gamma_5\gamma_4} &= -S_{du}^{\gamma_5\gamma_4} = w_u B_u C_d - w_d B_d C_u,\\ S_{ud}^{\gamma_5\vec\gamma\gamma_4} &= -S_{du}^{\gamma_5\vec\gamma\gamma_4} = A_u w_d B_d - A_d w_u B_u,\end{aligned} \tag{11}$$

and

$$\begin{aligned} X_a &= A_a^2\vec p^2 + B_a^2 w_a^2 + C_a^2,\\ Y &= C_u C_d + A_u A_d \vec p^2 + w_u w_d B_u B_d + D^2,\end{aligned} \tag{12}$$

$$H = X_u X_d + D^4 + 2D^2(C_u C_d + A_u A_d\vec p^2 + w_u w_d B_u B_d). \tag{13}$$

With these decomposition, the gap equation can be expressed as

$$\Sigma_{ij}(p) = \frac{4}{3}\int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q)\gamma_\mu S_{ij}(q)\gamma_\nu. \quad (14)$$

Since each term of Eq. (11) is nonzero at finite μ_I, more Lorentz structures with new scalar functions should also be considered in Σ_{ud} and Σ_{du} to guarantee the self-consistent treatment of the gap equation. However, introducing more Lorentz structures will complicate resolution of the gap equations. Just as the Lorentz tensor structure is not concerned in Σ_{aa} at the traditional treatment of DSE at finite (T, μ) [23], we suppose that $i\gamma_5 D$ is the leading order term of Σ_{ud} and Σ_{du} and other Lorentz structures have a small impact on the determination of quark self-energy. Because there are no structures $\gamma_5\vec{\gamma}$, $\gamma_5\gamma_4$, and $\gamma_5\vec{\gamma}\gamma_4$ in Σ_{ud} and Σ_{du}, the corresponding structures associated with Eq. (11) in S_{ud} and S_{du} are ignored in the following. At least, this is a good approximation in the case with small μ_I.

Because of the phenomenological nature of this effective theory, for simplicity, the Feynman-like gauge $g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu} g^2 D(p-q)$ was adopted in our calculation. With this approximation, the gap equation (14) is reduced to seven coupled integral equations, which are still complicated to solve. To get a qualitative understanding of the phase diagram and the structure of the ground state at finite μ_I, a pedagogical model first introduced by Munczek and Nemirovsky [24] for the modeling of confinement in QCD is favored in this study. The Munczek-Nemirovsky (MN) model has been extensively used to explore the properties of strong QCD both at zero (T, μ) and nonzero (T, μ) [21,25], which can always give qualitatively consistent results with the more sophisticated models. The effective gluon propagator of the MN model takes the form

$$g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu}\frac{3}{16}(2\pi)^4\eta^2\delta^4(p-q), \quad (15)$$

with the single parameter η determined by π and ρ masses in vacuum. The scale parameter η is related to the string tension of QCD, and in the more real world, it should be a function of T and μ. Using Eq. (15) simplifies the complete expressions of Eq. (14) as seven-coupled algebraic equations:

$$(A_u - 1) = \frac{1}{2}\eta^2[X_d A_u + D^2 A_d]/H,$$
$$(A_d - 1) = \frac{1}{2}\eta^2[X_u A_d + D^2 A_u]/H, \quad (16)$$

$$(B_u - 1) = \frac{1}{2}\eta^2\left[X_d B_u + \frac{w_d}{w_u}D^2 B_d\right]\Big/ H,$$
$$(B_d - 1) = \frac{1}{2}\eta^2\left[X_u B_d + \frac{w_u}{w_d}D^2 B_u\right]\Big/ H, \quad (17)$$

$$C_u - m = \eta^2[X_d C_u + D^2 C_d]/H,$$
$$C_d - m = \eta^2[X_u C_d + D^2 C_u]/H, \quad (18)$$

$$D = \eta^2 D[C_u C_d + A_u A_d \vec{p}^2 + w_u w_d B_u B_d + D^2]/H. \quad (19)$$

Equation (19) illustrates that there are two distinctive solutions to D: one characterized by $D \equiv 0$, which describes the normal phase, and the alternative, characterized by $D \neq 0$, which describes the pion superfluidity phase. The phase with small free energy is favored in nature.

