Correlations and fluctuations: Generalized factorial moments

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A systematic study of the relations between fluctuations of the extensive multiparticle variables and integrals of the inclusive multiparticle densities is presented. The generalized factorial moments are introduced and their physical meaning discussed. The effects of the additive conservation laws are analyzed.

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I. INTRODUCTION

Studies of multiparticle production is a complicated matter because of the large number of variables describing the system, and thus several methods were developed to simplify the problem. Two of them turned out particularly useful in providing information: (i) studies of correlations between particles by investigation of the inclusive distributions and (ii) studies of event-by-event fluctuations. Since both of them give, in principle, a complete description of the system, they must be related. This relation was first discussed in Ref. [1] and reformulated in Ref. [2]. In the present paper, we follow the arguments of Refs. [1,2] and investigate further the relation between these two ways of approaching multiparticle production.

To illustrate the problem, consider the well-known formula

$$F_k = \int dq_1 \dots dq_k \rho_k(q_1, \dots, q_k), \tag{1}$$

where $\rho_k(q_1, \dots, q_k)$ is the inclusive distribution of k particles and F_k is the factorial moment of the k-th order,

$$\begin{aligned} F_k &= \sum_N P(N)N(N-1)\dots(N-k+1) \\ &\equiv \langle N(N-1)\dots(N-k+1)\rangle, \end{aligned} \tag{2}$$

where N is the event multiplicity and $P(N)$ denotes the multiplicity distribution.

The main interest in formula (1) is that its left-hand side characterizes event-by-event fluctuations of the multiplicity while its right-hand side represents the measurement of particle distribution by a k-arm spectrometer. Thus Eq. (1) connects the two quantities which are defined in an entirely different manner. Needless to say, confronting such apparently unrelated quantities is often a source of a new insight into the problem. This is precisely the interest in investigating further relations of this type.

Another feature of Eq. (1) that makes this relation very useful in practical applications is that it connects the factorial moment of order k with the inclusive density of the same order. Thus it can be used even if one does not have a complete knowledge of the system (which is of course practically always the case).[1]

For $k = 2$, the extension of Eq. (1) to quantities other than multiplicity was proposed in Ref. [3]. It was discussed and applied by several authors (see, e.g., Refs. [4–7] and the review in Ref. [8]). The possibility of generalization to $k > 2$ was suggested in Ref. [3] and considered in Ref. [9].

The purpose of the present paper is twofold. First, we derive an explicit formula, valid for arbitrary k, which extends the relation (1) to fluctuations of quantities other than multiplicity. This is obtained by introducing the *generalized* factorial moments [2], defined in terms of measurements of event-by-event fluctuations of measurable *extensive* quantities. Second, we investigate the physical meaning of the generalized factorial moments along the lines developed in Ref. [10], as described below.

As first discussed in Ref. [10], the factorial moments have a rather simple physical interpretation. It follows from the observation that for any distribution of multiplicity which can be represented as a superposition of Poisson distributions

$$P(N) = \int d\bar{N} W(\bar{N})\, e^{-\bar{N}} \frac{\bar{N}^N}{N!}, \tag{3}$$

one has

$$\begin{aligned} F_k &= \sum_N (N(N-1)\dots(N-k+1)P(N) \\ &= \int d\bar{N} W(\bar{N})\bar{N}^k. \end{aligned} \tag{4}$$

Thus the *factorial* moment of the actual multiplicity distribution $P(N)$ measures the *standard* moment of the underlying distribution $W(\bar{N})$. One can say [10] that the factorial moment removes the statistical noise (represented by the Poisson distribution) from the corresponding moment of the distribution $W(\bar{N})$. This observation is easily generalized to local multiplicity fluctuations [10].

The present paper will show that in absence of other constraints, the generalized factorial moments can be used as well to remove the statistical noise from the data. This is, of course, to be expected. The problem must be treated more

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[1] This feature may be contrasted with the relation between F_k and $P(N)$. They must be related because they both describe completely the multiplicity distribution. However, as seen from Eq. (2), to calculate F_k one needs to know *all* $P(N)$ (and vice versa).

carefully, however, in the presence of an additive conservation law. Indeed, the additive conservation law implies that the statistical noise cannot be entirely random, and therefore the simple argument of Ref. [10] must be modified. We consider in some detail the effects of conservation of charge and of transverse momentum.

The problem of removing the statistical noise from fluctuations of various quantities observed in multiparticle production is a long-standing one. The most popular method remains the Φ measure, introduced in Ref. [11]. Other possibilities and their relation to the Φ measure were discussed in Refs. [5–7]. They are now commonly used in data analysis [12–17].

Our treatment, using the generalized factorial moments, goes beyond the previous studies in two points. On the theoretical side, our method applies to correlations of any order (until now, mostly $k = 2$ was discussed[2]). On the practical side, including the effects of additive conservation laws makes the control over the reliability of the experimental results much more solid.[3]

In the next section, the generalized factorial moments are introduced and their relation to integrals of inclusive multiparticle densities explained. Their physical interpretation in the case of random noise is explained in Sec. III. Modifications due to conservation of a discrete quantum number are discussed in the example of charge distributions in Sec. IV. The effects of conservation of transverse momentum are considered in Sec. V. Discussion and conclusions are given in the last section. Appendices A and B explain the details of the algebra.

II. GENERALIZED FACTORIAL MOMENTS

Generalizations of Eq. (1) to fluctuations of extensive quantities other than multiplicity were found in Ref. [1] and reformulated in Ref. [2]. Here we follow the argument used in these two papers.

Consider a single-particle variable $x = x(p)$, where p is the particle momentum. Consider, furthermore, the set of extensive quantities

$$X_l(q_1, \ldots, q_N) = \sum_{n=1}^{N} [x(q_n)]^l, \tag{5}$$

where N is the multiplicity of the event.

