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Coulomb phase shift calculation in momentum space

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By using a new boundary condition model, we calculate Coulomb phase shifts in "momentum space." The K-matrix equation for the auxiliary potential is introduced by a Lemma under the boundary condition. The equation permits one to avoid many prohibited rules for the long-range property. Our numerical phase shift shows good agreement with the analytical phase shift. The usual renormalized phase shifts are also presented for comparison.

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Recently, one of the authors (S.O.) introduced a Coulomb boundary condition in momentum space where the screened Coulomb potential has an energy dependent screening range parameter [1] (hereafter the article will be referred as "A"). In this method, the two-potential theory is used [2,3]. Furthermore, the unitary relation for the given amplitudes was proven not only for the single channel integral equation but also the multichannel one. It was shown that the method is applicable for the three-body Faddeev equation [4,5]. On the other hand, there are some prior numerical calculations [6–12] based on the Mainz-Bonn-Tübingen (MBT) model [7–9]. The MBT-model used a screened Coulomb potential (SCP) with a phase shift renormalization which was investigated in two historical articles [13,14]; also, application to three-body problems [15–17] was explored. However, there are essential handicaps in the theory which was pointed out in A. For the negative energy case, the Coulomb problem is not very difficult, because the binding energy has a limit when the screening range reaches infinity. The positive energy cases are completely different from the usual scattering theory [18–37]. There are many investigations of two-and three-body Coulomb problems in momentum space regarding what kind of special properties exist [38–46].

In this Rapid Communication we obtain the two-body Coulomb phase shift in momentum space, in a manner in which the two-potential theory is extensively used with respect to the SCP(V_l^R) and the "auxiliary potential" (AP: $V^\phi = V_l^C - V_l^R$). However, the direct application should be avoided because the long range nature still exists in the formulae. In the process, the *Lemma* for the auxiliary amplitude with respect to the auxiliary potential, which was required in A, is essential to carry out practical calculations. Therefore, we calculate the "K-matrix" which has no "overlapping singularity" where the Green's function pole coincides with the logarithmic singularity of the long-range potential. The *Lemma* is only satisfied for a boundary condition at the special range R_{cl} of SCP. The effects of the on- and half-shell auxiliary amplitudes in the two-potential formulas will vanish. Consequently, the off-shell Coulomb amplitude is correctly given by the two-potential theory. We would like to guide the reader thoroughly through the process to obtain the Coulomb phase shift, because there are many "prohibited rules" for the usual scattering theory.

The two-potential theory has been used for the scattering problem governed by two potentials which consist of a primary potential and a secondary one. One example is the nuclear scattering problem in which the short range nuclear part is distorted by the long-range Coulomb interaction. However, the boundary condition is not the same for both potentials, i.e., one is for the plane wave and the other is for the Coulomb wave function. Therefore, we have to start from the SCP plus the nuclear potential to introduce the LS equation. Let us take up the Coulomb part with an SCP which is given by a parameter $\lambda \neq 0$. Then the Lippmann-Schwinger (LS) equation for the partial wave l could be expressed for the plane wave boundary by

$$T_l^C(p, p', \lambda; z) = V_l^C(p, p', \lambda) + \int_0^\infty V_l^C(p, p'', \lambda) G_0(p''; z) T_l^C(p'', p', \lambda; z) dp'', \quad (1)$$

where the SCP becomes the Coulomb potential (CP), when $\lambda \to 0$. In an abbreviated form, the LS equation can be written as

$$T_l^C(\lambda; z) = V_l^C(\lambda) + V_l^C(\lambda) G_0(z) T_l^C(\lambda; z). \quad (2)$$

In this paper we concentrate on obtaining the Coulomb phase shift in the momentum space representation. In order to carry this out, we adopt again the two potential theory in which two potentials are defined by the SCP as a short-range potential (SP: V_l^R) and an auxiliary potential (AP: V_l^ϕ),

$$V_l^C(\lambda) = V_l^R + \left(V_l^C(\lambda) - V_l^R\right) \equiv V_l^R + V_l^\phi(\lambda). \quad (3)$$

