Chiral properties of the QCD vacuum in ultrastrong magnetic fields: A Nambu-Jona-Lasinio model with a semiclassical approximation

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(Received 5 November 2005; revised manuscript received 23 May 2006; published 16 January 2007)

The breaking of chiral symmetry of light quarks at zero temperature in the presence of strong quantizing magnetic field is studied using a Nambu-Jona-Lasinio (NJL) model with a Thomas-Fermi-type semiclassical formalism. It is found that the dynamically generated light quark mass can never become zero if the Landau levels are populated and increases with the increase of magnetic field strength.

DOI: 10.1103/PhysRevC.75.015805 PACS number(s): 26.60.+c, 24.85.+p, 97.60.Jd

I. INTRODUCTION

The theoretical investigation of properties of compact stellar objects in the presence of strong quantizing magnetic field have gotten a new life after the recent discovery of a few magnetars [1–4]. These exotic stellar objects are believed to be strongly magnetized young neutron stars. Their surface magnetic fields are observed to be $\geqslant 10^{15}$ G. Then it is quite possible that the fields at the core region may go up to 10^{18} G. The exact source of strong magnetic field is of course yet to be known. These objects are also supposed to be the possible sources of anomalous X-ray and soft gamma emissions (AXP and SGR). Now, if the magnetic field is really so strong, in particular at the core region, it must affect most of the important physical properties of these stellar objects and also some of the physical processes, e.g., the rates/cross sections of elementary processes, in particular the weak and the electromagnetic decays/reactions taking place at the core region.

The strong magnetic field affects the equation of state of dense neutron star matter. As a consequence the gross properties of neutron stars [5–8], e.g., mass-radius relation, moment of inertia, rotational frequency, etc., should change significantly. In the case of compact neutron stars, the phase transition from neutron matter to quark matter which may occur at the core region is also affected by a strong quantizing magnetic field. It has been shown that a first order phase transition initiated by the nucleation of quark matter droplets is absolutely forbidden if the magnetic field strength $\sim 10^{15}$ G at the core region [9,10]. However, a second order phase transition is allowed, provided the magnetic field strength $<10^{20}$ G. This is of course too high to achieve at the core region. The study of time evolution of nascent quark matter, produced at the core region through some higher order phase transition, shows that in the presence of strong magnetic field it is absolutely impossible to achieve chemical equilibrium (β-equilibrium) configuration among the constituents of the quark phase if the magnetic field strength is as low as $B_m \sim 10^{14}$ G.

The elementary processes, in particular, the weak and the electromagnetic decays/reactions taking place at the core region of a neutron star are strongly affected by such ultrastrong magnetic fields [11,12]. Since the cooling of neutron stars is mainly controlled by neutrino/antineutrino emissions, the presence of a strong quantizing magnetic field should affect the thermal history of strongly magnetized neutron stars. Further, the electrical conductivity of neutron star matter which mainly comes from free electron gas, directly controls the evolution of the neutron star magnetic field, should also change significantly [12].

Similar to the study of quark-hadron deconfinement transition inside a neutron star core in the presence of a strong quantizing magnetic field, a lot of investigations have also been done on the effect of ultrastrong magnetic field on chiral symmetry breaking. In those studies, quantum field theoretic formalisms were mainly used [13–19]. In Ref. [20], Inagaki *et al.* studied the chiral symmetry violation with the Nambu-Jona-Lasinio (NJL) model using quantum field theoretic approach in the presence of a strong quantizing magnetic field. In many of these papers, the effect of curvature with or without external magnetic field on chiral symmetry violation have been investigated. In Refs. [13,21], Gusynin *et al.* have thoroughly investigated the chiral symmetry breaking in the presence of a strong external quantizing magnetic field. They have used the NJL model in $2+1$ and also in $3+1$ dimensions. It has been shown that the external magnetic field acts as a catalyst to generate fermion mass dynamically. In the first paper [13] they have studied it in a $2+1$ dimension and showed how the external magnetic field generated a dynamical mass of fermion and broke the dynamical flavor symmetry. They have further shown by using the NJL model that chiral symmetry breaks dynamically even if the attractive interaction between the fermions is extremely weak. In the second paper [21] they have extended the calculation to a $3+1$ dimension. In another work, Lee *et al.* [17] have studied the breaking of chiral symmetry for fermions in the presence of an external magnetic field. It has also been shown in this work that the symmetry is broken dynamically and further the effect of finite density and the temperature of the system on the chiral properties of the fermions have been investigated thoroughly in this paper in the presence of a strong magnetic field. It has been reported in this

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paper that there exists a critical density (or chemical potential) above which the chiral symmetry is again restored (which actually indicates the restoration of chiral symmetry at high enough density) and if it is treated as a chiral phase transition, the order will be of first order in nature. On the other hand the chiral symmetry is again restored at high temperature above some critical value. In this case the transition is of second order in nature. In an extensive review work [22], Klevansky has reported the dynamical chiral symmetry breaking in the presence of a strong external quantizing magnetic field using the NJL model in SU(2) and SU(3) flavor space. In this paper the effect of density (i.e., finite chemical potential) and temperature of the system on chiral symmetry restoration have also been reviewed. In some very recent work Wang *et al.* have critically examined the consistency and gauge independence of bare vertex approximation that has been extensively used in truncating the Schwinger-Dyson equation to obtain the dynamical mass of fermions generated through chiral symmetry breaking in a strong magnetic field. They inferred that the gauge independence approach poses a serious question on the validity of the results and conclusions obtained in earlier studies [23].