We want to study fluctuations of $X \equiv X_1$. One possible method is to consider the moments

$$\langle X^k \rangle = \sum_N \int dq_1 \ldots dq_N P(q_1, \ldots, q_N; N)(x_1 + \cdots + x_N)^k, \tag{6}$$

where $P(q_1, \ldots, q_N; N)$ is the probability to find an event with N particles at momenta $(q_1, \ldots, q_N)$ and where we have introduced the shorthand

$$x_j \equiv x(q_j), \tag{7}$$

which will be used henceforth. It was shown in Ref. [1] that the moments (6) can be expressed by linear combinations of the integrals

$$R(k_1, \ldots, k_s; s) = \int dq_1 \ldots dq_s \rho_s(q_1, \ldots, q_s) x_1^{k_1} \ldots x_s^{k_s}. \tag{8}$$

These relations are, however, fairly complicated.

To find a more elegant formulation, we observe that when one takes $x \equiv 1$, one has $X \equiv N$, and thus the simple relation (1) must hold. It is thus clear that one has to find a generalization of the factorial moments (2). Such a generalization was proposed in Ref. [2]:

$$F_k[x] = \langle [X - (k-1)\hat{x}][X - (k-2)\hat{x}] \ldots [X - \hat{x}] X \rangle, \tag{9}$$

where $\hat{x}$ is the operator, acting on a product $[X_{l_1} \ldots X_{l_m}]$ as follows[4]

$$\hat{x}\big[X_{l_1} \ldots X_{l_m}\big] = \frac{1}{m} \sum_{s=1}^{m} \big[X_{l_1} \ldots X_{l_s+1} \ldots X_{l_m}\big], \tag{10}$$

where the indices $[l_1, \ldots, l_m]$ need not be different. It follows from Eq. (10) that, in particular,

$$\hat{x} X_l = X_{l+1}, \quad \hat{x}[X_l]^m = X_{l+1}[X_l]^{m-1}. \tag{11}$$

From Eq. (10) one can also deduce how $\hat{x}$ acts on a product $x_{j_1} \ldots x_{j_k}$. The result is

$$\hat{x}\big[x_{j_1} \ldots x_{j_k}\big] = \big(x_{j_1} \ldots x_{j_k}\big) \frac{1}{k} \sum_{s=1}^{k} x_{j_s}, \tag{12}$$

where, again, the indices $j_1, \ldots, j_k$ need not be different.

Using Eqs. (9) and (10), we obtain for $k = 1, 2, 3$,

$$\begin{aligned} F_1[X] &= \langle X \rangle, \quad F_2[X] = \langle X^2 \rangle - \langle \hat{x} X \rangle = \langle X^2 \rangle - \langle X_2 \rangle, \\ F_3[X] &= \langle X^3 - 2\hat{x} X^2 - X \hat{x} X + 2[\hat{x}]^2 X \rangle \\ &= \langle X^3 - 3 X X_2 + 2 X_3 \rangle. \end{aligned} \tag{13}$$

We shall now prove that

$$\begin{aligned} &[X - (k-1)\hat{x}][X - (k-2)\hat{x}] \ldots [X - \hat{x}] X \\ &= \sum_{i_1=1}^{N} \ldots \sum_{i_k=1}^{N} x_{i_1} \ldots x_{i_k}, \end{aligned} \tag{14}$$

where all indices are different from each other.

The proof goes by induction. For $k = 2$, we have $X^2 - X_2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j$ with $i \neq j$. Thus Eq. (14) is satisfied. Suppose now that it is valid for a given k. Multiplying both sides of Eq. (14) by $(X - k\hat{x})$, we thus have

$$\begin{aligned} &[X - k\hat{x}][X - (k-1)\hat{x}] \ldots [X - \hat{x}] X \\ &= (X - k\hat{x}) \sum_{i_1=1}^{N} \ldots \sum_{i_k=1}^{N} x_{1_1} \ldots x_{i_k} \end{aligned} \tag{15}$$

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[2] A related method of removing the statistical noise, applicable to arbitrary k, was described in Ref. [18] and studied in Ref. [19].

[3] For $k = 2$, the effects of charge conservation were discussed in Ref. [6]. An interesting proposal to deal with energy-momentum conservation can be found in Ref. [20].

[4] The definition of the operator $\hat{x}$ given in Ref. [2] was incomplete. I would like to thank Andrzej Kotanski for pointing out the error.

But

$$X\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}x_{1_1}\ldots x_{i_k}=\sum_{j=1}^{N}x_j\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}x_{1_1}\ldots x_{i_k}$$
$$=\sum_{s=1}^{k}\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}x_{1_1}\ldots[x_{i_s}]^2\ldots x_{i_k}$$
$$+\sum_{s=k+1}^{N}\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}x_{1_1}\ldots x_{i_k}x_{i_s},\quad (16)$$

where in the first term in the right-hand side, one of the variables from X is identical to one of $x_{i_1}\ldots x_{i_k}$, whereas in the second term, all variables differ from each other. Noting that, as seen from Eq. (12), the first term is identical to

$$k\hat{x}\left[\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}x_{1_1}\ldots x_{i_k}\right],\quad (17)$$

one sees that it cancels in Eq. (15), and thus right-hand side of Eq. (15) equals the second term in the right-hand side of Eq. (16). But this term is just what is needed to complete the proof.