Therefore, two-potential theory is applicable to these potentials, and we obtain

$$T_l^C(\lambda; z) = T_l^{R\phi}(\lambda; z) + T_l^\phi(\lambda; z) \quad (4)$$

$$= \overline{\omega}_l^\phi(\lambda; z) t_l^{R\phi}(\lambda; z) \omega_l^\phi(\lambda; z) + T_l^\phi(\lambda; z), \quad (5)$$

$$t_l^{R\phi}(\lambda; z) = V_l^R + V_l^R G^\phi(\lambda; z) t_l^{R\phi}(\lambda; z), \quad (6)$$

$$G^\phi(\lambda; z) = G_0(z) + G_0(z) T_l^\phi(\lambda; z) G_0(z) \quad (7)$$

$$\equiv G_0(z) \overline{\omega}_l^\phi(\lambda; z) \equiv \omega_l^\phi(\lambda; z) G_0(z), \quad (8)$$

$$\overline{\omega}_l^\phi(\lambda; z) \equiv 1 + T_l^\phi(\lambda; z) G_0(z) \quad (9)$$

$$\omega_l^\phi(\lambda; z) \equiv 1 + G_0(z) T_l^\phi(\lambda; z). \quad (10)$$

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Then, we can define the LS equation ($\lambda \neq 0$) for the AP by the time reversal invariance in the electromagnetic interaction,

$$T_l^{\phi}(\lambda; z) = V_l^{\phi}(\lambda) + V_l^{\phi}(\lambda) G_0(z) T_l^{\phi}(\lambda; z) = V_l^{\phi}(\lambda) + T_l^{\phi}(\lambda; z) G_0(z) V_l^{\phi}(\lambda). \quad (11)$$

Until this point the method is the same as the two potential theory with short-range potentials.

In Eq. (11), the on-shell amplitude could be easily obtained for $\lambda \neq 0$. However, it is impossible to calculate the on-shell t-matrix for the case $\lambda \to 0$ in momentum space. It is the same reason why the Coulomb t-matrix cannot be calculated in momentum space. Even so, it is obviously recognized that one can obtain the phase shift $\phi_l(z) = 0$ in the Schrödinger equation for the potential $V_l^{\phi}(r)$, if one chooses an appropriate range $R = R_{cl}$ for the SCP at the particular energy of interest. We know that $\lim_{\lambda\to 0} T_l^{\phi}(p, p'; \lambda) \neq T_l^{\phi}(p, p')$ by the same reason as $\lim_{\lambda\to 0} T_l^{C}(p, p'; \lambda) \neq T_l^{C}(p, p')$. However, if and only if, one demands $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_l^{\phi}(k, k; \lambda)\} = 0$ at such a range "R_{cl}" where $\phi_l(z) = n\pi (n = 1, 2, \ldots)$ is satisfied, then the *Lemma*-1 exists, as was proven in article A, i.e.,

$$\lim_{\lambda\to 0} T_l^{\phi}(k, k; \lambda) = \lim_{\lambda\to 0} T_l^{\phi}(p, k; \lambda) = \lim_{\lambda\to 0} T_l^{\phi}(k, p'; \lambda) = 0, \quad (12)$$

at $R = R_{cl}$, or simply we presented them by

$$T_l^{\phi}(k, k) = T_l^{\phi}(p, k) = T_l^{\phi}(k, p') = 0. \quad (13)$$

Therefore, one could say that Eq. (12) or Eq. (13) indicates the border where the screened Coulomb amplitude and the pure Coulomb amplitude are synchronized by taking $\lim \lambda \to 0$, otherwise $\lim_{\lambda\to 0} T_l^{\phi}(p, p'; \lambda) \neq T_l^{\phi}(p, p')$ or $\lim_{\lambda\to 0} T_l^{C}(p, p'; \lambda) \neq T_l^{C}(p, p')$ in general.