In the present article we shall study the effect of a strong quantizing magnetic field on the chiral properties of QCD vacuum with the help of the NJL model following a semiclassical Thomas-Fermi-type mean field approach in the presence of a strong quantizing external QED magnetic field. Now in the NJL model, there is no built-in mechanism of color confinement, however, it can produce two chirally distinct phases—appropriate for confined quark matter within the bag and the matter outside the bag. These phases are also known as the Wigner phase and spontaneously broken chiral phase, respectively. Therefore, in this formalism, if one reformulates the NJL model in the presence of a strong quantizing magnetic field, it is quite possible to obtain the effect of quantizing magnetic field on these two chirally distinct phases and hence obtain the effect of magnetic field on chiral symmetry breaking. Further, it is also possible to obtain bag pressure from the difference of vacuum energy densities of these two phases and hence its variation with a strong magnetic field. Assuming that the confinement and spontaneously broken chiral symmetry are synonymous, Bhaduri *et al.* obtained some estimate of the bag constant from the difference of energy densities [24] for the conventional case. In the present article we shall modify these original calculations of Bhaduri *et al.* [24] and Providência *et al.* [25] to study the breaking of chiral symmetry of light quarks in the presence of strong magnetic fields and show that the chiral symmetry always remains broken in the presence of a strong quantizing magnetic field if the Landau levels for quarks are populated. Our motivation in this work was to study the effect of a strong quantizing magnetic field on two chirally distinct phases and then obtain the vacuum pressure as a function of strong external magnetic field. Unfortunately, we have noticed that the Wigner phase does not exist if the Landau levels of quarks are populated and in this formalism, there is no way, either by controlling the chemical potential, i.e., the density of matter (which is meaningless in our investigation since we have considered QCD vacuum state) or temperature of the system to restore chiral symmetry (we have considered $T = 0$). Our present investigation is therefore basically an application of the formalism developed recently to study the equation of state of dense fermionic matter of astrophysical interest in the presence of a strong quantizing magnetic field [26]. Recently we have also used that formalism incorporating ρ-meson exchange in dense neutron star matter and shown that the self-energies of both neutron and proton become complex in nature in the presence of a strong quantizing magnetic field, even if it is a mean field approximation [27].

II. BASIC FORMALISM

We start with the density matrix $\rho(x, x')$, defined by

$$\rho(x, x') = \sum_{\text{spin}, p} \psi(x)\psi^\dagger(x')\theta(\Lambda - |p_z|), \tag{1}$$

where ψ and $\psi^\dagger$ are, respectively, the negative energy Dirac spinor and the corresponding adjoint, satisfying the equation

$$h\psi = E_-\psi \tag{2}$$

(and similarly for $\psi^\dagger$) with the single particle Hamiltonian

$$h = \gamma_5 \vec{\Sigma}.(\vec{p} - q_f\vec{A}) + \beta m \tag{3}$$

with

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{4}$$

γ_5 and β are the usual Dirac matrices, Λ is the infrared cutoff in the momentum integral over p_z (since we are considering vacuum, unlike a many body fermionic statistical system we have to put the cutoff by hand. Further, the z-component of momentum ranges from $-\infty$ to 0. We have therefore used $-\Lambda$ as the infrared cutoff for p_z-integrals.) and $\vec{A}$ is the electromagnetic field three vector corresponding to the external constant magnetic field of strength B_m along the z-axis. Here the light quark mass m is assumed to be generated dynamically. Now in the presence of a strong quantizing magnetic field along the z-direction obtained from the choice of gauge $A^\mu \equiv (0, 0, xB_m, 0)$, the up and down spin negative energy spinor solutions are therefore given by

$$\psi(x) = \frac{1}{(L_yL_z)^{1/2}} \exp[i(E_\nu t - p_y y - p_z z)]v_-^{(\uparrow,\downarrow)}, \tag{5}$$

where

$$v_-^{(\uparrow)} = \frac{1}{[2E_-(E_- - m)]^{1/2}} \begin{pmatrix} p_z I_\nu \\ -i(2\nu q_f B_m)^{1/2} I_{\nu-1} \\ (E_- - m)I_\nu \\ 0 \end{pmatrix} \tag{6}$$

and

$$v_-^{(\downarrow)} = \frac{1}{[2E_-(E_- - m)]^{1/2}} \begin{pmatrix} i(2\nu q_f B_m)^{1/2} I_\nu \\ -p_z I_{\nu-1} \\ 0 \\ (E_- - m)I_{\nu-1} \end{pmatrix}, \tag{7}$$

where $E_- = -(p_z^2 + m^2 + 2\nu q_f B_m)^{1/2} = -E_\nu$, is the single particle energy eigenvalue, $\nu = 0, 1, 2...$, are the Landau quantum numbers, q_f is the magnitude of the charge carried by fth flavor and

$$I_\nu = \left(\frac{q_f B_m}{\pi}\right)^{1/4} \frac{1}{(\nu!)^{1/2}} 2^{-\nu/2} \exp\left[-\frac{1}{2} q_f B_m \left(x - \frac{p_y}{q_f B_m}\right)^2\right] \times H_\nu\left[(q_f B_m)^{1/2}\left(x - \frac{p_y}{q_f B_m}\right)\right] \quad (8)$$

with H_ν is the well-known Hermite polynomial of order ν, and L_y, L_z are, respectively, length scales along the Y and Z directions. Now it can very easily be shown that $\nu = 0$ state is singly degenerate, whereas all other states are doubly degenerate. We now express the density matrix, as the modified version of the Wigner transform in the presence of a strong quantizing magnetic field, in the following form:

$$\rho(x, x') = \sum \rho(x, x', p_y, p_z, \nu) \exp[i\{(t - t')E_- - (y - y')p_y - (z - z')p_z\}], \quad (9)$$

where the sum is over the momentum components p_y, p_z and the Landau quantum number ν. Since the momentum variables are continuous, the sum over momentum components will be replaced by the corresponding integrals. Now we have from Eq. (9)

$$\rho(x, x', p_y, p_z, \nu) = \sum_{\text{spin}=-1/2}^{+1/2} v(x, p_y, p_z, \nu) v^\dagger(x', p_y, p_z, \nu) \quad (10)$$

and on substituting the negative energy up and down spinors states, we get

$$\rho(x, x', p_y, p_z, \nu) = \frac{1}{2E_-}[E_- A - p_z \gamma_z \gamma_0 A + m\gamma_0 A - p_\perp \gamma_y \gamma_0 B]\theta(\Lambda - |p_z|) \quad (11)$$