It follows from Eq. (14) that

$$F_k[X]\equiv\sum_{N=0}^{\infty}\int dq_1\ldots dq_N P(q_1,\ldots,q_N;N)$$
$$\times[X-(k-1)\hat{x}]\ldots[X-\hat{x}]X$$
$$=\int\rho_k(q_1,\ldots,q_k)x(q_1)\ldots x(q_k)dq_1\ldots dq_k,\quad (18)$$

where $\rho_k(q_1,\ldots,q_k)$ is the k-particle inclusive density. This follows almost directly from the definition of inclusive density

$$\rho_k(q_1,\ldots,q_k)=\sum_N N(N-1)\ldots(N-k+1)$$
$$\times\int dq_{k+1}\ldots dq_N P(q_1,\ldots,q_N;N)\quad (19)$$

if one notices that the number of terms in the right-hand side of Eq. (14) equals $N(N-1)\ldots(N-k+1)$.

Equation (18) represents a generalization of Eq. (1), which is obtained from Eq. (18) by putting $x(q)\equiv 1$. It has the same two attractive features: it connects the fluctuations of the extensive variables X_l with the integral of the inclusive density and it involves only moments of finite order.

III. PHYSICAL INTERPRETATION OF THE GENERALIZED FACTORIAL MOMENTS

To discuss the physical interpretation of the generalized factorial moments, we again consider the momentum space split into an s bin and the rest. To determine the generalized factorial moment in the s bin, we need the distribution of the number of particles and of the variable x in it. Following the lines of Ref. [10], we demand that in each bin, particles are distributed as randomly as possible. In the absence of any constraints, this implies that in the s bin, the distribution is

$$p_s(q_1,\ldots,q_n,n;\bar{n},\eta)dq_1\ldots dq_n=e^{-\bar{n}}\frac{[\bar{n}]^n}{n!}\prod_{l=1}^{n}[g(q_l;\eta)\,dq_l],\quad (20)$$

where $\bar{n}$ is the average multiplicity in the s bin and $g(q;\eta)$ is the momentum distribution of one particle on this bin, with η representing a collection of parameters on which this distribution may depend.

To obtain the actual distribution, we have to weight Eq. (20) by the underlying probability distribution $W_s(\bar{n},\eta)$:

$$P_s[q_1,\ldots,q_n,n]=\int d\bar{n}d\eta_m W_s(\bar{n},\eta)$$
$$\times p_s(q_1,\ldots,q_n,n;\bar{n},\eta).\quad (21)$$

From Eq. (21) we deduce the inclusive particle density in the s bin as

$$\rho_k(q_1,\ldots q_k)=\sum_{n_s}n_s(n_s-1)\ldots(n_s-k+1)$$
$$\times\int dq_{k+1}\ldots dq_n P_s(q_1,\ldots,q_n;n).\quad (22)$$

Introducing Eq. (22) into Eq. (18) and using Eq. (20), we obtain for the generalized factorial moment in the s bin

$$F_k[X]=\sum_n n(n-1)\ldots(n-k+1)$$
$$\times\int dq_1\ldots dq_n P_s(q_1,\ldots,q_n;n)x(q_1)\ldots x(q_k)$$
$$=\int d\bar{n}d\eta W_s(\bar{n},\eta)[\bar{n}\bar{x}]^k\equiv\langle[\bar{n}\bar{x}]^k\rangle=\langle\bar{X}^k\rangle,\quad (23)$$

where

$$\bar{x}=\bar{x}(\eta)=\int g(q;\eta)x(q)\,dq\quad (24)$$

is the average of $x(q)$ at fixed η, and $\bar{X}$ is the average of X at fixed $\bar{n}$ and η.

Equation (23) is the generalization we were seeking. It shows the physical interpretation of the generalized factorial moments: they remove the statistical noise from the moments of $\bar{n}\bar{x}=\bar{X}$ of the underlying distribution W_s. For $x\equiv 1$, we of course recover the well-known result [10].

IV. FLUCTUATIONS OF CHARGE

As already explained in the introduction, the argument present in the previous section fails when the variable x satisfies an additive conservation law. In this section, we discuss fluctuations of charge. The discussion applies to any discrete, additive quantum number.

We again select an s bin from the momentum space and consider the distribution of particles inside and outside of this bin.

The first problem is to define the distribution describing the statistical noise. Following the idea of Ref. [10], we take the normalized product of Poisson distributions (expressing the independent particle emission and thus introducing no correlations). This is supplemented by the Kronecker δ symbol to satisfy the conservation law:

$$p_s = \frac{1}{A_Q} e^{-\hat{n}^+} \frac{[\hat{n}^+]^{n^+}}{n^+!} e^{-\hat{n}^-} \frac{[\hat{n}^-]^{n^-}}{n^-!} e^{-\hat{N}^{*+}} \frac{[\hat{N}^{*+}]^{N^{*+}}}{N^{*+}!} e^{-\hat{N}^{*-}} \times \frac{[\hat{N}^{*-}]^{N^{*-}}}{N^{*-}!} \delta_{n^+ + N^{*+} - n^- - N^{*-} - Q_0}, \tag{25}$$

where n^+, n^- denote the number of produced charges in the s bin, and N^{*+}, N^{*-} those outside of the s bin. Q_0 is the net charge in the initial state. We take $Q_0 \geqslant 0$. The normalization factor A_Q is given by (see Appendix A)

$$A_Q = e^{-\hat{N}} \omega^{-Q_0} I_{Q_0}(\tilde{N}), \tag{26}$$

where I_{Q_0} denotes the Bessel function,

$$\hat{N} = \hat{n}^+ + \hat{N}^{*+} + \hat{n}^- + \hat{N}^{*-} \equiv \hat{N}^+ + \hat{N}^-, \tag{27}$$

and

$$\tilde{N} = 2\sqrt{\hat{N}^+ \hat{N}^-}; \quad \omega = \frac{2\hat{N}^-}{\tilde{N}} = \frac{\tilde{N}}{2\hat{N}^+}. \tag{28}$$