III. THERMAL POTENTIAL AND CONDENSATES

In the GCM/DSE formalism, whether the normal phase or the pion superfluidity phase is stable is determined by evaluating the (μ, μ_I)-dependent pressure difference

$$\delta P(\mu, \mu_I) = P[\mu, \mu_I, S[D \neq 0]] - P[\mu, \mu_I, S[D = 0]], \quad (20)$$

where the pressure is calculated by using a steepest-descent approximation [26]:

$$P[T, \mu, \mu_I, S] = -\Omega[S] = \frac{1}{\beta V}\mathrm{Tr}\ \ln[\beta S^{-1}] - \frac{1}{\beta V}\mathrm{Tr}\ln[\Sigma S]. \quad (21)$$

Using the technique

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det(A)\det(B)\det(C)\det(C^{-1}DB^{-1} - A^{-1}), \quad (22)$$

one can express the pressure at finite (μ, μ_I) as

$$\begin{aligned} P[S] = &\int_p 2\ln[H] + 2N_c\int_p \big[D^2[\vec{p}^2(A_u + A_d) \\ &+ w_u w_d(B_u + B_d)] + X_u\big[\vec{p}^2 A_d + w_d^2 B_d\big] \\ &+ X_d\big[\vec{p}^2 A_u + w_d^2 B_u\big] + m[X_d C_u \\ &+ X_u C_d + D^2(C_u + C_d)]\big]/H, \end{aligned} \quad (23)$$

where $\int_p = \int \frac{d^4p}{(2\pi)^4}$. Note that a constant term has been ignored in Eq. (23). Though the pressure (or thermal potential) calculated through Eq. (23) is ultraviolet divergent, the pressure difference or the "bag constant" δP [16] is finite.

From the GCM generating functional, it is straightforward to calculate the vacuum expectation value (VEV) of any quark operator with the forms

$$\mathcal{O}_n \equiv \big(\bar{q}_{j_1}\Lambda^{(1)}_{j_1 i_1} q_{i_1}\big)\big(\bar{q}_{j_2}\Lambda^{(2)}_{j_2 i_2} q_{i_2}\big)\cdots\big(\bar{q}_{j_n}\Lambda^{(n)}_{j_n i_n} q_{i_n}\big) \quad (24)$$

in the mean-field vacuum. Here $\Lambda^{(i)}$ stands for an operator in Dirac, flavor, and color space. The VEV of the operator $\mathcal{O}_n$ has the form [27]

$$\langle\mathcal{O}_n\rangle = (-1)^n\sum_p(-)^p\big[\Lambda^{(1)}_{j_1 i_1}\cdots\Lambda^{(n)}_{j_n i_n} S_{i_1 j_{p(1)}}\cdots S_{i_n j_{p(n)}}\big], \quad (25)$$

where p stands for a permutation of the n indices. Based on formula (25), the low-dimensional condensates, such as chiral condensate and pion condensate can been expressed as

$$\begin{aligned}\langle\bar{u}u\rangle &= -\mathrm{Tr}_{D,C}[S_{uu}(x,x) - \sigma_{UV}(x,x)] \\ &= -N_c\int_p \mathrm{Tr}_D[S_{uu}(p) - \sigma_{UV}(p)], \end{aligned} \quad (26)$$

$$\begin{aligned}\langle\bar{d}d\rangle &= -\mathrm{Tr}_{D,C}[S_{dd}(x,x) - \sigma_{UV}(x,x)] \\ &= -N_c\int_p \mathrm{Tr}_D[S_{dd}(p) - \sigma_{UV}(p)], \end{aligned} \quad (27)$$

$$\langle \bar{q} i\gamma_5\tau_1 q\rangle = -\mathrm{Tr}_{D,C,F}[i\gamma_5\tau_1 S(x,x)] = -N_c\int_p \mathrm{Tr}_D[i\gamma_5 S_{ud}(p) + i\gamma_5 S_{du}(p)]. \quad (28)$$