(see Appendix for detail derivation) where the matrices A and B are given by

$$A = \begin{pmatrix} I_\nu I'_\nu & 0 & 0 & 0 \\ 0 & I_{\nu-1} I'_{\nu-1} & 0 & 0 \\ 0 & 0 & I_\nu I'_\nu & 0 \\ 0 & 0 & 0 & I_{\nu-1} I'_{\nu-1} \end{pmatrix}, \quad (12)$$

$$B = \begin{pmatrix} I_{\nu-1} I'_\nu & 0 & 0 & 0 \\ 0 & I_\nu I'_{\nu-1} & 0 & 0 \\ 0 & 0 & I_{\nu-1} I'_\nu & 0 \\ 0 & 0 & 0 & I_\nu I'_{\nu-1} \end{pmatrix}, \quad (13)$$

where the primes indicate the functions of x'. Now in the evaluation of vacuum energy, we have noticed that it would be more convenient to define a quantity μ_f (to be more specific, it should be $-\mu_f$), similar to the chemical potential for the fth flavor in a multiquark statistical system in the presence of a strong quantizing magnetic field (strictly speaking we are not considering a multiquark statistical system and μ_f is therefore not the quark chemical potential. However, its minimum value should be m and not zero, i.e., in this simplified model, just like the dynamical mass m, this quantity is also treated as a parameter and we evaluate numerically μ_f and m and then obtain the upper limit of Landau quantum number $[\nu_{\max}^{(f)}]$ and the cutoff Λ. Then it is very easy to write

$$\Lambda = \left({\mu_f}^2 - m^2 - 2\nu q_f B_m\right)^{1/2}. \quad (14)$$

Since $\Lambda > 0$, it is also possible to express the upper limit of ν, which is the maximum value of Landau quantum number of the levels occupied by fth flavor, and is given by

$$\nu_{\max}^{(f)} = \left[\frac{\mu_f^2 - m^2}{2q_f B_m}\right], \quad (15)$$

where [ ] indicates the nearest integer but less than the actual number. Now to obtain the energy density of the vacuum, we consider the NJL (chiral) Hamiltonian, given by

$$H = \sum_{i=i}^{N} t(i) + \frac{1}{2}\sum_{i\neq j} V(i, j) \quad (16)$$

$$= \sum_{i=1}^{N} \gamma_5(i)\vec{\Sigma}(i).(\vec{p}_i - q_f \vec{A}) - \frac{1}{2}(2g)\sum_{i\neq j} \delta(\vec{x}_i - \vec{x}_j)[\beta(i)\beta(j) - \beta(i)\gamma_5(i)\beta(j)\gamma_5(j)] \quad (17)$$

where Σ, γ_5, and β are usual 4×4 matrices and $2g$ is the effective coupling. Here we have used the formulation of da Providência *et al.* [25] for the mean field density matrix to describe the Dirac vacuum, thereby employing the Thomas-Fermi semiclassical method instead of formal field theory. As we have noticed, the physics of condensation energy is more transparent in this method than the formal field theoretic technique. The energy of the vacuum is then given by

$$\epsilon_v = \sum_{p_{1_z}, \nu_1} \int dx_1 tr_1[\{\gamma_5 \vec{\Sigma}.(\vec{p}_1 - q_f \vec{A})\}\rho_{p_1}] + \epsilon_v^{(I)}, \quad (18)$$

where ρ_{p_1} is given by Eq. (11), the first term is the kinetic energy part and $\epsilon_v^{(I)}$ indicates the interaction term, including the exchange interaction. To evaluate the vacuum energy, we first calculate the kinetic energy term in Eq. (18). This quantity is proportional to the trace defined by $\mathrm{Tr}(\rho h)$, which can easily be evaluated by using ρ from Eq. (11) and the single particle Hamiltonian h from Eq. (17). Now using the orthonormality relations for the Hermite polynomials at the time of the evaluation of the integral over dx and also using the anticommutation relations for γ-matrices, we have the first term at zero temperature (see Appendix)

$$\epsilon_v^{(0)} = 2N_c \sum_{f=u}^{d} \frac{q_f B_m}{2\pi^2} \sum_{\nu=0}^{[\nu_{\max}^{(f)}]} (2 - \delta_{\nu 0}) \int_0^\Lambda dp_z \frac{\vec{p}^2}{E_-}, \quad (19)$$

where $\vec{p}^2 = p_z^2 + 2\nu q_f B_m$, $N_c = 3$, the number of colors, and $E_- = -E_\nu$.

In the evaluation of all the traces in this paper we have used the following important relation:

$$\mathrm{Tr}\,(\gamma^\mu \gamma^\nu A_1 A_2..B_1 B_2..) = \mathrm{Tr}\,(A_1 A_2..B_1 B_2..) g^{\mu\nu}, \quad (20)$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma A_1A_2..B_1B_2..) = \mathrm{Tr}\,(A_1A_2..B_1B_2..)(g^{\mu\nu}g^{\sigma\lambda} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}), \quad (21)$$

Tr (product of odd γs with A and/or B) = 0, etc. The other interesting aspects of A and B matrices are

(i) $k_{1\mu}k^{2\mu}\mathrm{Tr}\,(A_1A_2) = (E_1E_2 - k_{1z}k_{2z})\mathrm{Tr}\,(A_1A_2)$,
(ii) $k_{1\mu}k^{2\mu}\mathrm{Tr}\,(B_1B_2) = \vec{k}_{1\perp}.\vec{k}_{2\perp}\,\mathrm{Tr}\,(B_1B_2)$,
(iii) $k_{1\mu}k^{2\mu}\mathrm{Tr}\,(A_1B_2) = k_{1\mu}k^{2\mu}\mathrm{Tr}\,(B_1A_2) = 0$,
(iv) $p_{1\mu}k^{1\mu}p_{2\nu}k^{2\nu}\mathrm{Tr}\,(A_1B_2) \neq 0 = (E_{\nu_1}E_{\nu_2'} - p_{1z}k_{1z})\,\vec{p}_{2\perp}.\vec{k}_{2\perp}\,\mathrm{Tr}\,(A_1B_2)$.

