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# Sum rule for the polarized structure function $g_2$ corresponding to the moment at $n = 0$

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In the small $Q^2$ region, the sum rule for the polarized structure function $g_2$ corresponding to the moment at $n = 0$ is derived. This sum rule shows that there is a tight connection among the resonances, the elastic and the continuum in the $g_2$. Further, the Born term contribution in this sum rule is proportional to $Q^2$ and very small compared with that in the corresponding sum rule for the polarized structure function $g_1$. However, the Born term contribution divided by $Q^2/2$ which also appears in the Schwinger sum rule for the $g_2$ corresponding to the moment at $n = 1$ has a very similar behavior with that in the sum rule for the $g_1$ corresponding to the moment at $n = 0$.

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The polarized structure functions $g_1$ and $g_2$ at low energy in the small $Q^2$ region attract great interest recently. The $\Delta(1232)$ gives the large negative contribution in this region and it can explain the sign difference between Ellis-Jaffe sum rule [1] and Gerasimov-Drell-Hearn sum rule [2,3]. This $\Delta(1232)$ contribution also invalidate a naive application of Bloom-Gilman duality to the small $Q^2$ region [4]. Now, in this region, we also have the continuum contribution and the large elastic contribution. Recently, the sum rule for the $g_1$ in the small $Q^2$ region has been derived, and it has been shown that there exists the tight connection among the resonances, the elastic and the continuum in the $g_1$ [5,6]. In this paper we show that a similar sum rule exists for the $g_2$.

According to Ref. [7], fixed-mass sum rules based on the canonical quantization on the null-plane gives us

$$\int_0^1 \frac{dx}{x} g_1^{[ab]}(x, Q^2) = -\frac{1}{16} f_{abc} \int_{-\infty}^{\infty} d\alpha \left[A_c^5(\alpha, 0) + \alpha \bar{A}_c^5(\alpha, 0)\right], \tag{1}$$

and

$$\int_0^1 \frac{dx}{x} g_2^{[ab]}(x, Q^2) = -\frac{1}{16} f_{abc} \int_{-\infty}^{\infty} d\alpha \epsilon(\alpha) \alpha \bar{A}_c^5(\alpha, 0), \tag{2}$$

where $x = Q^2/2\nu$ and $\nu = p \cdot q$, and $A_c^{5\beta}(x|0)$ ($x$ in this expression is the space-time variable) is the antisymmetric bilocal current, and its matrix element is defined as

$$\begin{aligned}\langle p, s|A_c^{5\beta}(x|0)|p, s\rangle_c = s^\mu A_c^5(p \cdot x, x^2) \\ + p^\mu (x \cdot s) \bar{A}_c^5(p \cdot x, x^2) \\ + x^\mu (x \cdot s) \tilde{A}_c^5(p \cdot x, x^2).\end{aligned} \tag{3}$$

Similar sum rules can be derived from the current anticommutation relation on the null-plane [6]. These sum rules are for the symmetric combination under the interchange of superscript $a$ and $b$. The basic difference between the sum rules based on the current commutation relation and the current anticommutation relation is that the former ones are based on the operator relation while the latter ones are based on the connected matrix element between the one particle stable hadron. In this sense, the former sum rules are more general than the latter ones. However, in the latter case, the sum rules are directly applied to the structure functions in the electroproduction, while in the former case, it is for the isovector photon. The sum rules for the $g_1$ are given in Refs. [5,6] based on the fact that the right-hand side of Eq. (1) is $Q^2$ independent. The sum rule for the $g_2$ can be derived by the same kind of reasoning that the right-hand side of Eq. (2) is $Q^2$ independent as

$$\int_0^1 \frac{dx}{x} g_2^{ab}(x, Q^2) = \int_0^1 \frac{dx}{x} g_2^{ab}\left(x, Q_0^2\right), \tag{4}$$

where the superscript $ab$ is kept. In the current commutator case, it takes the ones corresponding to the charged photon as in Ref. [5], and in the current anticommutator case, it takes the ones corresponding to the usual electromagnetic current as in Ref. [6]. Now since we have

$$\begin{aligned}\triangle\sigma^{ab}(\nu, Q^2) &= \sigma_{3/2}^{ab}(\nu, Q^2) - \sigma_{1/2}^{ab}(\nu, Q^2) \\ &= -\frac{8\pi^2\alpha_{em}}{K}\left(\frac{g_1^{ab}(x, Q^2)}{\nu} - \frac{m_N^2 Q^2 g_2^{ab}(x, Q^2)}{\nu^3}\right),\end{aligned} \tag{5}$$

where $K = (1 - \frac{Q^2}{2\nu})$, we have the following relation at $Q^2 = 0$:

$$\frac{g_1^{ab}(x, 0)}{\nu} = -\frac{1}{8\pi^2\alpha_{em}} \triangle\sigma^{ab}(\nu, 0). \tag{6}$$