To get a convergent condensate integral, a subtracting term $\sigma_{\rm UV}(p)$, which simulates the ultraviolet behavior of the quark propagator, is introduced in the definition of the quark condensate. In the case with nonzero current quark mass and zero chemical potential, for MN model one has, with $s = p^2$,

$$A(s) = B(s) = 1 + \frac{1}{2s}, \quad C(s) = m\left(1+\frac{2}{s}\right), \quad (29)$$

with corrections of high order in $(1/s)$. From these approximate expressions one can construct

$$\sigma_{\rm UV}(s) = \frac{C(s)}{sA(s)^2 + C(s)^2}. \quad (30)$$

Since $\sigma_{\rm UV}(s)\to m/s$ as $s\to\infty$ and $\sigma_{\rm UV}(s)\to 0$ as $s\to 0$, this prescription will provide an absolutely convergent result with no need for a cutoff. This definition can be generalized to the case with nonzero quark chemical potential. There is no need to introduce a subtracting term in the definition of the pion condensate since $D(p)$ remains zero in the large-momentum region in the MN model. The formula for evaluating the six-dimensional four-quark condensates can also be directly derived from Eq. (25), which is consistent with the vacuum saturation approximation at zero quark chemical potential. At finite isospin density, the new type four-quark condensates, such as $\langle \bar{q} i\gamma_5\tau_1 q\bar{q}q\rangle$, will appear in OPE. It is easy to prove that $\langle \bar{q} i\gamma_5\tau_1 q\bar{q}q\rangle \sim \langle \bar{q} i\gamma_5\tau_1 q\rangle\langle \bar{q}q\rangle$ with the approximation that only γ_5 structure holds in $\Sigma_{ud(du)}$ and $S_{ud(du)}$, with $\langle\bar{q}q\rangle = \langle \bar{u}u + \bar{d}d\rangle$.

Since the functional integration over the gluon field A_μ^a is quadratic in the framework of GCM, one can perform the integration over the gluon field analytically. By using the technique introduced by Meissner [20], through the following integral formulas:

$$\begin{aligned}
\int \mathcal{D}A e^{-\frac{1}{2}AD^{-1}A + jA} &= e^{\frac{1}{2}jDj},\\
\int \mathcal{D}A A e^{-\frac{1}{2}AD^{-1}A + jA} &= (jD)e^{\frac{1}{2}jDj},\\
\int \mathcal{D}A A^2 e^{-\frac{1}{2}AD^{-1}A + jA} &= [D + (jD)^2]e^{\frac{1}{2}jDj},\\
&\cdots
\end{aligned} \quad (31)$$

the gluon fields vacuum average can be replaced by the quark current j_μ^a with the effective gluon propagator $D(x-y)$. At the mean-field level, according to Eq. (25), one can in principle obtain the VEVs for any gluon fields. This technique provides a feasible way to calculate the VEVs of operators with low-dimensional gluon fields such as the traditional mixed quark-gluon condensate and the isospin-density-induced pion mixed quark-gluon condensate. Since the number of terms produced by Eq. (25) will increase rapidly with the number of gluonic fields, this technique is not suitable for the evaluation of the VEV of the operator involving high powers of gluonic field A. For instance, for the gluon condensate $\langle GG\rangle$, which contains an A^4 term, the calculation already gets rather involved.

Applying the method just described, we obtain the expression

$$\begin{aligned}
g\left\langle \bar{q} i\gamma_5\tau_1\sigma_{\mu\nu}G_{\mu\nu}^a\frac{\lambda_c^a}{2}q\right\rangle
&= -2iN_c\int_y \frac{4}{3}[\partial_\mu^x g^2 D(y-x)]\mathrm{Tr}_{D,F}\\
&\quad\times[S(y-x)i\gamma_5\tau_1\sigma_{\mu\nu}S(x-y)\gamma_\nu]\\
&\quad + 4iN_c\int_y\int_z g^2D(y-x)g^2D(z-x)\mathrm{Tr}_{D,F}\\
&\quad\times[S(z-x)i\gamma_5\tau_1\sigma_{\mu\nu}S(x-y)\gamma_\mu S(y-z)\gamma_\nu].
\end{aligned} \quad (32)$$

A similar expression for evaluating the traditional quark-gluon condensate $g\langle\bar{q}\sigma_{\mu\nu}G_{\mu\nu}^a\frac{\lambda_c^a}{2}q\rangle$ can be obtained by replacing the structure $i\gamma_5\tau_1\sigma_{\mu\nu}$ in Eq. (32) with $\sigma_{\mu\nu}$.