The observed distribution of particles is thus

$$P(n^+, n^-, N^{*+}, N^{*-}; Q_0) = \int d\hat{n}^+ d\hat{n}^- d\hat{N}^{*+} d\hat{N}^{*-} W(\hat{n}^+, \hat{n}^-; \hat{N}^{*+}, \hat{N}^{*-}) \times p_s(n^+, n^-; N^{*+}, N^{*-}; Q_0; \hat{n}^+, \hat{n}^-; \hat{N}^{*+}, \hat{N}^{*-}), \tag{29}$$

where W is the "dynamical" distribution, free of the statistical noise.[5]

It is important to observe that, contrary to the naive expectation, the parameters $\hat{n}^+, \hat{n}^-; \hat{N}^{*+}, \hat{N}^{*-}$ do not represent the average values of $n^+, n^-; N^{*+}, N^{*-}$, respectively. The explicit formulas for these average values are given in Appendix A. In particular, the average values of the number of produced positive and negative particles in the the s bin are given by

$$\langle \bar{n}^\pm \rangle = \left\langle \hat{n}^\pm \omega^{\pm 1} \frac{I_{Q_0 \mp 1}(\tilde{N})}{I_{Q_0}(\tilde{N})} \right\rangle = \left\langle \frac{\hat{n}^\pm}{\hat{N}^\pm} \frac{\tilde{N}}{2} \frac{I_{Q_0 \mp 1}(\tilde{N})}{I_{Q_0}(\tilde{N})} \right\rangle, \tag{30}$$

where we have denoted by $^-$ the average at fixed $\hat{n}^+, \hat{n}^-; \hat{N}^{*+}, \hat{N}^{*-}$, and by $\langle \ldots \rangle$ the average over $\hat{n}^+, \hat{n}^-; \hat{N}^{*+}, \hat{N}^{*-}$ (with the probability distribution W).

The explicit formulas for the generalized factorial moments of the charge F_k^- and the (standard) factorial moments of the multiplicity F_k^+ in the s bin are derived in Appendix A. They read

$$F_k^\pm = \left\langle \left(\frac{\tilde{N}}{2}\right)^k \sum_{m=0}^{k} \frac{k!}{(k-m)!m!} \left[\frac{\hat{n}^+}{\hat{N}^+}\right]^{k-m} \times \left[\pm \frac{\hat{n}^-}{\hat{N}^-}\right]^m \frac{I_{Q_0+2m-k}(\tilde{N})}{I_{Q_0}(\tilde{N})} \right\rangle. \tag{31}$$

Equations (30) and (31) simplify substantially in the interesting limit of a very large number of produced particles, $\bar{N} \approx \tilde{N} \rightarrow \infty$. In this limit, using the asymptotic expansion of the two Bessel functions, one obtains up to the terms of order $\bar{N}^{-1}$

$$F_k^\pm = \left\langle [\bar{n}^+ \pm \bar{n}^-]^k - \frac{k(k-1)}{2\bar{N}} [\bar{n}^+ \pm \bar{n}^-]^{k-2} [\bar{n}^+ \mp \bar{n}^-]^2 \right\rangle. \tag{32}$$

The first term corresponds to the standard interpretation of the generalized factorial moments, as expressed by Eq. (23). One sees from Eq. (32) that the correction to this result (induced by the conservation law) vanishes, at fixed $\bar{n}^\pm$, with the inverse power of the total multiplicity of produced particles. In this case, the correction is expected to be small at high energies (particularly for the central heavy ion collisions).[6] We thus conclude that the generalized factorial moments can indeed provide a useful tool for eliminating the statistical noise from the event-by-event fluctuations of multiplicity (F_k^+) and charge (F_k^-) even in the presence of the conservation law.

V. FLUCTUATIONS IN TRANSVERSE MOMENTUM

In this section, we discuss fluctuations of the transverse momentum, as an example a continuous variable subject to an additive conservation law.

Selecting one bin in rapidity (s bin) and dividing the available momentum phase space into this bin and the rest, we write the transverse momentum distribution in the form

$$P[q_1, \ldots, q_n, n; q_1^*, \ldots, q_{N^*}^*, N^*; q_0] = \int d\hat{n} d\hat{q}\, dD\, d\hat{N}^* d\hat{q}^* dD^* W(\hat{n}, \hat{q}, d, ; \hat{N}^*, \hat{q}^*, D^*) \times e^{-\hat{n}} \frac{[\hat{n}]^n}{n!} \prod_{l=1}^{n} \left[\frac{e^{-(q_l - \hat{q})^2/2D^2}}{2\pi D^2}\right] \times e^{-\hat{N}^*} \frac{[\hat{N}^*]^{N^*}}{N^*!} \prod_{l=1}^{N^*} \left[\frac{e^{-(q_l^* - \hat{q}^*)^2/2(D^*)^2}}{2\pi (D^*)^2}\right] \times \frac{1}{A_\perp} \delta(q_1 + \cdots + q_n + q_1^* + \cdots + q_{N^*}^* - q_0). \tag{33}$$

where q denotes the two-dimensional transverse momentum vector and q_0 is the total transverse momentum of the considered phase-space region. The uncorrelated statistical noise in the form of a Poisson distribution for multiplicity multiplied by a Gaussian distribution of transverse momenta is supplemented by the δ function ensuring the transverse momentum conservation[7] and by the normalization factor $A_\perp$.

[5]The selection of statistical noise in the form of Eq. (25) is not unique. Another natural possibility is to consider independent emission of particle pairs (supplemented by independent emission of Q_0 positive particles). The two methods differ mostly by the treatment of the initial charge Q_0. Without further information about dynamics it is difficult to decide which of them describes better the physics of the problem.