This set of relations was derived in a very recent publication by us [26]. Since γ matrices are traceless and both A and B matrices are diagonal with identical blocks, it is very easy to evaluate the above traces of the product of γ-matrices multiplied with any number of A and/or B, from any side with any order.

To evaluate the interaction term, we first consider the direct part which is proportional to $\mathrm{Tr}\,(\beta\rho_{p_1})\mathrm{Tr}\,(\beta\rho_{p_2})$ and it is very easy to show that $\mathrm{Tr}\,(\beta\gamma_5\rho) = 0$ (see Appendix). Then using the orthonormality relations for Hermite polynomials and the anticommutation relations for the γ-matrices, we have the direct term

$$V_{\mathrm{dir}} = -4\,\mathrm{gm}^2[\mathcal{V}(\Lambda, m)]^2 \quad (22)$$

(the four fermion coupling is included in $\mathcal{V}$, it is $\sim V(1,2)\rho_1\rho_2$) where

$$\mathcal{V}(\Lambda, m) = \frac{N_c}{2\pi^2}\sum_{f=u}^{d} e_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^\Lambda \frac{dp_z}{\left(p_z^2 + m_{\nu,f}^2\right)^2}, \quad (23)$$

where $m_{\nu,f} = (m^2 + 2\nu q_f B_m)^{1/2}$ (see Appendix for derivation).

To evaluate the exchange term, we first calculate $\mathrm{Tr}\,((\beta\rho_{p_1})(\beta\rho_{p_2}))$. Now

$$\beta\rho = \frac{1}{2E_-}[E_-\beta A + p_zA\gamma_z + mA - p_\perp B\gamma_y]. \quad (24)$$

Then to obtain the trace of the product of $\beta\rho_1$ and $\beta\rho_2$ it is very easy to show that only the direct product terms are nonzero whereas the cross product terms do not contribute. Therefore

$$\beta\rho_1\beta\rho_2 = \frac{1}{4E_1E_2}[E_1\beta A + p_{1z}A\gamma_z + mA - p_{1\perp}B\gamma_y] \times [E_2\beta A' + p_{2z}A'\gamma_z + mA' - p_{2\perp}B'\gamma_y]. \quad (25)$$

Then using the orthonormality relations for Hermite polynomials at the time of integration over dx_1 and dx_2, the above trace reduces to

$$\int_{-\infty}^{+\infty}(\beta\rho_1\beta\rho_2)dx = \frac{1}{4E_1E_2}[4E_1E_2 + 4p_{1z}p_{2z} + 4m^2] = \left[1 + \frac{p_{1z}p_{2z}}{E_1E_2} + \frac{m^2}{E_1E_2}\right], \quad (26)$$

where both E_1 and E_2 are negative. Then in the energy contribution, after integrating over p_{1z} and p_{2z}, the first term gives

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\Lambda\right)^2. \quad (27)$$

Similarly the contribution from second term is given by

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\left(\Lambda^2 + m_{\nu,f}^2\right)^{1/2}\right)^2 \quad (28)$$

and finally, the third term is given by

$$m^2\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\ln\left[\frac{\Lambda + \left(\Lambda^2 + m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]\right)^2 \quad (29)$$

(see Appendix for derivation of these expressions)

To obtain the next term in the exchange part, we evaluate the trace $\mathrm{Tr}\,((\beta\gamma_5\rho_{p_1})(\beta\gamma_5\rho_{p_2}))$, which unlike the direct case, gives nonzero contribution. Using the anticommutation relations of γ-matrices and as usual with the help of orthonormality relations for Hermite polynomials, we finally arrive to the following result:

$$-\left[1 + \frac{p_{1z}p_{2z}}{E_1E_2} + \frac{m^2}{E_1E_2} + m\left(\frac{1}{E_1} + \frac{1}{E_2}\right)\right] \quad (30)$$

(see Appendix for derivation). The contribution to the interaction energy will again be obtained if we integrate over p_{1z} and p_{2z} (done in a similar manner as has been done for the direct case). Then the first term is given by

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\Lambda\right)^2. \quad (31)$$

The second term is given by

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\left(\Lambda^2 + m_{\nu,f}^2\right)^{1/2}\right)^2. \quad (32)$$

The third term is given by

$$m^2\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\ln\left[\frac{\Lambda + \left(\Lambda^2 + m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]\right)^2 \quad (33)$$

and finally the fourth and fifth terms, which are identical, are given by

$$m\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\ln\left[\frac{\Lambda + \left(\Lambda^2 + m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]\right) \times \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\Lambda\right). \quad (34)$$