Using Eq. (14) and the formulas

$$\mathrm{Tr}_c\left[\frac{\lambda_c^a}{2}\frac{\lambda_c^b}{2}\frac{\lambda_c^c}{2} - \frac{\lambda_c^a}{2}\frac{\lambda_c^c}{2}\frac{\lambda_c^b}{2}\right] = \frac{i}{2}f^{abc}, \quad f^{abc}f^{abc} = N_c\delta^{aa}, \quad (33)$$

we can simplify the expression for the pion mixed quark-gluon condensate as

$$g\left\langle \bar{q} i\gamma_5\tau_1\sigma_{\mu\nu}G_{\mu\nu}^a\frac{\lambda_c^a}{2}q\right\rangle = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7, \quad (34)$$

where

$$\begin{aligned}
I_1 &= -72\int_p \frac{D}{H}Y\big[(A_u + A_d - 2)\vec{p}^2\\
&\quad + (B_u - 1)w_u^2 + (B_d - 1)w_d^2\big],
\end{aligned} \quad (35)$$

$$I_2 = 36\int_p \frac{D}{H}[X_dA_u + D^2A_d + X_uA_d + D^2A_u]\vec{p}^2, \quad (36)$$

$$\begin{aligned}
I_3 &= 36\int_p \frac{D}{H}\big[X_dB_uw_u^2 + D^2w_uw_dB_d\\
&\quad + X_uB_dw_d^2 + D^2w_uw_dB_u\big],
\end{aligned} \quad (37)$$

$$I_4 = \frac{81}{2}\int_p \frac{D}{H}Y[D^2 - (C_u - m)(C_d - m)], \quad (38)$$

$$\begin{aligned}
I_5 &= \frac{81}{2}\int_p \frac{D}{H}[(C_u - m)(X_dC_u + D^2C_d)\\
&\quad + (C_d - m)(X_uC_d + D^2C_u)],
\end{aligned} \quad (39)$$

$$\begin{aligned}
I_6 &= \frac{81}{2}\int_p \frac{D}{H}[(A_u - 1)(X_dA_u + D^2A_d)\\
&\quad + (A_d - 1)(X_uA_d + D^2A_u)]\vec{p}^2,
\end{aligned} \quad (40)$$

$$\begin{aligned}
I_7 &= \frac{81}{2}\int_p \frac{D}{H}\big[(B_u - 1)\big(X_dB_uw_u^2 + D^2B_dw_uw_d\big)\\
&\quad + (B_d - 1)\big(X_uB_dw_d^2 + D^2B_uw_uw_d\big)\big].
\end{aligned} \quad (41)$$

Note that the scalar functions A_a, B_a, C_a, and D are all momentum dependent in the DSE formalism. These integral expressions explicitly show that the pion mixed quark-gluon condensate is a order parameter for the pion superfluidity phase transition. It should be mentioned that the expression in

Eq. (34) for pion mixed quark-gluon condensate is only valid in Feynman-like gauge.

IV. NUMERICAL RESULTS AND DISCUSSION

To get a qualitative understanding of the finite μ_I effects on the ground state, the numerical studies in the following are all based on the MN model. Because of the defect of the supposed Lorentz structure of quark self-energy, the isospin chemical potential has been limited within the range $|\mu_I/2| < 0.2$ GeV. In this paper, for simplicity, we only concern ourselves with the case of zero temperature and baryon chemical potential.

A. The critical isospin chemical potential for pion condensate

Effective field theory arguments [1] indicate that critical isospin chemical potential μ_I^c for the pion superfluidity phase is exactly the vacuum pion mass m_π at $T = \mu_B = 0$. Whether the MN model can algebraically reproduce this result within our formalism is discussed next.

For the solution with $D \neq 0$, the gap equation (19) is reduced to

$$H = \eta^2[C_u C_d + A_u A_d \vec{p}^2 + B_u B_d w_u w_d + D^2]. \quad (42)$$

In the neighborhood of μ_{Ic}, one can probably neglect the D^2 and D^4 in this expression. According to gap equations (16) and (17), $A_{u(d)}$ and $B_{u(d)}$ are identical for $D = 0$. Therefore, the gap equation (42) can be further reduced to

$$X\left(p+\frac{P'}{2}\right)X\left(p-\frac{P'}{2}\right) = \eta^2\left[C\left(p+\frac{P'}{2}\right)C\left(p-\frac{P'}{2}\right) + A\left(p+\frac{P'}{2}\right)A\left(p-\frac{P'}{2}\right) \times \left(p+\frac{P'}{2}\right)\cdot\left(p-\frac{P'}{2}\right)\right], \quad (43)$$

where $P' = (\vec{0}, i\mu_I^c)$ and the flavor subscript has been ignored. Within the Feynman-like gauge and only considering the γ_5 structure, the Bethe-Salpeter equation (BSE) for the vacuum pseudoscalar amplitude $\Gamma_\pi^j(p, P)$ in GCM takes the form [17]