[6]For F_k^- the *relative* correction may be large if $\bar{n}_+ + \bar{n}_- \gg \bar{n}_+ - \bar{n}_-$.

[7]This is a commonly used form of the statistical noise (see, e.g., Refs. [20,21]). Note that, since $W(\hat{n}, \hat{q}, d, ; \hat{N}^*, \hat{q}^*, D^*)$ is a general (positive) function, the assumed Gaussian forms in Eq. (33) do not restrict seriously the observed distribution of particles (as both the average values, $\hat{q}, \hat{q}^*$ and dispersions, D, D^* of the Gaussians can fluctuate according to the distribution W).

It is also natural to take

$$\hat{n}\hat{q} + \hat{N}^*\hat{q}^* = q_0. \quad (34)$$

Replacing the δ function in Eq. (33) by its Fourier transform, we obtain for the inclusive distribution in the s bin

$$\rho_k(q_1, \dots q_k) = \int d\hat{N} d\hat{q} dD\, d\hat{N}^* d\hat{q}^*\, dD^* W \times (\hat{n}, \hat{q}, D; \hat{N}^*, \hat{q}^*, D^*) e^{\hat{N}^*[e^{i\hat{q}^* y - D^{*2}y^2/2} - 1]} \times \frac{1}{(2\pi)^2 A_\perp} \int dy e^{-iyq_0} \sum_n n(n-1)\dots \times (n-k+1) e^{-\hat{n}} \frac{[\hat{n}]^n}{n!} \int dq_{(k+1)} \dots dq_n \times \prod_{l=1}^{n} \left[\frac{1}{2\pi D^2} e^{iq_l y} e^{-(q_l - \hat{q})^2/2D^2} \right]. \quad (35)$$

Using Eq. (18), we thus have for the generalized factorial moment in the s bin

$$F_k[X; q_0] = \int d\hat{n}\, d\hat{q} dD d\hat{N}^* d\hat{q}^* dD^* W \times (\hat{n}, \hat{q}, D; \hat{N}^*, \hat{q}^*, D^*) \frac{S_{k\perp}}{S_{0\perp}} \equiv \left\langle \frac{S_{k\perp}}{S_{0\perp}} \right\rangle, \quad (36)$$

with

$$S_{k\perp} = \frac{1}{(2\pi)^2} \int dy e^{-iyq_0} [\hat{n}\phi(y)]^k e^{\hat{n}[e^{i\hat{q}y - D^2y^2/2} - 1]} \times e^{\hat{N}^*[e^{i\hat{q}^* y - D^{*2}y^2/2} - 1]} \quad (37)$$

(note that $A_\perp = S_{0\perp}$), and

$$\phi(y) = \frac{1}{2\pi D^2} \int d^2q e^{-(q-\hat{q})^2/2D^2} e^{iqy} x(q); \quad \phi(0) = \bar{x}, \quad (38)$$

where $\bar{x}$ is the average value of x at fixed $\hat{q}$ and D^2. The function $\phi(y)$ depends obviously on the choice of the variable $x(q)$. Generally, if $x(q)$ is a polynomial in q of order l, then $\phi(y)$ is a polynomial in y of the same order[8] multiplied by $e^{i\hat{q}y - y^2D^2/2}$.

If the number of particles is large, the integral over d^2y can be evaluated by the saddle point method. We write

$$S_{k\perp} = \frac{1}{(2\pi)^2} \int d^2y\, [\hat{n}\phi(y)]^k\, e^{\Psi(y)}, \quad (39)$$

with

$$\Psi(y) = -iq_0 y + \hat{n}[e^{i\hat{q}y - D^2y^2/2} - 1] + \hat{N}^*[e^{i\hat{q}^* y - D^{*2}y^2/2} - 1]. \quad (40)$$

The saddle point equation $\Psi' = 0$ gives $y_0 \sim q_0 - \hat{n}\hat{q} - (\hat{N}^*\hat{q}^*)$, and thus Eq. (34) implies $y_0 = 0$. Furthermore, $\Psi''(0) = -\hat{N}(\tilde{D}^2 + \tilde{q}^2) \equiv -\hat{N}\Delta^2$, where $\hat{N} = \hat{n} + \hat{N}^*$ and

$$\tilde{D}^2 = \frac{\hat{n}D^2 + \hat{N}^* D^{*2}}{\hat{N}}\,; \quad \tilde{q}^2 = \frac{\hat{n}\hat{q}^2 + \hat{N}^*\hat{q}^{*2}}{\hat{N}}. \quad (41)$$

[8] E.g., for $x(q) \equiv q$ we have $\phi(y) = (\hat{q} + iyD^2)e^{i\hat{q}y - y^2D^2/2}$, for $x(q) = q^2$ we obtain $\phi(y) = [2D^2 + (\hat{q} + iyD^2)^2]e^{i\hat{q}y - y^2D^2/2}$.

Consequently, we obtain

$$\frac{S_{k\perp}}{S_{0\perp}} \approx \frac{\hat{N}\Delta^2}{2\pi} \int d^2y [\hat{n}\phi(y)]^k e^{-y^2\hat{N}\Delta^2/2}. \quad (42)$$

To evaluate this integral in the limit of large $\hat{N}$, one can expand $\phi(y)$ in powers of y and observe that since the coefficient in the exponent increases with increasing $\hat{N}$, the dominant term at large $\hat{N}$ is that with the lowest power of y^2 under the integral (the terms with odd powers of y do not contribute). A detailed discussion is given in Appendix B. Here we only summarize the results.

If $\phi(0) = \bar{x} \neq 0$, one obtains

$$F_k[X] = \langle [\bar{n}\bar{q}]^k \rangle, \quad (43)$$

i.e., we recover the formula (23). Thus, if the average value of $x(q)$ in the s bin does not vanish, the standard interpretation of the generalized factorial moments remains valid, even in the presence of the conservation law. The corrections to this result vanish as $1/\langle N \rangle$. They are given in Eq. (B5) of Appendix B.