In order to remind the fact, we repeat the proof in the other method. In Eq. (11), let us separate the Green's function into the principal part and the δ-function part: $G_0 \equiv (G_0^P + iG_0^{\delta})$, and also the amplitude: $T_l^{\phi} = T_{Rl}^{\phi} + iT_{Il}^{\phi}$ are defined. The equality of both integrands in Eq. (11) gives, by the abbreviation of "λ" in T_l^{ϕ}, and V_l^{ϕ},

$$\left\langle V_l^{\phi}\left(G_0^P + iG_0^{\delta}\right)\left(T_{Rl}^{\phi} + iT_{Il}^{\phi}\right)\right\rangle_{pp'} = \left\langle \left(T_{Rl}^{\phi} + iT_{Il}^{\phi}\right)\left(G_0^P + iG_0^{\delta}\right)V_l^{\phi}\right\rangle_{pp'}, \quad (14)$$

where p and p' are the initial and final momenta, respectively. Thus, the real and the imaginary parts satisfy

$$\left\langle V_l^{\phi}\left(G_0^P T_{Rl}^{\phi} - G_0^{\delta} T_{Il}^{\phi}\right)\right\rangle_{pp'} = \left\langle \left(T_{Rl}^{\phi} G_0^P - T_{Il}^{\phi} G_0^{\delta}\right)V_l^{\phi}\right\rangle_{pp'}, \quad (15)$$

$$\left\langle V_l^{\phi}\left(G_0^P T_{Il}^{\phi} + G_0^{\delta} T_{Rl}^{\phi}\right)\right\rangle_{pp'} = \left\langle \left(T_{Rl}^{\phi} G_0^{\delta} + T_{Il}^{\phi} G_0^P\right)V_l^{\phi}\right\rangle_{pp'}. \quad (16)$$

By putting $p \neq k = p'$, we have for the real part equation

$$\left\langle V_l^{\phi} G_0^P T_{Rl}^{\phi} - T_{Rl}^{\phi} G_0^P V_l^{\phi}\right\rangle_{pk} = \left\langle V_l^{\phi} G_0^{\delta} T_{Il}^{\phi} - T_{Il}^{\phi} G_0^{\delta} V_l^{\phi}\right\rangle_{pk}$$

or, in the specific form, it gives for $\lim \lambda \to 0$,

$$\lim_{\lambda\to 0}\int_0^{\infty}\left[V_l^{\phi}(p, p''; \lambda) G_0^P(p''; z) T_{Rl}^{\phi}(p'', k, \lambda; z) - T_{Rl}^{\phi}(p, p'', \lambda; z) G_0^P(p''; z) V_l^{\phi}(p'', k; \lambda)\right] dp'' = \lim_{\lambda\to 0}\rho(k)\left[V_l^{\phi}(p, k; \lambda) T_{Il}^{\phi}(k, k; \lambda) - T_{Il}^{\phi}(p, k; \lambda) V_l^{\phi}(k, k; \lambda)\right], \quad (17)$$

where $\rho(k) = -\nu k/2\pi$ and $k = \sqrt{2\nu z}$, respectively.

Now, if we demand $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_l^{\phi}(k, k; \lambda)\} = 0$, then the first term of the right hand side (r.h.s.) of Eq. (17) vanishes. While, $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} V_l^{\phi}(k, k; \lambda)\} \to \infty$. Since, the principal value integral of the left hand side (l.h.s.) of Eq. (17) should take a finite value, then $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_{Il}^{\phi}(p, k; \lambda)\} \to 0$ would be verified. In the imaginary part (16), we obtain

$$\lim_{\lambda\to 0}\int_0^{\infty}\left[V_l^{\phi}(p, p''; \lambda) G_0^P(p''; z) T_{Il}^{\phi}(p'', k, \lambda; z) - T_{Il}^{\phi}(p, p'', \lambda; z) G_0^P(p''; z) V_l^{\phi}(p'', k; \lambda)\right] dp'' = \lim_{\lambda\to 0}\rho(k)\left[-V_l^{\phi}(p, k; \lambda) T_{Rl}^{\phi}(k, k, \lambda; z) + T_{Rl}^{\phi}(p, k, \lambda; z) V_l^{\phi}(k, k; \lambda)\right], \quad (18)$$

then, we can conclude $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_{Rl}^{\phi}(p, k, \lambda; z)\} \to 0$ by the same reason which is mentioned above. However, it should be stressed that the principal value integrals with respect to $T_{Rl}^{\phi}(p'', k, \lambda; z)$ and $T_{Il}^{\phi}(p'', k, \lambda; z)$ never vanish because of $\lambda \neq 0$ in the integrand. After the integral is carried out, $\lim \lambda \to 0$ is taken. Therefore, such integrals have finite values in general.