$$\Gamma_{\pi_j}(p, P) = -\frac{2}{9}\int \frac{d^4q}{(2\pi)^4} D(p-q)\mathrm{Tr}_{D,C,F}\left[i\gamma_5\tau_j{}^+ \times S\left(q+\frac{P}{2}\right)i\gamma_5\tau_j S\left(q-\frac{P}{2}\right)\right]\Gamma_{\pi_j}(q, P), \quad (44)$$

where $P^2 = -m_\pi^2$. By using $\mathrm{Tr}_F[\tau_j^+\tau_j] = 2$, the BSE (44) is greatly simplified in the MN model:

$$X\left(p+\frac{P}{2}\right)X\left(p-\frac{P}{2}\right) = \eta^2\left[A\left(p+\frac{P}{2}\right)A\left(p-\frac{P}{2}\right) \times \left(p+\frac{P}{2}\right)\cdot\left(p-\frac{P}{2}\right) + C\left(p+\frac{P}{2}\right)C\left(p-\frac{P}{2}\right)\right], \quad (45)$$

which has the same form as the gap equation (43). Therefore, if the BSE (44) can produce the vacuum pion mass, we algebraically prove $-P'^2 = m_\pi^2$ and get the conclusion $\mu_I^c = m_\pi$ at $T = \mu = 0$.

However, this proof is only true for the chiral limit case since the MN model does not support the pion bound state beyond the chiral limit if only the γ_5 structure is considered in the pseudoscalar meson Bethe-Salpeter amplitude [24]. Therefore, to produce the result $\mu_I^c = m_\pi$ beyond the chiral limit in the MN model, other Dirac amplitudes beyond $\gamma_5 D$ should also be included in the off-diagonal term of the inverse quark propagator, which will make solving the gap equation more involved.

Note that for other improved effective gluon propagators such as those used in Refs. [20,28,29], the pion Bethe-Salpeter amplitude with only γ_5 structure is a good approximation to obtain the vacuum pion mass. In principle, using these improved effective gluon propagators to explore the pion superfluidity within the DSE formalism should give more quantitatively reasonable results in contrast with the simple MN model. However, the set of seven coupled algebraic gap equations (16)–(19) will be replaced by a set of seven coupled integral equations, which leads to a more difficult numerical calculation.

For simplicity and to get a qualitatively understanding of the pion superfluidity within the DSE formalism, we still use the MN model in the following that only contains the γ_5 structure in the off-diagonal term of the inverse quark propagator.

B. The chiral limit

In the chiral limit, there are four possible solutions according to the gap equations (16)–(19):

$$\begin{aligned} &C_u = C_d = 0, D = 0; \qquad C_u \neq 0, C_d \neq 0, D = 0; \\ &C_u = C_d = 0, D \neq 0; \qquad C_u \neq 0, C_d \neq 0, D \neq 0; \end{aligned} \quad (46)$$

these characterize four possible phases of QCD at finite isospin density, respectively. However, the solution with both nonzero quark condensate and nonzero pion condensate is not found in our numerical study. By comparing the corresponding free energies of the former three possible phases, it is found that the pion superfluidity phase is favored in the chiral limit at nonzero isospin chemical potential. It seems that this is a universal result and has been confirmed by many former studies [1,2]. In this case, chiral symmetry is not broken and both quark condensate and mixed quark-gluon condensate disappear in OPE. In contrast with these vanishing chiral condensates, new condensates induced by isospin density—such as the pion condensate and the pion mixed quark-gluon condensate—appear. Note that the new four-quark condensates, such as $\langle\bar{q}i\gamma_5\tau_1 q\bar{q}q\rangle$, also vanish in this case because these condensates factorize at the mean-field level within our formalism with the approximation used in Sec. II.

The μ_I dependence of the pressure difference between the pion superfluid phase and the normal phase, pion condensate, and pion mixed quark-gluon condensate is shown in Fig. 1.