Then combining all these terms we finally obtain the vacuum energy density. Since the mass m, which is assumed to be the same for both u and d quarks, is generated dynamically, we obtain this quantity by minimizing the total vacuum energy density with respect to m, i.e., by putting $d\epsilon_v/dm = 0$. Simplifying this nonlinear equation, we finally get

$$\frac{d\epsilon_v}{dm} = -P + 2gQR = 0, \tag{35}$$

where

$$P = \frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\rm max}^{(f)}]}(2-\delta_{\nu 0}) \times \left[\frac{2m^3\Lambda}{m_{\nu,f}^2}\frac{1}{\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}} - 2mX\right], \tag{36}$$

$$Q = \frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\rm max}^{(f)}]}(2-\delta_{\nu 0}) \times \left[X - \frac{m^2}{m_{\nu,f}^2}\frac{\Lambda}{\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}}\right], \tag{37}$$

$$R = \frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\rm max}^{(f)}]}(2-\delta_{\nu 0})\,[\Lambda - 4mX] \tag{38}$$

with

$$X = \ln\left[\frac{\Lambda + \left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]. \tag{39}$$

It is therefore obvious from Eq. (35) that the trivial solution $m = 0$ is not possible in this particular situation, or in other words, the gap equation given by

$$m = 4g\mathcal{V}m \tag{40}$$

cannot exist. On the other hand in a nonmagnetic case, or for the magnetic field strength less than the quantum critical value, Eq. (35) reduces to the gap equation as written above [Eq. (40)]. Here $\mathcal{V}$ is the overall contribution of interaction terms. Hence it is obvious that $m = 0$, the trivial solution exists in this nonmagnetic or the conventional scenario, investigated by Bhaduri *et al.* [24]. The phase with $m = 0$ is the Wigner phase and $m \neq 0$ is the so-called Goldstone phase. Now Eq. (40) further gives

$$4g\mathcal{V} = 1 \tag{41}$$

which is nothing but the well-known gap equation used in BCS theory. The gap equation therefore does not exist in the presence of a strong quantizing magnetic field if the Landau levels for u and d quarks are populated.

III. CONCLUSION

The nonexistence of trivial solution ($m = 0$) indicates the spontaneously broken chiral symmetry in the presence of a strong quantizing magnetic field. Therefore as soon as the Landau levels are populated for light quarks in the presence of a strong external magnetic field, the chiral symmetry gets broken, the quarks become massive, and the mass m (assumed to be same for both u and d quarks) is generated dynamically.

Therefore we may conclude here that the Wigner phase does not exist in the case of relativistic Landau diamagnetic system. Further, if the deconfinement transition and restoration of chiral symmetry occur simultaneously, or in other words, if the chiral symmetry remains restored within the bag, as it is generally assumed, then it puts a big question mark whether the idea of bag model is applicable at all in the presence of a strong quantizing magnetic field. Questions may also arise, that if the Wigner phase still exists inside the bag, then whether the external quantizing magnetic field can penetrate the bag boundary, if not, what is the underlying physics which prevents the external magnetic field from entering the periphery of the bag.

Now to illustrate the variation of dynamical quark mass with magnetic field, we consider the relation

$$m_\pi^2 = -\frac{m_0}{f_\pi^2}\langle\psi\bar{\psi}\rangle, \tag{42}$$

where m_π is the pion mass, m_0 is the quark current mass and f_π is the pion decay constant. Using the spinor solutions given by Eqs. (6) and (7) we get

$$m_\pi^2 = \frac{2m_0 m}{f_\pi^2}\frac{N_c}{2\pi^2}\sum_{f=u}^{d}\sum_{\nu=0}^{[\nu_{\rm max}^{(f)}]}(2-\delta_{\nu 0})\ln\left[\frac{\Lambda+\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]. \tag{43}$$

We have now solved Eqs. (35) and (43) numerically to obtain Λ and m for various values of magnetic field strength. In Fig. 1, we have shown the variation of m with the strength of the magnetic field. In the actual numerical work we have solved self-consistently for the dynamical mass m and the parameter μ_f using Eq. (15) for $\nu_{\rm max}^{(f)}$ and then from Eq. (14) we get the infrared momentum cutoff Λ. In our calculation we have always used μ_f instead of Λ which allows us to obtain $\nu_{\rm max}^{(f)}$. So we cannot compare our result with those obtained with zero chemical potential, since in our calculation it is just

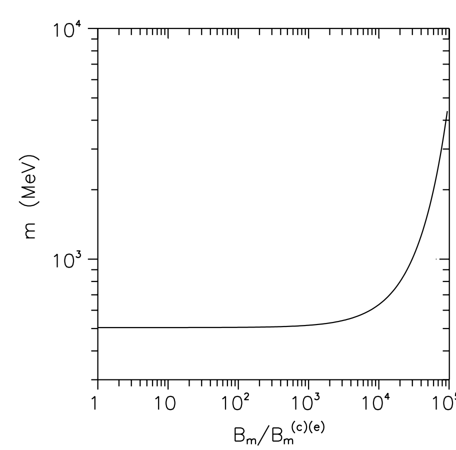

FIG. 1. The variation of dynamically generated quark mass with the strength of magnetic field (expressed in terms of $B_m^{(c)(e)} = 4.4 \times 10^{13}$ G).

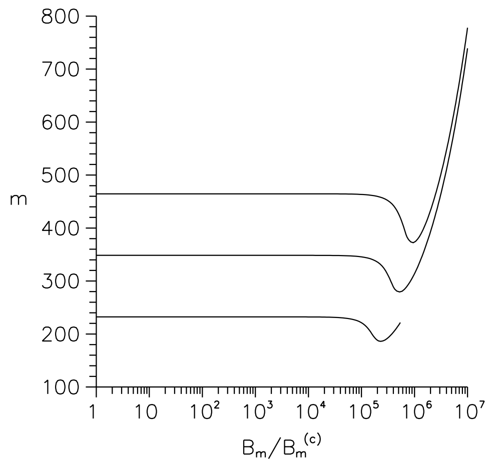

FIG. 2. The variation of constituent quark mass with the strength of magnetic field (expressed in terms of $B_m^{(c)(e)} = 4.4 \times 10^{13}$ G) from [21]. The lower curve is for $\Lambda = 200$ MeV, middle one for $\Lambda = 400$ MeV, and the top one for $\Lambda = 550$ MeV.