By putting $p = k \neq p'$, we could also conclude that $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_{Rl}^{\phi}(k, p', \lambda; z)\} \equiv T_{Rl}^{\phi}(k, p') \to 0$ and $\lim_{\lambda\to 0}\{\lim_{R\to R_{cl}} T_{Il}^{\phi}(k, p', \lambda; z)\} \equiv T_{Il}^{\phi}(k, p') \to 0$, then Eq. (12) is proved.

On the other hand, substituting T_l^{ϕ} into the principal part of the Green's function term: G_0^P of Eq. (11), we obtain the higher order series by iteration,

$$T_l^{\phi}(p, p', \lambda; z) = K_l^{\phi}(p, p', \lambda; z) + iK_l^{\phi}(p, k, \lambda; z)\rho(k) T_l^{\phi}(k, p', \lambda; z), \quad (19)$$

$$T_l^{\phi}(p, p', \lambda; z) = K_l^{\phi}(p, p', \lambda; z) + iT_l^{\phi}(p, k, \lambda; z)\rho(k) K_l^{\phi}(k, p', \lambda; z), \quad (20)$$

where $K_l^{\phi}(p, p', \lambda; z)$ is the K-matrix which is defined by

$$K_l^{\phi}(p, p', \lambda; z) = \left(V_l^{\phi} + V_l^{\phi} G_0^P V_l^{\phi} + V_l^{\phi} G_0^P V_l^{\phi} G_0^P V_l^{\phi} + \cdots\right)_{pp'} = V_l^{\phi}(p, p'; \lambda) + \int_0^{\infty} V_l^{\phi}(p, p''; \lambda) G_0^P(p'') K_l^{\phi}(p'', p'; \lambda) dp''. \quad (21)$$

These amplitudes will converge very quickly for the Coulomb potential with the *em* coupling constant: $\alpha = 1/137.0$. In a practical calculation, the kernels of these K-matrix equations

have no overlapping singularities by the reason that both of the singularities are killed between G_0^P and the logarithmic type potential singularity due to the abnormal integral type $\varepsilon \ln \varepsilon \to 0$, and $\varepsilon(\ln \varepsilon)^2 \to 0$ for $\varepsilon \to 0$.

In Eqs. (19) and (20), for the limit $\lambda \to 0$ and with the aid of the *Lemma*-1, we obtain the *Lemma*-2,

$$\lim_{\lambda\to 0}\lim_{R\to R_{cl}} T_l^{\phi}(p,p',\lambda;z) = \begin{cases} 0 & (p=k, \text{and/or} p'=k) \\ \lim_{\lambda\to 0}\lim_{R\to R_{cl}} K_l^{\phi}(p,p',\lambda;z) & (p\neq k,\ p'\neq k). \end{cases} \tag{22}$$

Here, we should recall that $T_l^{\phi}(p,p',\lambda=0;z)$ cannot be obtained by solving a LS-equation but by the K-matrix equation for the AP.

In order to calculate Eq. (21) at the limit of $\lambda \to 0$, we have to calculate the abnormal integral where the logarithmic singularity exists. The singularity comes from $V_l^{\phi}(p,p')$ at $p=p'$. However, we know that the integral value has a limit with the types of terms like $\varepsilon \ln \varepsilon \to 0$ or $\varepsilon(\ln \varepsilon)^2 \to 0$ for $\varepsilon \to 0$. Therefore the integral including $V_l^{\phi}(p,p')$ as an integrand is harmless at $p=p'$. This fact suggests that we can neglect $p=p'$ from the AP without any change for the integral value. Therefore, we introduce a null-diagonal AP with the following definition:

$$\overline{V}_l^{\phi}(p,p',\lambda) \equiv \begin{cases} 0 & (p=p') \\ V_l^C(p,p',\lambda) - V_l^R(p,p') & (p\neq p'), \end{cases} \tag{23}$$

where we take

$$V_l^C(p,p',\lambda) \equiv \begin{cases} V_l^R(p,p') & (p=p') \\ V_l^C(p,p') & (p\neq p'). \end{cases} \tag{24}$$