Thus the method to use the photoreaction as the regularization point cannot be applied directly to the $g_2$. Though we can take one particular reaction at small $Q^2$ as a regularization point, the relation with the real photon reaction is interesting in itself, since the real and the virtual photon is essentially different. Further, if we can derive a similar sum rule as the $g_1$, we can consider the $g_1$ and the $g_2$ at the same footing. Now if we differentiate Eq. (5) by $Q^2$ and take the limit $Q^2 \to 0$, we obtain the relation

$$\begin{aligned}\frac{g_2^{ab}(x, 0)}{\nu} &= \frac{g_1^{ab}(x, 0)}{2m_N^2} + \frac{\nu}{m_N^2} \left.\frac{\partial g_1^{ab}(x, Q^2)}{\partial Q^2}\right|_{Q^2=0} \\ &+ \frac{\nu^2}{8\pi^2 m_N^2 \alpha_{em}} \left.\frac{\partial \triangle\sigma^{ab}(\nu, Q^2)}{\partial Q^2}\right|_{Q^2=0}.\end{aligned} \tag{7}$$

 

All the quantities on the right-hand side are experimentally measurable. Hence we can relate $g_2^{ab}(x,0)/\nu$ to the experimentally measurable quantity. Then, by setting $Q^2=0$ on the right-hand side of Eq. (4), we can rewrite the sum rule (4) by the same method as in the sum rule for the $g_1^{ab}$ [6]. We first separate the Born term contribution and then cut off the integral of the continuum part at some value in $E$ where $E$ is defined in the laboratory frame as $\nu = p\cdot q = m_N E$. We denote this cutoff value as $E_c$. Then we define the threshold value for the continuum as $E_0$, and $E_0(Q)=E_0+Q^2/2m_N$, $E_c(Q)=E_c+Q^2/2m_N$, $x_c(Q)=Q^2/(2m_N E_c(Q))$, and take $E_c = 2$ (GeV). In this way, we obtain the sum rule

$$\int_{x_c(Q)}^1 \frac{dx}{x} g_2^{ab}(x,Q^2) = B_2^{ab}(Q^2) + \int_{E_0}^{E_c}\frac{dE}{E} g_2^{ab}(x,0) + K_2^{ab}(E_c,Q^2), \tag{8}$$

where $B_2^{ab}(Q^2)$ is the Born term at $Q^2=0$ minus the Born term at $Q^2$ and $K_2^{ab}(E_c,Q^2)$ is given as

$$K_2^{ab}(E_c,Q^2) = \int_{E_c}^{\infty}\frac{dE}{E} g_2^{ab}(x,0) - \int_{E_c(Q)}^{\infty}\frac{dE}{E} g_2^{ab}(x,Q^2), \tag{9}$$

and the quantities on the right-hand side in Eq. (7) is substituted for $g_2^{ab}(x,0)/E$ in Eqs. (8) and (9). Further, through the regularization of the sum rule explained in Ref. [6], the integral in Eq. (9) is taken after the subtraction of the high energy behavior. Note that the integral on the left hand side of Eq. (8) is restricted below $x_0(Q)=Q^2/2m_N E_0(Q)$ since the Born term is separated out, where $E_0(Q)$ is determined by the threshold of the pion electroproduction as $2m_N E_0(Q) = (m_N+m_\pi)^2 - m_N^2 + Q^2$.

Now, in case of the proton target, the sum rule for the current commutation relation with $a=(1+i2)/\sqrt{2}$, $b=a^\dagger$ is given by taking $g_2^{ab}(x,Q^2)$ and $B_2^{ab}(Q^2)$ which we denote $g_2^{+-}(x,Q^2)$ and $B_2^{+-}(Q^2)$ respectively as

$$g_2^{+-}(x,Q^2) = 2g_2^{1/2}(x,Q^2) - g_2^{3/2}(x,Q^2), \tag{10}$$

where the superscript 1/2 or 3/2 means the quantity in the reaction (isovector photon) + (proton) → (states of isospin I) where $I = 1/2, 3/2$, and

$$B_2^{+-}(Q^2) = \frac{Q^2}{16m_p^2}\frac{1}{1+\frac{Q^2}{4m_p^2}} G_M^+(Q^2)(G_M^+(Q^2) - G_E^+(Q^2)), \tag{11}$$