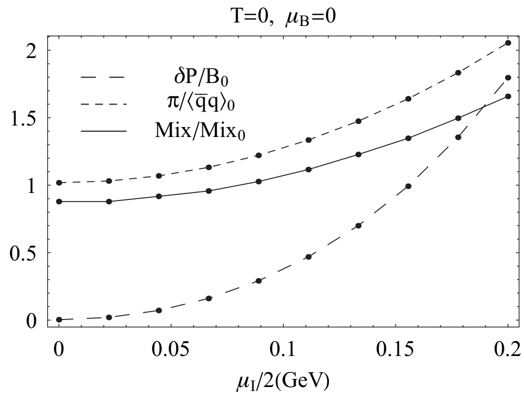

FIG. 1. The pressure difference δP, pion condensate π, and pion mixed quark-gluon condensate Mix in the chiral limit. $B_0 = (0.1\eta)^4$, $\langle \bar{q}q\rangle_0 = \langle \bar{u}u + \bar{d}d\rangle_0$, and $\mathrm{Mix}_0 = 310\ (\mathrm{MeV})^5$ are the bag constant, chiral condensate, and mixed quark-gluon condensate of the vacuum obtained from the MN model with $m = 0$ and $\eta = 1.06$ GeV, respectively.

For the case with $\mu_I = 0$, the zero pressure difference of δP suggests that the nonzero pion condensate and nonzero quark condensate correspond to equivalent but distinct vacua, which is guaranteed by chiral symmetry (i.e., a small quark mass will destabilize the superfluidity phase). In contrast with this event, for the whole domain of nonzero μ_I concerned, the positive pressure difference δP suggests that the pion superfluidity phase is the stable ground state in the chiral limit for two-flavor QCD. Figure 1 shows that the magnitudes of both induced condensates and the pressure difference are monotonically increasing functions of μ_I.

Figure 2 shows the μ_I-dependent behavior of the ratio of pion mixed quark-gluon condensate to pion condensate from the MN model. In Fig. 2, the ratio ranges from 1.56 to 1.66 $(\mathrm{GeV})^2$, which suggests that the induced mixed quark-gluon condensate has the same magnitude of the traditional mixed condensate in the domain of μ_I of concern. (In the MN model, the ratio of the mixed quark-gluon condensate and the chiral condensate is 1.92 in the vacuum [21].) Note that numerical study suggests that the pion mixed quark-gluon condensate defined in Eq. (32) is positive, which is consistent with the sign of the traditional mixed quark-gluon condensate.

C. Finite current mass

With finite current quark mass $m = 12$ MeV [24], apparently, only two types of solutions exist: the normal phase with nonzero quark condensate and zero pion condensate and the pion superfluidity phase with nonzero pion condensate and nonzero quark condensate. In contrast to the chiral limit, for finite current mass, there is no solution with zero quark condensate to the gap equations owing to the explicit chiral-symmetry-breaking term in the Lagrangian.

According to the numerical study, in the small isospin chemical potential region $\mu_I < 64$ MeV, only the normal phase solution exists. For the region $\mu_I > 64$ MeV, there also exists a solution corresponding to the pion superfluidity phase. The stable ground state in the region $\mu_I > 64$ MeV is again determined by the difference of the pressure δP in Eq. (23).

It is shown in Fig. 3 that at the point $\mu_I = 64$ MeV, the scaled pressure difference of the two solutions is close to zero and for the region $\mu_I > 64$ MeV, the pressure difference is positive and monotonically increases with increasing isospin chemical potential, which suggests that the pion superfluidity phase is favored in the $\mu_I > 64$ MeV region. From the μ_I-dependent behavior of both pressure difference δP and the two order parameters, pion condensate and pion mixed quark-gluon condensate, one can judge that the phase transition from normal phase to pion superfluidity phase is second order. This conclusion is consistent with the result obtained from lattice simulation and other model studies.

Note that at zero μ, the scalar functions $A(B, C)_u$ in S_{uu} and $A(B, C)_d$ in S_{dd} at the same point $(\vec{p}, w_u, w_d)$ are complex conjugates; therefore the relation $\langle \bar{u}u\rangle = \langle \bar{d}d\rangle$ always holds. Figure 3 shows that the magnitude of quark condensate monotonically increases with an increase of μ_I in the normal phase region, which is consistent with the dependence of the chiral condensate on the baryon chemical potential obtained within DSE formalism [23]. In the pion superfluidity phase,

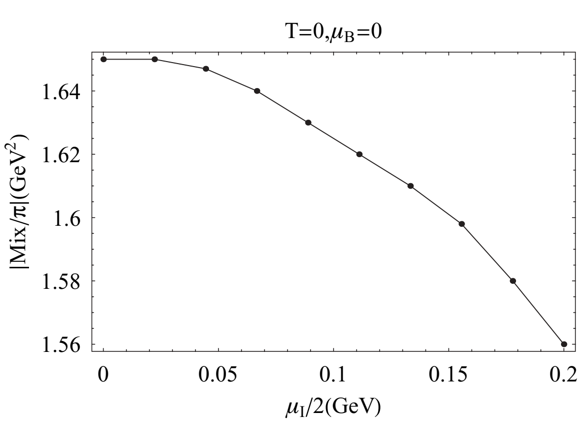

FIG. 2. The isospin chemical potential dependence of the ratio between pion mixed quark-gluon condensate and pion condensate from the MN model with $m = 0$ and $\eta = 1.06$ GeV.