This is not the case if $\bar{x} = 0$ and $\overline{qx} \neq 0$. For k even, $k = 2p$, one obtains

$$F_{2p}[X] = (-1)^p p! \left\langle [\bar{n}\overline{qx}]^p \left(\frac{2\bar{n}\overline{qx}}{\hat{N}\Delta^2} \right)^p \right\rangle, \quad (44)$$

which shows that now the factorial moments are related to the average value $\overline{qx}$ rather than to $\bar{x}$. For a fixed $\bar{n}$ and large $\bar{N}$ these moments tend to zero. For a finite ratio $\bar{n}/\bar{N}$, however, they may be large. The corrections to Eq. (44) and the formula for k odd are given in Appendix B .

VI. DISCUSSION

We have reconsidered relations between the event-by-event fluctuations of extensive multiparticle variables and the integrals of the inclusive distributions [1]. Our main result is the formula (18) which expresses the integral of the k-particle inclusive distribution in terms of the generalized factorial moment of the k-th order.

A discussion of the physical meaning of the generalized factorial moments shows that they remove the random uncorrelated statistical noise from the data, a result already known from previous investigations [2,10]. When an additive conservation law is at work, however, the statistical noise cannot be uncorrelated and, consequently, its removal can at best be approximate. The exact formulas were derived for charge and transverse momentum conservation. The corrections were evaluated in the limit of a very large total number of produced particles, relevant for collisions at high incident energy.

In short, the results presented in this paper formulate a systematic approach to investigation of fluctuations of extensive variables. They seem to be particularly useful for studies aiming to uncover the *local* structure of the multiparticle system. To take just one example, interpretation of the data on transverse momenta (or energies) in small rapidity bins will be much more transparent if presented in terms of the generalized factorial moments.

It was shown in Ref. [1] that the relation between fluctuations and correlations can be extended to other extensive variables. This, however, goes beyond the scope of the present investigation.

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APPENDIX A

Replacing the Kronecker δ symbol by its integral representation, one obtains

$$P(n^+,n^-;N^{*+},N^{*-};Q_0)=\langle w_Q(n^+,n^-;N^{*+},N^{*-};Q_0)\rangle \tag{A1}$$

where

$$\begin{aligned}w_Q(n^+,n^-;N^{*+},N^{*-};Q_0)\\ =\frac{1}{A_Q}\frac{1}{2\pi}\int_0^{2\pi}df e^{-iQ_0f}e^{-\hat{n}^+}\frac{[\hat{n}^+e^{if}]^{n^+}}{n^+!}e^{-\hat{n}^-}\frac{[\hat{n}^-e^{-if}]^{n^-}}{n^-!}\\ \times e^{-\hat{N}^{*+}}\frac{[\hat{N}^{*+}e^{if}]^{N^{*+}}}{N^{*+}!}e^{-\hat{N}^{*-}}\frac{[\hat{N}^{*-}e^{-if}]^{N^{*-}}}{N^{*-}!},\end{aligned} \tag{A2}$$

and A_Q is the normalization factor.

Thus the generating function for the distribution w_Q is

$$\begin{aligned}\Phi_Q(z^+,z^-,Z^+,Z^-)\equiv&\sum_{n^\pm,N^{\pm *}}w_Q(n^+,n^-;N^{*+},N^{*-};Q_0)\\ &\times(z^+)^{n^+}(z^-)^{n^-}(Z^+)^{N^{+*}}(Z^-)^{N^{-*}}\\ =&\frac{1}{2\pi}\int_0^{2\pi}df e^{-iQ_0f}e^{\hat{n}^+(z^+e^{if}-1)}\\ &\times e^{\hat{n}^-(z^-e^{-if}-1)}e^{\hat{N}^{+*}(Z^+e^{if}-1)}\\ &\times e^{\hat{N}^{-*}(Z^-e^{-if}-1)}.\end{aligned} \tag{A3}$$

From this it is not difficult to derive, by the standard methods, the average multiplicities. One obtains

$$\begin{aligned}\bar{n}^\pm&=\frac{1}{2\pi A_Q}\int_0^{2\pi}df e^{-iQ_0f}e^{\hat{N}^+(e^{if}-1)}e^{\hat{N}^-(e^{-if}-1)}\hat{n}^\pm e^{\pm if}\\ &=\frac{e^{-\hat{N}}}{2\pi i A_Q}\oint\frac{dz}{z}e^{\tilde{N}(z+1/z)/2}(\omega z)^{-Q_0}\hat{n}^\pm(\omega z)^{\pm 1}\\ &=\frac{e^{-\hat{N}}}{A_Q}\hat{n}^{\pm 1}\omega^{\pm 1-Q_0}\sum_{j=-\infty}^{\infty}I_j(\tilde{N})\frac{1}{2\pi i}\oint\frac{dz}{z}z^jz^{\pm 1-Q_0}\\ &=\hat{n}^\pm\omega^{\pm 1}\frac{I_{Q_0\mp 1}(\tilde{N})}{I_{Q_0}(\tilde{N})},\end{aligned} \tag{A4}$$

where ω and $\tilde{N}$ are defined in Eq. (28).