where

$$\begin{aligned} G_E^+(Q^2) &= G_E^p(Q^2) - G_E^n(Q^2),\\ G_M^+(Q^2) &= G_M^p(Q^2) - G_M^n(Q^2), \end{aligned} \tag{12}$$

and Sachs form factors $G_E^p(Q^2)$, $G_M^p(Q^2)$ are normalized as $G_E^p(0)=1$, $G_M^p(0)=\mu_p=2.793$. It should be noted that the Born term contribution is proportional to $Q^2$, and hence its contribution is zero at $Q^2=0$. Further,we denote $K_2^{ab}(E_c,Q^2)$ as $K_2^{+-}(E_c,Q^2)$.

In case of the current anticommutation relation for the proton target, we get the sum rules for the structure function in the electroproduction, hence we denote $g_2^{ab}(x,Q^2)$ and $B_2^{ab}(Q^2)$ in this case as $g_2^{ep}(x,Q^2)$ and $B_2^{ep}(Q^2)$, respectively. Further, we denote $K_2^{ab}(E_c,Q^2)$ in this case as $K_2^{ep}(E_c,Q^2)$. The explicit form of the Born term contribution is

$$B_2^{ep}(Q^2) = \frac{Q^2}{8m_p^2}\frac{1}{1+\frac{Q^2}{4m_p^2}} G_M^p(Q^2)\big(G_M^p(Q^2) - G_E^p(Q^2)\big). \tag{13}$$

Combined with a similar sum rule for the $g_1^{ep}$ in the previous paper [6] given as

$$\int_{x_c(Q)}^1 \frac{dx}{x} g_1^{ep}(x,Q^2) = B_1^{ep}(Q^2) - \frac{m_p}{8\pi^2\alpha_{em}}\int_{E_0}^{E_c} dE\big\{\sigma_{3/2}^{\gamma p} - \sigma_{1/2}^{\gamma p}\big\} + K_1^{ep}(E_c,Q^2), \tag{14}$$

where $B_1^{ep}(Q^2)$ is given as

$$\begin{aligned} B_1^{ep}(Q^2) &= \frac{1}{2}\big\{F_1^p(0)\big[F_1^p(0)+F_2^p(0)\big] - F_1^p(Q^2)\big[F_1^p(Q^2)+F_2^p(Q)\big]\big\}\\ &= \frac{1}{2}\left\{\mu_p - \frac{1}{1+\frac{Q^2}{4m_p^2}}\left[G_M^p(Q^2)\left(G_E^p(Q^2) + \frac{Q^2}{4m^2}G_M^p(Q^2)\right)\right]\right\}, \end{aligned} \tag{15}$$

and $K_1^{ep}(E_c,Q^2)$ as

$$K_1^{ep}(E_c,Q^2) = \frac{m_p}{8\pi^2\alpha_{em}}\int_{E_c}^{\infty} dE\big\{\sigma_{1/2}^{\gamma p} - \sigma_{3/2}^{\gamma p}\big\} - \int_{E_c(Q)}^{\infty}\frac{dE}{E} g_1^{ab}(x,Q^2), \tag{16}$$

we obtain the sum rule for the $(g_1^{ep}+g_2^{ep})$ as

$$\begin{aligned} &\int_{x_c(Q)}^1 \frac{dx}{x}\big(g_1^{ep}(x,Q^2)+g_2^{ep}(x,Q^2)\big)\\ &= B_1^{ep}(Q^2) + B_2^{ep}(Q^2) + \int_{E_0}^{E_c}\frac{dE}{E}\big(g_1^{ep}(x,0) + g_2^{ep}(x,0)\big) + K_1^{ep}(E_c,Q^2) + K_2^{ep}(E_c,Q^2). \end{aligned} \tag{17}$$

The explicit form of the Born term is

$$B_1^{ep}(Q^2) + B_2^{ep}(Q^2) = \tfrac{1}{2}\big(\mu_p - G_M^p(Q^2)G_E^p(Q^2)\big). \tag{18}$$