It should be noted that the new potential has meaning only inside of the integral. Now, the reader should be reminded that we can calculate Eq. (21) by using Eq. (23) except for the first term: V_l^{ϕ} which contains a logarithmic singularity. Furthermore, since the two-potential theory includes the K-matrix as the integrand, then the V_l^{ϕ} can also be replaced by Eq. (23) without any change. These K-matrices satisfy

$$\overline{K}_l^{\phi}(\lambda;z) = \overline{V}_l^{\phi}(\lambda) + \overline{V}_l^{\phi}(\lambda) G_0^P(z) \overline{K}_l^{\phi}(\lambda;z), \tag{25}$$

$$K_l^{\phi}(\lambda;z) = V_l^{\phi}(\lambda) + \overline{V}_l^{\phi}(\lambda) G_0^P(z) \overline{K}_l^{\phi}(\lambda;z). \tag{26}$$

Now, the fully off-shell amplitude could be defined by

$$\begin{aligned} \lim_{\lambda\to 0} T_l^C(p,p',\lambda;z) &= \lim_{\lambda\to 0}\lim_{R\to R_{cl}} \left\{ T_l^{R\phi}(p,p',\lambda;z) + T_l^{\phi}(p,p',\lambda;z) \right\} \\ &= \lim_{\lambda\to 0}\lim_{R\to R_{cl}} \left\{ \int_0^{\infty}\int_0^{\infty} dp''dp''' \right. \\ &\quad \times \overline{\omega}_l^{\phi}(p,p'',\lambda;z) t_l^{R\phi}(p'',p''',\lambda;z) \omega_l^{\phi}(p''',p',\lambda;z) \\ &\quad \left. + T_l^{\phi}(p,p',\lambda;z) \right\}, \end{aligned} \tag{27}$$

where it should be reminded that the half-shell Møller operators $\overline{\omega}_l^{\phi}(p,k,\lambda;z)$ and $\omega_l^{\phi}(k,p',\lambda;z)$ in the integrand are never reduced as Eq. (38) of article A, because they commit to the integral by $\lambda \neq 0$. Moreover, since the Møller operators in Eq. (27) exist in the integrand, then $\lambda \to 0$ could be performed by putting $V_l^{\phi} \to \overline{V}_l^{\phi}$ and being integrated.

As a consequence, the logarithmic singularity and the Green's function pole could be safely treated in the K-matrix formulation only in the integrand. Here, it should be noted that the new AP potential has meaning only inside of the integral.

The on-shell part of Eq. (27) is reduced by using Eqs. (12), (22), and $G_0(p'';z) = G_0^P(p'';z) + i\rho(k)\delta(k-p'')$, by omitting z,

$$\begin{aligned} \lim_{\lambda\to 0} T_l^C(k,k,\lambda) &= \lim_{\lambda\to 0}\lim_{R\to R_{cl}} \int_0^{\infty}\int_0^{\infty} dp''dp''' \\ &\quad \times \left[\delta(k-p'') + T_l^{\phi}(k,p'',\lambda) G_0^P(p'') \right] t_l^{R\phi}(p'',p''',\lambda) \\ &\quad \times \left[\delta(p'''-k) + G_0^P(p''') T_l^{\phi}(p''',k,\lambda) \right], \end{aligned} \tag{28}$$

where $T_l^{\phi}(k,p'',\lambda)$, and $T_l^{\phi}(p''',k,\lambda)$ do not vanish in the integrand. Substituting Eq. (20) for $T_l^{\phi}(k,p'',\lambda)$, and Eq. (19) for $T_l^{\phi}(p''',k,\lambda)$, Eq. (28) is rewritten by using Eq. (12),