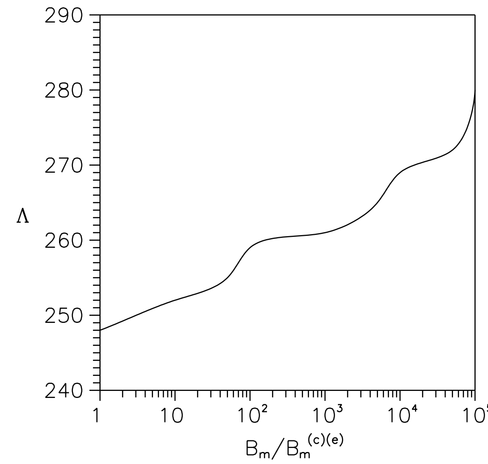

FIG. 3. The variation of magnitude of infrared cutoff Λ in MeV with the strength of magnetic field (expressed in terms of $B_m^{(c)(e)} = 4.4 \times 10^{13}$ G).

a parameter, it has no resemblance with chemical potential in a finite density quark (u, d) matter system. We were forced to to do this to obtain m, Λ and also $\nu_{\rm max}^{(f)}$ self-consistently from the numerical solutions of Eqs. (35) and (43). In doing numerical calculations, we have considered the following sets of numerical values for the parameters. The current quark mass $m_0 = 10$ MeV, pion mass $m_\pi = 140$ MeV, pion decay constant $f_\pi = 93$ MeV, coupling constant $g = 10$ GeV^{-2} and electron mass $m_e = 0.5$ MeV. In Fig. 1 we have shown the variation of dynamically generated quark mass with the strength of magnetic field. As it is evident that the dynamical quark mass never goes to zero and diverges beyond $B_m \approx 10^{17}$ G.

In Fig. 2, we have plotted for the sake of comparison the variation of constituent quark mass with magnetic field strength after solving numerically Eq. (17) of Ref. [21]. Here also for low and moderate values of magnetic field strength, the constituent mass does not change and for very high field strength, it diverges. Three curves correspond to $\Lambda = 200$ MeV (lower one), $\Lambda = 400$ MeV (middle one), and $\Lambda = 550$ MeV (upper one). We have noticed that solution of this equation does not exist for $\Lambda \geqslant 600$ MeV.

For the sake of the clarification of some of our findings, we would now like to add the following note. To explain, why we have used μ_f, instead of a constant momentum cutoff (infrared) Λ, we consider the following points:

(i) Actual range of p_z is from $-\infty$ to 0.

(ii) Energy eigenvalue $E_- = -E_\nu = -(p_z^2 + 2\nu q_f B_m + m^2)^{1/2}$, is negative and corresponds to νth Landau level.

(iii) Now for a positive energy fermion at $T = 0$ in a statistical system, the maximum energy and the corresponding momentum are μ_f and p_f, respectively. In this case, however, no such limits exist.

(iv) Further the presence of B_m which is greater than the quantum critical value, breaks the spherical symmetry of the momentum space and makes it cylindrical in nature, with $-\infty \leqslant p_z \leqslant +\infty$ (in the present case it is $-\infty \leqslant p_z \leqslant 0$) and $p_| = 2\nu q_f B_m$ (which is negative in the present case) and changes in a discrete manner (quantized).

(v) Therefore we are forced to introduce two different cutoff values to fix these momentum components.

(vi) From the knowledge of relativistic version of Landau diamagnetism, we know that the maximum value of the Landau quantum number is a decreasing function of B_m. Beyond some upper limit of B_m, the maximum value of the Landau quantum number ($[\nu_{\rm max}^{(f)}]$) becomes zero. Which means that for such strong B_m the zeroth Landau level will only be occupied.

(vii) To know $[\nu_{\rm max}^{(f)}]$ for a given B_m, it is therefore absolutely necessary to introduce some energy cutoff, which is $-\mu_f$ in this particular case, otherwise it is impossible to evaluate the upper limit of ν (which is further associated with the transverse component of quark momentum). We are therefore forced to use $-\mu_f$ as an infrared energy cutoff for the fth flavor.

(viii) It has been noticed that with our present model Λ cannot be a constant. It is a slowly varying function of B_m. It increases slowly with B_m as shown in Fig. 3. On the other hand for very high B_m, $[\nu_{\rm max}^{(f)}] \to 0$ and quarks become too massive. In fact, in ultrastrong magnetic field, quarks behave like massive particles moving along z-direction only, with finite momentum $\leqslant \Lambda$.

(ix) Finally, unless the above formalism is followed, we will not be able to evaluate any physical quantities, e.g., energy, etc., associated with the system. In our opinion the use of a magnetic field dependent infrared cutoff μ_f for energy eigenvalue, instead of a fixed Λ, is a draw back of this model. Which of course is not necessary in other kinds of models to study chiral symmetry breaking with strong magnetic field.

(x) Further, it has been observed that the model calculation we are doing here has no scope to study the effect of a strong quantizing magnetic field on the scalar excitation of QCD vacuum. We are investigating it as a separate problem.