The magnitude of the Born term contributions in the moment at $n=0$ for the $g_1^{ep}$ and the $(g_1^{ep}+g_2^{ep})$ are very similar, but that of the $g_2^{ep}$ is very small compared with these since it is proportional to $Q^2$. However, if this Born term is divided by $Q^2/2$, it has a finite limit as $Q^2\to 0$, and has an interesting behavior. These quantities are the ones which appear in the Schwinger sum rule for the $g_2^{ep}$ given as [8]

$$\frac{-1}{4m_p^2+Q^2} G_M^p(Q^2)\big(G_M^p(Q^2) - G_E^p(Q^2)\big) + \int_{\nu_0(Q)}^{\infty} d\nu G_2^{ep}(\nu,Q^2) = 0, \tag{19}$$

where we separate the Born term in this sum rule. At large $Q^2$, because of the Burkhart-Cottingham (BC) sum rule [9] for the inelastic reaction, we have the relation

$$I(Q^2) = \int_{\nu_0(Q)}^{\infty} d\nu G_2^{ep}(\nu, Q^2) = \frac{2}{Q^2}\int_0^1 dx g_2^{ep}(x, Q^2) = 0. \tag{20}$$

Thus we can consider the main contribution in the continuum part in the Schwinger sum rule (19) comes from a relatively low energy region. Therefore, in the sum rule given as

$$\int_{\nu_0(Q)}^{\infty} d\nu G_2^{ep}(\nu, Q^2) - \int_{\nu_0}^{\infty} d\nu G_2^{ep}(\nu, 0) = B_S^{ep}(Q^2), \tag{21}$$

where

$$B_S^{ep}(Q^2) = \frac{1}{4m_p^2 + Q^2} G_M^p(Q^2)\big(G_M^p(Q^2) - G_E^p(Q^2)\big) - \frac{\mu_p(\mu_p - 1)}{4m_p^2}, \tag{22}$$

the main contribution on the left hand side comes from the low $Q^2$ region. Since the Born term contribution $B_S(Q^2)$ changes rapidly in this region, the left hand side of the sum rule also changes rapidly. Since we have the relation $\nu = Q^2/2$ at the elastic point, $B_S^{ep}(Q^2)$ is related to $B_2^{ep}(Q^2)$ as

$$B_S^{ep}(Q^2) = \frac{2}{Q^2} B_2^{ep}(Q^2) - \left\{\frac{2}{Q^2} B_2^{ep}(Q^2)\right\}\bigg|_{Q^2=0}. \tag{23}$$

Now the contribution to the quantity

$$\int_{x_c(Q)}^1 \frac{dx}{x} g_2^{ep}(x, Q^2) - \int_{x_c}^1 \frac{dx}{x} g_2^{ep}(x, 0) \tag{24}$$

in the sum rule (8) comes from the low energy region and we can expect it roughly given by $B_2^{ep}(Q^2)$. Thus the sum rule (8) and the Schwinger sum rule gives us the same picture that the rapid behavior of the elastic is compensated by the rapid behavior of the resonance and the continuum. Now if we plot the Born term contributions $B_1^{ep}(Q^2)$, $B_1^{ep} + B_2^{ep}(Q^2)$, and $-B_S^{ep}(Q^2)$, we find that these three functions behave very similarly. As is shown in Fig. 1 the difference between $B_1^{ep}(Q^2)$ and $-B_S^{ep}(Q^2)$ is very small and moreover the difference is almost constant.

Though the moments which give $B_S^{ep}(Q^2)$ and $B_1^{ep}(Q^2)$ are different, we see that the behavior of the integral of $\{-2g_2^{ep}(x, Q^2)/Q^2 + (2g_2^{ep}(x, Q^2)/Q^2)|_{Q^2=0}\}$ and that of $\{g_1^{ep}(x, Q^2)/x - (g_1^{ep}(x, Q^2)/x)|_{Q^2=0}\}$ in the small $Q^2$ region is very similar. Since the latter is related to the sign change of the generalized Gerasimov-Drell-Hearn sum, this fact may suggest that the $g_2^{ep}$ is related to this phenomena [10]. However, in our approach, we have no direct relation between the $g_1^{ep}$ and the $g_2^{ep}$.

Concerned with this, we should point out that the seeming relation between the $g_1^{ab}$ and the $g_2^{ab}$ in Eq. (7). This relation does not mean that the $g_1^{ab}$ is related to the $g_2^{ab}$. However, if we substitute the experimental values for the quantities on the right-hand side of Eq. (7), the $g_2^{ab}(x, 0)$ determined by this relation depends on these values. In this sense, the dependence on the $g_1^{ab}$ enters. Since the relation (7) depends on the $Q^2$ dependence of $K$, and since we can extract an experimental