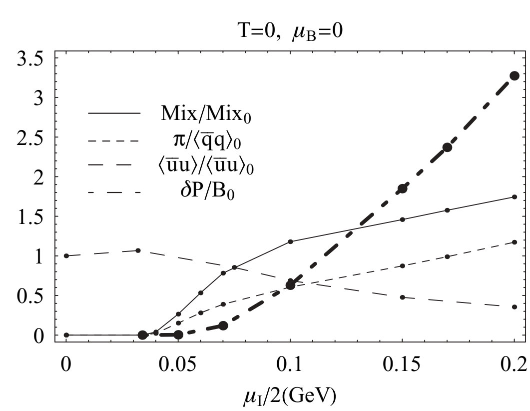

FIG. 3. The pressure difference δP (plotted only in the region $\mu_I \geqslant \mu_I^C$), quark condensate $\langle \bar{u}u\rangle$, pion condensate π, and pion mixed quark-gluon condensate Mix obtained from the MN model with $m = 12$ MeV and $\eta = 1.06$ GeV.

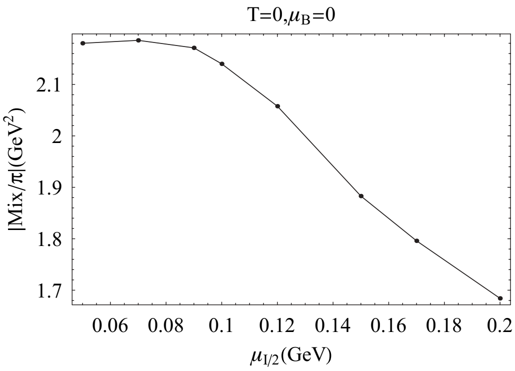

FIG. 4. The μ_I dependence of the ratio between pion mixed quark-gluon condensate and pion condensate obtained from the MN model with $m = 12$ MeV and $\eta = 1.06$ GeV.

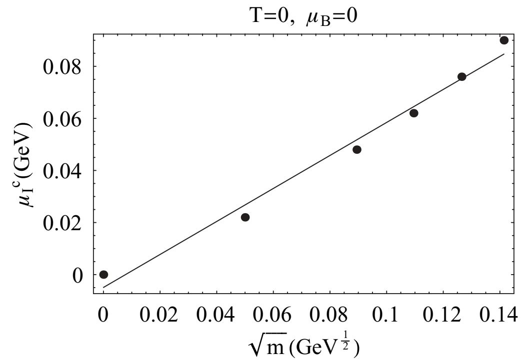

FIG. 5. The dependence of the critical isospin potential μ_I^c on the current quark mass from the MN model with $\eta = 1.06$ GeV. The current quark mass ranges from 0 to 20 MeV.

the magnitude of quark condensate monotonically decreases with an increase of μ_I, which is an anticipated result due to the monotonically increasing behavior of pion condensate with respect to μ_I. This behavior is similar to the dependence of the chiral condensate on the baryon chemical potential for the appearance of diquark condensate [23]. It is expected that the traditional mixed quark-gluon condensates, $g\langle\bar{u}\sigma Gu\rangle$ and $g\langle\bar{d}\sigma Gd\rangle$, have similar μ_I-dependent behavior. Figure 3 manifests a competitive relationship between the induced condensates and their corresponding traditional partners.

In Fig. 4, we display the numerical result of the isospin chemical potential dependence of the ratio of the pion mixed quark-gluon condensate to pion condensate. Such a value ranges from 2.2 to 1.7 $(\mathrm{GeV})^2$, which is also close to the ratio of the traditional mixed quark-gluon condensate to quark condensate obtained in the vacuum [21]. The large magnitude of the ratio suggests that the induced mixed quark-gluon condensate is an important parameter within the QCD sum rules at finite μ_I. In addition, in contrast with the chiral limit case, the nonzero pion condensate and nonzero quark condensate suggests that the new four-quark condensates also have nonzero value in the superfluidity phase, even at the mean-field level.