For the sake of comparison with the existing results, particularly those published in Ref. [21], we would like to add the following note: The formalism developed in Ref. [21], as cited in this manuscript, to obtain the constituent mass of quarks generated dynamically is basically a NJL model with field theoretic (path integral) approach. Whereas in our paper, we have followed the Thomas-Fermi-type semiclassical formalism. In our model, we have treated the dynamically generated quark mass as a parameter and is evaluated numerically by minimizing the total energy of the system with respect to this parameter for a given magnetic field strength. In Ref. [21], on the other hand, obtained two different fields, σ (the scalar part) and π (the pseudoscalar part). These are derived from the fermion fields using the standard form of definitions. Neglecting the pseudo-scalar part and assuming that $\rho^2 = \sigma^2 + \pi^2$ remains invariant, a nonlinear form of potential for the scalar field is obtained. The scalar field, or equivalently ρ is obtained by minimizing the potential $V(\rho)$ with respect to this field variable. The dynamically generated quark constituent mass is then obtained just by equating $\sigma = \rho = m_{\rm dyn}$. The similarity with our work is that we have minimized the total energy with respect to the dynamically generated mass, which we have treated as a parameter, whereas in the work of Ref. [21], they have minimized the potential $V(\rho)$ with respect to ρ which is just the dynamically generated quark constituent mass. In Fig. 2 we have plotted this dynamically generated quark constituent mass against the strength of magnetic field for various values of the cut off parameter Λ. We have solved Eq. (17) of Ref. [21] numerically to obtain the mass for a given magnetic field strength. It is found that for low and moderate values of magnetic field strength, the dynamical mass of quarks is almost insensitive to magnetic field strength and diverges at the ultrastrong limit. The result is almost identical with our findings. However, the constant part of dynamical quark mass as obtained from the numerical solution from Ref. [21] increases with the increase of the value of cutoff parameter Λ. Whereas, in our case, since we have obtained both Λ and the dynamical mass self-consistently for a given magnetic field strength, we cannot change the value of Λ by hand to check the variation of the constant portion of dynamical quark mass. In Fig. 3 we have plotted the variation of Λ with the strength of the magnetic field. Further, we have noticed that for $\Lambda =$ 560 Mev, there cannot exist any solution of Eq. (17) of Ref. [21], which is also consistent with our result (obviously from Fig. 3). In our case it is about 300 MeV and this is the reason why we have obtained a relatively small value of constituent mass for relatively low magnetic field strength.

APPENDIX A: EVALUATION OF DENSITY MATRIX

To obtain the density matrix [Eq. (11)], we use Eqs. (6) and (7), then

$$v^\uparrow v^{\uparrow\dagger} = \frac{1}{2E(E-m)} \begin{pmatrix} p_z I_\nu \\ -i(2\nu q_f B_m)^{1/2} I_{\nu-1} \\ (E-m) I_\nu \\ 0 \end{pmatrix} (p_z I'_\nu i(2\nu q_f B_m)^{1/2} I'_{\nu-1} (E-m) I'_\nu 0) \tag{A1}$$

$$= \frac{1}{2E(E-m)} \begin{pmatrix} p_z^2 I_\nu I'_\nu & i p_z (2\nu q_f B_m)^{1/2} I_\nu I'_{\nu-1} & p_z (E-m) I_\nu I'_\nu & 0 \\ -i p_z (2\nu q_f B_m)^{1/2} I_{\nu-1} I'_\nu & 2\nu q_f B_m I_{\nu-1} I'_{\nu-1} & -i (2\nu q_f B_m)^{1/2} (E-m) I_{\nu-1} I'_\nu & 0 \\ p_z (E-m) I_\nu I'_\nu & i(2\nu q_f B_m)^{1/2} (E-m) I_\nu I'_{\nu-1} & (E-m)^2 I_\nu I'_\nu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A2}$$

and

$$v^\downarrow v^{\downarrow\dagger} = \frac{1}{2E(E-m)} \begin{pmatrix} i(2\nu q_f B_m)^{1/2} I_{\nu-1} \\ -p_z I_{\nu-1} \\ 0 \\ (E-m) I_{\nu-1} \end{pmatrix} \left(-i(2\nu q_f B_m)^{1/2} I_\nu - p_z I'_{\nu-1} 0 (E-m) I'_{\nu-1}\right) \tag{A3}$$

$$= \frac{1}{2E(E-m)} \begin{pmatrix} (2\nu q_f B_m) I_\nu I'_\nu & -i p_z (2\nu q_f B_m)^{1/2} I_\nu I'_{\nu-1} & 0 & i(2\nu q_f B_m)^{1/2} (E-m) I_\nu I'_{\nu-1} \\ + i p_z (2\nu q_f B_m)^{1/2} I_{\nu-1} I'_\nu & p_z^2 I_{\nu-1} I'_{\nu-1} & 0 & -(E-m) p_z I_{\nu-1} I'_{\nu-1} \\ 0 & 0 & 0 & 0 \\ - i(2\nu q_f B_m)^{1/2} (E-m) I_{\nu-1} I'_\nu & -p_z (E-m) I_{\nu-1} I'_{\nu-1} & 0 & (E-m)^2 I_{\nu-1} I'_{\nu-1} \end{pmatrix}. \tag{A4}$$

Adding

$$v^{\uparrow}v^{\uparrow\dagger}+v^{\downarrow}v^{\downarrow\dagger}=\frac{1}{2E}\begin{pmatrix}(E+m)I_\nu I'_\nu & 0 & p_zI_\nu I'_\nu & i(2\nu q_fB_m)^{1/2}I_\nu I'_{\nu-1}\\ 0 & (E+m)I_{\nu-1}I'_{\nu-1} & -i(2\nu q_fB_m)^{1/2}I_{\nu-1}I'_\nu & -p_zI_{\nu-1}I'_{\nu-1}\\ p_zI_\nu I'_\nu & i(2\nu q_fB_m)^{1/2}I_\nu I'_{\nu-1} & (E-m)I_\nu I'_\nu & 0\\ -i(2\nu q_fB_m)^{1/2}(E-m)I_{\nu-1}I'_\nu & -p_zI_{\nu-1}I'_{\nu-1} & 0 & (E-m)I_{\nu-1}I'_{\nu-1}\end{pmatrix} \tag{A5}$$

which may easily be simplified after a little algebra and reduces to

$$\rho(x,x',p_y,p_z,\nu)=\frac{1}{2E_-}[E_-A-p_z\gamma_z\gamma_0A+m\gamma_0A-p_\perp\gamma_y\gamma_0B]\theta(\Lambda-|p_z|), \tag{A6}$$

where the matrices A and B are given by Eqs. (12) and (13).