$$\begin{aligned} \lim_{\lambda\to 0} T_l^C(k,k,\lambda) &= \lim_{\lambda\to 0}\lim_{R\to R_{cl}} \int_0^{\infty}\int_0^{\infty} dp''dp''' \\ &\quad \times \left[\delta(k-p'') + \overline{K}_l^{\phi}(k,p'',\lambda) G_0^P(p'') \right] t_l^{R\phi}(p'',p''',\lambda) \\ &\quad \times \left[\delta(p'''-k) + G_0^P(p''';z) \overline{K}_l^{\phi}(p''',k,\lambda) \right]. \end{aligned} \tag{29}$$

And again, substituting Eq. (19) into $T_l^{\phi}(k,p'',\lambda)$, and Eq. (20) into $T_l^{\phi}(p''',k,\lambda)$, we obtain

$$\begin{aligned} \lim_{\lambda\to 0} T_l^C(k,k,\lambda) &= \lim_{\lambda\to 0}\lim_{R\to R_{cl}} \int_0^{\infty}\int_0^{\infty} dp''dp''' \\ &\quad \times \left[\delta(k-p'') + \{ \overline{K}_l^{\phi}(k,p'',\lambda) \right. \\ &\quad \left. + i\rho(k) \overline{K}_l^{\phi}(k,k,\lambda) T_l^{\phi}(k,p'',\lambda) \} G_0^P(p'') \right] \\ &\quad \times t_l^{R\phi}(p'',p''',\lambda) \left[\delta(p'''-k) + G_0^P(p''') \left\{ \overline{K}_l^{\phi}(p''',k) \right. \right. \\ &\quad \left. \left. + i\rho(k) T_l^{\phi}(p''',k,\lambda) \overline{K}_l^{\phi}(k,k,\lambda) \right\} \right]. \end{aligned} \tag{30}$$

By comparison with Eq. (29), we conclude from Eq. (30)

$$\overline{K}_l^{\phi}(k,k,\lambda;z) = 0. \tag{31}$$

This "request" gives the critical range $R_{cl}(k)$ in the momentum space. By adopting $R = \exp[\gamma + \pi n/\eta(k)]/2k$ or $\eta(k)[\ln 2kR - \gamma] = \pi n$ (with $n = 0, 1, 2, \ldots$), $\overline{K}_0^{\phi}(k,k,\lambda;z)$ $(l=0, n=0)$ takes the values from 10^{-3} to 10^{-30} in the energy region which is concerned. Therefore, $1/\lambda = \exp[\gamma]/2k$, which we call the "MBT range," is sufficient to satisfy our *request*: $\overline{K}_0^{\phi}(k,k,\lambda;z) = 0$, although the MBT range takes a nonzero λ. Then our first trial or the first approximation is done for the case $R_{c0} = \exp[\gamma]/2k$ (for $l=0$) in the Yukawa-type SCP which was already predicted in article A.

Consequently, with new modified Møller operators in Eq. (29), the on-shell Coulomb amplitude is obtained by

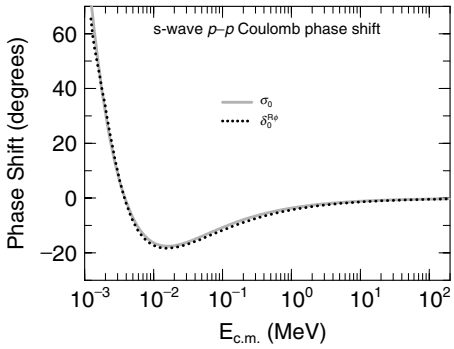

FIG. 1. The S-wave proton-proton Coulomb phase shift: $\delta_0^{R\phi}$ of "our theory (dotted-line)" is plotted along with the analytic result: σ_0 (gray solid-line) in the energy range from keV region to several hundred MeV region. Here, a simple approximation for the boundary range parameter: $R_{c0} = \exp[\gamma]/2k$ is adopted [1], which corresponds to $R_{c0} = 181.4$ fm for the energy $z = 1$ keV. Here, γ stands for the Euler constant for the Yukawa-type screened Coulomb potential: $V^R(r) = V^C(r)\exp[-(r/R)]$.