The critical chemical potential of 64 MeV is relatively small in contrast with the vacuum pion mass. As previously mentioned, the main reason for this discrepancy arises from the fact that only the γ_5 structure is considered in the off-diagram part of the inverse quark propagator, whereas the MN model does not support the pseudoscalar bound state beyond the chiral limit when only the γ_5 structure is contained in the pseudoscalar meson Bethe-Salpeter amplitude. One can expect that this discrepancy will become small when we either adopt the other improved effective gluon propagators in the calculation or include other allowed Dirac structures such as $\gamma_5\vec{\gamma}\cdot\vec{p}$ in the off-diagonal part of the inverse quark propagator within the MN model.

The dependence of the critical isospin potential μ_I^c on the current quark mass is plotted in Fig. 5, which shows that μ_I^c monotonically increases with the value of the current quark mass. Though μ_I^c obtained in the MN model with only the γ_5 structure considered is markedly smaller than m_π, the critical point is still roughly proportional to the square root of the current quark mass in the range 0–20 MeV.

V. SUMMARY AND REMARKS

Using a pedagogical confining model within the framework of GCM, we have qualitatively investigated the phase structure and condensates of two-flavor QCD at finite isospin density with zero temperature and baryon chemical potential. By solving the quark gap equation through the DSE formalism, we obtained that the truncated DSE-type model supports the pion superfluidity phase transition at high enough isospin chemical potential. In contrast with the previous model studies, the obtained gaps responsible for both the chiral condensate and pion condensate are all momentum dependent within the DSE formalism, which are closer to the real world. In addition, some new types of low-dimensional condensates of QCD induced by finite isospin chemical potential, such as pion mixed quark-gluon condensate and mixed four-quark condensate, are proposed and investigated in this paper.

In the chiral limit with finite isospin chemical potential, the normal phase is unfavored and the pion superfluidity phase is the stable ground state. For the case with finite current quark mass, only the solution corresponding to the normal phase is found in the gap equations in the region $\mu_I < \mu_I^c$, whereas for the region $\mu_I > \mu_I^c$, the normal phase is unfavored and the pion superfluidity phase is the stable ground state. The distinctly different phase structure between the chiral limit and the finite current quark mass suggests that the value of the critical isospin chemical is closely related to the pion mass. Even though for the simplicity of numerical study, the obtained critical point μ_I^c in this paper is not exactly the pion mass for the case beyond the chiral limit, we point out that the improved (more involved) calculation within the DSE formalism should confirm $\mu_I^c = m_\pi$.

Furthermore, our calculation shows that in the chiral limit with finite isospin chemical potential, both quark condensate and traditional mixed quark-gluon condensate vanish in OPE with the appearance of isospin-density-induced pion condensate and pion mixed quark-gluon condensate. In the real world, for $\mu_I < \mu_I^c$, quark condensate and mixed quark-gluon condensate exist in OPE with the vanishing of pion and pion mixed quark-gluon condensate; for $\mu_I > \mu_I^c$, the magnitudes of both isospin-density-induced condensates increase with increasing isospin chemical potential, whereas the magnitude of quark condensate decreases with increasing isospin chemical potential. (It is expected that the mixed quark-gluon condensate has similar behavior.) Meanwhile, numerical calculations suggest that the induced pion condensate and pion mixed quark-gluon condensate have the same signs as their corresponding traditional chiral condensates. We also obtained that the magnitude of the ratio of pion mixed quark-gluon condensate to pion condensate is close to the one of traditional quark-gluon condensate to quark condensate in the vacuum, both for the chiral limit and for the real world in the pion superfluidity phase.

Since there is no fermion sign problem at finite isospin chemical potential with zero baryon chemical potential, in principle, the evaluation of the induced mixed quark-gluon condensate and four-quark condensate can be investigated through the lattice Monte Carlo method. The effect of these isospin chemical potential induced condensates on the hadron properties can be investigated in the framework of QCD sum rules. However, such effects on the hadron properties can more directly be explored using suitable Bethe-Salpeter equations in conjunction with the solutions of the quark gap equation. A natural extension of the present work is to investigate the two-flavor QCD phase diagram, condensates, and hadron properties at finite isospin chemical potential for both nonzero temperature and baryon chemical potential.

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