Similarly, one obtains

$$\bar{N}^\pm=\hat{N}^\pm\omega^{\pm 1}\frac{I_{Q_0\mp 1}(\tilde{N})}{I_{Q_0}(\tilde{N})}. \tag{A5}$$

It follows that

$$\hat{n}^\pm\omega^{\pm 1}=\frac{\tilde{N}}{2}\frac{\hat{n}^\pm}{\hat{N}^\pm}=\frac{\tilde{N}}{2}\frac{\bar{n}^\pm}{\bar{N}^\pm}. \tag{A6}$$

The inclusive distribution in the s bin is

$$\rho_k(k^+,k^-)=\left\langle\frac{1}{2\pi A_Q}\int_0^{2\pi}df e^{-iQ_0f}H^Q(k^+,k^-;f)\right\rangle, \tag{A7}$$

where

$$\begin{aligned}H^Q(k^+,k^-,f)=&e^{\hat{N}^{*+}(e^{if}-1)}e^{\hat{N}^{*-}(e^{-if}-1)}\\ &\times\sum_{n^+,n^-}V(n^+,n^-;k^+,k^-)e^{-\hat{n}^+}\frac{[\hat{n}^+e^{if}]^{n^+}}{n^+!}\\ &\times e^{-\hat{n}^-}\frac{[\hat{n}^-e^{-if}]^{n^-}}{n^-!}\end{aligned} \tag{A8}$$

and

$$\begin{aligned}V(n^+,n^-;k^+,k^-)&=\delta_{k-k^+-k^-}\frac{k!}{k^+!k^-!}\frac{n^+!}{(n^+-k^+)!}\frac{n^-!}{(n^--k^-)!}\\ &=\frac{1}{2\pi}\int_0^{2\pi}dh\,e^{-ihk}k!\frac{n^+!}{k^+!(n^+-k^+)!}\\ &\quad\times e^{ihk^+}\frac{n^-!}{k^-!(n^--k^-)!}e^{ihk^-}\end{aligned} \tag{A9}$$

are the number of ways one can select k^+ out of n^+ positive and k^- out of n^- negative particles in a given order.

To calculate the generalized factorial moment F_k^- in the s bin

$$F_k^-=\sum_{k^++k^-=k}(-1)^{k^-}\rho_k(k^+,k^-), \tag{A10}$$

we observe that

$$\begin{aligned}&\sum_{k^+}\sum_{k_-}(-1)^{k^-}V(n^+,n^-;k^+,k^-)\\ &\quad=\frac{k!}{2\pi}\int_0^{2\pi}dhe^{-ihk}[1+e^{ih}]^{n^+}[1-e^{ih}]^{n^-},\end{aligned} \tag{A11}$$

and thus

$$\begin{aligned}\sum_{k_+,k_-}(-1)^{k^-}H^Q(k^+,k^-,f)&=e^{\hat{N}^+(e^{if}-1)}e^{\hat{N}^-(e^{-if}-1)}\frac{k!}{2\pi}\\ &\quad\times\int_0^{2\pi}dhe^{-ihk}e^{[\hat{n}^+e^{if}-\hat{n}^-e^{-if}]e^{ih}}\\ &=e^{\hat{N}^+(e^{if}-1)}e^{\hat{N}^-(e^{-if}-1)}\\ &\quad\times[\hat{n}^+e^{if}-\hat{n}^-e^{-if}]^k.\end{aligned} \tag{A12}$$

Now integration over df gives

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}df e^{-iQ_0f}\sum_{k^++k^-=k}H^Q(k^+,k^-;f)\\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}df e^{-iQ_0f}e^{\hat{N}^+(e^{if}-1)}e^{\hat{N}^-(e^{-if}-1)}\\ &\qquad\times[\hat{n}^+e^{if}-\hat{n}^-e^{-if}]^k\\ &\quad=\frac{e^{-\hat{N}}}{2\pi i}\oint\frac{dz}{z}e^{\tilde{N}(z+1/z)/2}(\omega z)^{-Q_0}[\hat{n}^+\omega z-\hat{n}^-/(\omega z)]^k\end{aligned}$$

$$= e^{-\hat{N}}\omega^{-Q_0}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}[-\hat{n}^-/\omega]^m[\hat{n}^+\omega]^{k-m}$$
$$\times\sum_{j=-\infty}^{\infty}I_j(\tilde{N})\frac{1}{2\pi i}\oint\frac{dz}{z}z^j z^{k-2m-Q_0}$$
$$= e^{-\hat{N}}\omega^{-Q_0}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}[-\hat{n}^-/\omega]^m$$
$$\times[\omega\hat{n}^+]^{k-m}I_{Q_0+2m-k}(\tilde{N}), \quad (A13)$$

where $\hat{N}^{\pm}$, $\tilde{N}$ and ω are defined in Eqs. (27) and (28).

The normalization factor A_Q is obtained from Eq. (A13) by putting $k=0$. Inserting Eqs. (A13) and (A6) into Eqs. (A7) and (A10), we obtain Eq. (31).

We also note that the standard factorial moment for the multiplicity can be obtained from Eq. (A13) simply by replacing $[-\hat{n}^-]^m$ by $[\hat{n}^-]^m$.

APPENDIX B

Here we evaluate the generalized factorial moments for variables related to transverse momentum, given by Eq. (42), in the limit of a very large total number of produced particles (but without any restriction on the ratio $\hat{n}/\hat{N}$, i.e., on the size of the s bin).

Developing the function $\phi(y)$ around $y=0$, we have

$$\phi(y)=\sum_{m=0}^{\infty}\phi_m\frac{(iy)^m}{m!}, \quad (B1)$$

where

$$\phi_m=\frac{1}{2\pi D^2}\int d^2q\,x(q)q^m e^{-q^2/2D^2}=\overline{x(q)q^m}. \quad (B2)$$

This implies that $[\phi(y)]^k$ can also be represented by a series

$$[\phi(y)]^k=\sum_{m=0}^{\infty}\phi_m^{(k)}y^m, \quad (B3)$$

where the coefficients $\phi_m^{(k)}$ can be explicitly evaluated when necessary.