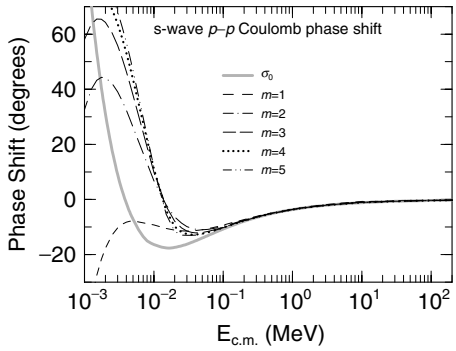

FIG. 2. The S-wave proton-proton Coulomb phase shifts by the "renormalization method" are plotted along with the analytic result (gray solid-line) in the energy range from keV region to several hundred MeV region. The calculated phase shifts for the screened Coulomb potential $V^R(r) = V^C(r)\exp[-(r/R)^m]$ plus the renormalization phase $\phi(k, R) = \eta(k)[\ln 2kR - \gamma/m]$ are illustrated. Here, $R = 200$ fm, and $m = 1$ (dashed line), 2 (dashed-dotted line), 3 (long-dashed line), 4 (dotted line), and 5 (dashed-dotted-dotted line) are adopted, respectively.

solving

$$T_l^{R\phi}(\lambda; z) \equiv \overline{\omega}_l^{\phi}(\lambda; z) t_l^{R\phi}(\lambda; z) \omega_l^{\phi}(\lambda; z)$$
$$= \overline{\omega}_l^{\phi}(\lambda; z) V_l^R \omega_l^{\phi}(\lambda; z)$$
$$+ \overline{\omega}_l^{\phi}(\lambda; z) V_l^R G_0(z) T_l^{R\phi}(\lambda; z). \qquad (32)$$

The Coulomb phase shift is obtained by $T_l^{R\phi}(\lambda; z)$ of Eq. (32).

Here, it should be mentioned that the procedure of $\lim_{\lambda\to 0}$ is performed by the "abnormal integral," because the result for $\lambda \neq 0$ is equivalent with the result for $\lambda = 0$. The calculated phase shift is illustrated for the case: $V^R(r) = V^C(r)\exp[-(r/R)]$ in Fig. 1. The result is a very good fit to the analytic phase shift in the energy range from the keV region to several hundred MeV.

In Fig. 2, the calculated phase shifts by the usual renormalization method are disclosed for the SCP: $V^R(r) = V^C(r)\exp[-(r/R)^m]$ in which the renormalization phase is given by $\phi(k, R) = \eta(k)[\ln 2kR - \gamma/m]$. However, $\delta_0^R(k) + \phi(k, R)$ could not represent the Coulomb phase shift in the energy region less than 100 keV for $m \leqslant 5$ and $R = 200$ fm. We adopted $R = 200$ fm for comparison with ours, because our longest range in this energy region is $R_{c0} = \exp[\gamma]/2k = 181.4$ fm which corresponds to the energy $z = 1$ keV. These renormalization calculations were done by us in the configuration space to preserve sufficient accuracy (see also [6–12,27]).

Finally, let us summarize our theory and the numerical method together with those of the article A.

(i) The LS equation for the Coulomb potential could not be solved.
(ii) The two potential theory for the short-range potential plus an auxiliary potential is one of the most useful methods.
(iii) The K-matrix equation for AP is introduced by the *Lemma* in article A, which has no overlapping singularities where the Green's function pole coincides with the logarithmic singularity of the Coulomb potential.
(iv) We introduce a "null-diagonal AP potential" which is only available in the integrand of the two-potential formalism. The K-matrix equation is safely solved by the potential.
(v) We found a formula, in this paper, which could induce the critical range R_{cl}.

The result is a very good fit to the exact one for a wide energy range. Details of the numerical method will be presented in another occasion. We also would like to postpone the discussions about the Coulomb plus nuclear potentials [20–22]. We believe that this work could be a new beginning in super low energy nuclear reaction problems.

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