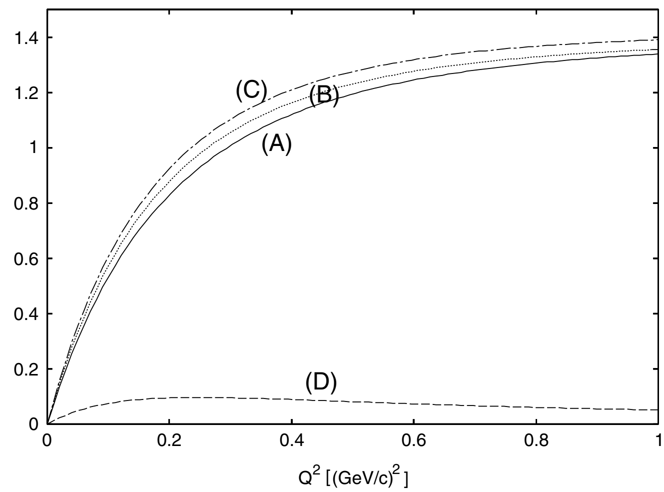


FIG. 1. The various Born term contributions. (A) is the $B_1^{ep}(Q^2)$ given in Eq. (15); (B) is the $B_1^{ep}(Q^2) + B_2^{ep}(Q^2)$ given in Eq. (18); and (C) is the $-B_S^{ep}(Q^2)$ given in Eq. (22). (D) is the difference between $B_1^{ep}(Q^2)$ and $(-B_S^{ep}(Q^2))$.

value even if we modify this flux factor, we can have another sum rule by changing this factor. For example, let us take $\bar{K}$ as

$$\bar{K} = 1 + b \cdot \frac{m_N^2 Q^2}{\nu^2}, \tag{25}$$

where $b$ is an arbitrary dimensionless number, and $\bar{K}$ must be 1 at $Q^2 = 0$ since $\triangle\bar{\sigma}^{ab}(\nu, Q^2)$ defined through $\bar{K}$

$$\triangle\bar{\sigma}^{ab}(\nu, Q^2) = \bar{\sigma}_{3/2}^{ab}(\nu, Q^2) - \bar{\sigma}_{1/2}^{ab}(\nu, Q^2) = -\frac{8\pi^2\alpha_{em}}{\bar{K}}\left(\frac{g_1^{ab}(x, Q^2)}{\nu} - \frac{m_N^2 Q^2 g_2^{ab}(x, Q^2)}{\nu^3}\right), \tag{26}$$

must becomes quantity in the photoproduction. Then Eq. (7) changes as

$$\frac{g_2^{ab}(x, 0)}{\nu} = -\frac{b g_1^{ab}(x, 0)}{\nu} + \frac{\nu}{m_N^2}\frac{\partial g_1^{ab}(x, Q^2)}{\partial Q^2}\bigg|_{Q^2=0} + \frac{\nu^2}{8\pi^2 m_N^2 \alpha_{em}}\frac{\partial\triangle\bar{\sigma}^{ab}(\nu, Q^2)}{\partial Q^2}\bigg|_{Q^2=0}. \tag{27}$$

In this case, $g_1^{ab}(x, 0)/\nu$ appears instead of $g_1^{ab}(x, 0)$. Then by using the sum rule (8) and the sum rule for the $g_1^{ab}$ given in Eq. (14) we obtain the sum rule for the $(bg_1^{ab} + g_2^{ab})$. This sum rule looks different from the sum rule (14) even if we take $b = 1$. This seeming difference is the artifact of the difference of the definition of $\triangle\bar{\sigma}^{ab}(\nu, Q^2)$ and $\triangle\sigma^{ab}(\nu, Q^2)$. Then we see that how we reach the $Q^2 = 0$ point we have many different forms of the sum rule which are essentially the same one.

In conclusion, in the small $Q^2$ region, we have derived the sum rule for the polarized structure function $g_2$ corresponding to the moment at $n = 0$, which is similar to the corresponding sum rule for the $g_1$. The $g_2$ at $Q^2 = 0$ is related to the experimentally measurable quantity, and it is shown that the sum rule in appearance depends on how we reach the $Q^2 = 0$ point but that these seeming different sum rules are essentially the same one. Then, independent of the $g_1$, we show that there is a tight connection among the resonances, the elastic and the continuum in the $g_2$. Since the Born term contribution is proportional to $Q^2$ and very small compared with that in the corresponding sum rule for the $g_1$, the change of the sum of the resonances and the continuum is small in this sum rule. However, if we divide the Born term contribution in the sum rule for the $g_2$ by $Q^2/2$, which also appears in the Schwinger sum rule for the $g_2$ corresponding to the moment at $n = 1$, the quantity obtained has a very similar behavior with the Born term contribution in the sum rule for the $g_1$ corresponding to the moment at $n = 0$. Whether this similarity is a mere happening or has a deep physical meaning is not yet clear and needs a further study.


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