Introducing this into Eq. (42), we have

$$\frac{S_{k\perp}}{S_{0\perp}}\approx\hat{n}^k\sum_{m=0}^{\infty}\phi_m^{(k)}\langle y^{2m}\rangle/m!=\hat{n}^k\sum_{p=0}^{\infty}p!\phi_{2p}^{(k)}\left(\frac{2}{\hat{N}\Delta^2}\right)^p. \quad (B4)$$

One sees that in the limit of very large $\hat{N}$, the first nonvanishing term dominates. If $\phi_0=\bar{x}\neq 0$, we have

$$F_{k\perp}[x(q)]\approx\left\langle\hat{n}^k\phi_0^{(k)}\right\rangle=\langle[\hat{n}\bar{x}]^k\rangle, \quad (B5)$$

and thus we recover the original formula (23). We conclude that in this case the standard interpretation of the generalized factorial moments remains valid even in the presence of the conservation law.

If $\bar{x}=0$, and $\phi_1=\overline{qx}\neq 0$, we have

$$[\phi(y)]^k=[iy\phi_1]^k\left[1+iy\frac{k\phi_2}{2\phi_1}+\cdots\right]$$
$$=[iy\phi_1]^k+(iy)^{k+1}\phi_1^{k-1}\phi_2/2+\cdots. \quad (B6)$$

For even k, when $k=2p$, the dominant term is

$$\frac{S_{k\perp}}{S_{0\perp}}\approx(-1)^p\hat{n}^{2p}[\phi_1]^{2p}\frac{\hat{N}\Delta^2}{2\pi}\int d^2y(y^2)^p e^{-y^2\hat{N}\Delta^2/2}$$
$$=(-1)^p[\hat{n}\overline{qx}]^p p!\left(\frac{2\hat{n}\overline{qx}}{\hat{N}\Delta^2}\right)^p, \quad (B7)$$

where we have used the relation (B2).

For odd k, when $k=2p-1$, we obtain

$$\frac{S_{k\perp}}{S_{0\perp}}\approx(-1)^p\frac{1}{2}\hat{n}^{2p-1}[\overline{qx}]^{2p-2}\overline{q^2x}\frac{\hat{N}\Delta^2}{2\pi}\int d^2y(y^2)^p e^{-y^2\hat{N}\Delta^2/2}$$
$$=(-1)^p\frac{1}{2}\hat{n}^{p-1}[\overline{qx}]^{p-2}\overline{q^2x}\,p!\left(\frac{2\hat{n}\overline{qx}}{\hat{N}\Delta^2}\right)^p. \quad (B8)$$

where again Eq. (B2) was used. Note that these moments are proportional to $\overline{q^2x}$ and thus vanish if the variable $x(q)$ is odd with respect to change $(q\leftrightarrow -q)$.

We conclude that the standard interpretation of the factorial moments, as expressed in Eq. (23), holds only when the average value $\bar{x}$ does not vanish. When $\bar{x}=0$, the factorial moments measure the average value $\overline{qx}$ rather than $\bar{x}$.

[1] A. Bialas and V. Koch, Phys. Lett. **B456**, 1 (1999).
[2] A. Bialas, Acta Phys. Pol. B **35**, 683 (2004); **37**, 3679(E) (2006).
[3] S. Voloshin and D. Seibert, Phys. Lett. **B249**, 321 (1990).
[4] D. Seibert and S. Voloshin, Phys. Rev. D **43**, 119 (1991); D. Seibert, Phys. Rev. C **44**, 1223 (1991).
[5] S. A. Voloshin, V. Koch, and H. G. Ritter, Phys. Rev. C **60**, 024901 (1999).
[6] C. Pruneau, S. Gavin, and S. Voloshin, Phys. Rev. C **66**, 044904 (2002); Nucl. Phys. **A715**, 661 (2003).
[7] S. Gavin, Phys. Rev. Lett. **92**, 162301 (2004); M. Abdel-Aziz and S. Gavin, Nucl. Phys. **A774**, 623 (2006).
[8] S. Jeon and V. Koch, in *Quark-Gluon Plasma 3* (World Scientific, Singapore, 2003), p. 430.
[9] S. A. Voloshin, nucl-th/0206052.
[10] A. Bialas and R. Peschanski, Nucl. Phys. **B273**, 703 (1986); **B308**, 857 (1988).
[11] M. Gazdzicki and S. Mrowczynski, Z. Phys. C **54**, 127 (1992).
[12] H. Appelshauser *et al.* (NA49 Collaboration), Phys. Lett. **B459**, 679 (1999).
[13] K. Adkox *et al.* (PHENIX Collaboration), Phys. Rev. C **66**, 024901 (2002).
[14] D. Adamova *et al.* (CERES Collaboration), Nucl. Phys. **A727**, 97 (2003).
[15] J. Adams *et al.* (STAR Collaboration), Phys. Rev. C **72**, 044902 (2005).
[16] M. Atayan *et al.* (NA222/EHs Collaboration), Phys. Rev. D **71**, 012002 (2005).

[17] W. Broniowski, B. Hiller, W. Florkowski, and P. Bozek, Phys. Lett. **B635**, 290 (2006); W. Florkowski, W. Broniowski, B. Hiller, and P. Bozek, presented at the Thirty-Sixth International Symposium on Multiparticle Dynamics, Parity, Brazil, 2–8 September, 2006 (unpublished), nucl-th/0610035.

[18] Fu Jinghua and Liu Lianshou, Phys. Rev. C **68**, 064904 (2003).

[19] Fu Jinghua, Gao Yuanning, and Cheng Jianping, Phys. Rev. C **72**, 017901 (2005).

[20] G. Odyniec, Acta Phys. Pol. B **30**, 385 (1999).

[21] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **62**, 034902 (2000).