APPENDIX B: EVALUATION OF KINETIC TERM

Free Hamiltonian

$$h=\gamma^5\vec{\Sigma}\cdot\vec{p}+\beta m \tag{B1}$$

substituting Σ from Eq. (4) and using the properties of γ-matrices we get

$$h=\vec{\alpha}\cdot\vec{p}+\beta m \tag{B2}$$

the usual one. In presence of external magnetic field with the gauge choice as mentioned before, we have

$$h=\gamma_5\vec{\Sigma}\cdot(\vec{p}-q_f\vec{A})+\beta m \tag{B3}$$

by the same technique as above, we have

$$h=\vec{\alpha}\cdot(\vec{p}-q_f\vec{A})+\beta m. \tag{B4}$$

Now to evaluate $\mathrm{Tr}\,(h\rho)$, where

$$h=(\vec{\gamma}\cdot(\vec{p}-q_f\vec{A})+m)\gamma_0 \tag{B5}$$

we take $A_y=q_fxB_m$, whereas all other components are zero. Then we can express the above Hamiltonian in the following form:

$$h=(\gamma_xp_x+\gamma_y(p_y-q_fxB_m)+\gamma_zp_z+m)\gamma_0 \tag{B6}$$

substituting

$$(q_fB_m)^{1/2}\left(\frac{p_y}{q_fB_m}-x\right)=-\zeta \tag{B7}$$

we have

$$h=(\gamma_xp_x-(q_fB_m)^{1/2}\gamma_y\zeta+\gamma_zp_z+m)\gamma_0. \tag{B8}$$

In evaluating the trace $\mathrm{Tr}\,(h\rho)$ we integrate over x-coordinate (i.e., on ζ), use orthonormality relations for $H_\nu(\zeta)$ and finally using the anticommutation relations for γ-matrices, we have the first term of the product $\propto 0$ [from the conclusion drawn just after Eq. (21)], second term $\propto 4p_z^2$, third term $\propto 4m^2$, and the fourth term $\propto 4p_\perp^2$. Finally after summation, we have $\mathrm{Tr}\,(\rho h)\propto 2E$. Since $E<0$, this is actually $\propto -2E$, where $E=(p_z^2+p_\perp^2+m^2)^{1/2}$. Then the kinetic energy density is given by

$$\begin{aligned}\epsilon_v^{(0)}&=2N_c\sum_{f=u}^{d}\frac{q_fB_m}{2\pi^2}\sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu0})\int_0^\Lambda dp_zE\\&=\frac{N_c}{4\pi^2}\sum_{f=u}^{d}q_fB_m\sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu0})\left[\Lambda\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}\right.\\&\quad\left.+m_{\nu,f}^2\ln\left|\frac{\Lambda+\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right|\right],\end{aligned} \tag{B9}$$

where $m_{\nu,f}^2=m^2+2\nu q_fB_m$ and $q_u=2/3e$, $q_d=1/3e$ and $N_c=3$.

APPENDIX C: INTERACTION ENERGY (DIRECT TERM)

Now

$$\beta\gamma_5\rho=\gamma_x\gamma_y\gamma_z\frac{1}{2E}[EA-p_z\gamma_z\gamma_0A+m\gamma_0Ar+p_\perp\gamma_y\gamma_0B]. \tag{C1}$$

Then it is very easy to check that the trace of the above expression is zero.

So to obtain the direct term of interaction we have to evaluate the $\mathrm{Tr}\,(\beta\rho)$ integrated over x-coordinate, i.e.,

$$\int_{-\infty}^{+\infty}\mathrm{Tr}\,(\beta\rho)dx=\frac{1}{2E}\int_{-\infty}^{+\infty}\mathrm{Tr}\,[E\beta A+p_zA\gamma_z+mA-p_\perp B\gamma_y]dx \tag{C2}$$

since the first, second, and fourth terms contain odd (single) number γ with A the contribution of those terms are zero. Only the third term contributes to the integral and is given by

$$\frac{m}{2E}\int_{-\infty}^{+\infty}\mathrm{Tr}\,(A)dx=\frac{m}{2E}\int_{-\infty}^{+\infty}2\left[I_nu^2(x)+I_{\nu-1}^2(x)\right]dx. \tag{C3}$$

Using the orthonormality relation for Hermite polynomials the above integral reduces to $2m/E$. Since there is an identical expression with integral over x', finally we get

$$V_{\rm dir}=-4\,{\rm g}m^2[\mathcal{V}(\Lambda,m)]^2. \tag{C4}$$

APPENDIX D: MOMENTUM INTEGRAL

First term:

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} dp_{1z}\right) \times \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} dp_{2z}\right) = \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\Lambda\right)^2. \quad \text{(D1)}$$

Second term:

$$\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} \frac{p_{1z}dp_{1z}}{E_1}\right) \times \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} \frac{p_{2z}dp_{2z}}{E_2}\right) = \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2} m_{\nu,f}\right)^2. \quad \text{(D2)}$$

Third term:

$$m^2\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} \frac{dp_{1z}}{E_1}\right) \times \left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0})\int_0^{\Lambda} \frac{dp_{2z}}{E_2}\right) = m^2\left(\frac{N_c}{2\pi^2}\sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu_{\max}^{(f)}]}(2-\delta_{\nu 0}) \times \ln\left[\frac{\Lambda+\left(\Lambda^2+m_{\nu,f}^2\right)^{1/2}}{m_{\nu,f}}\right]\right)^2. \quad \text{(D3)}$$

Interaction energy (exchange terms):

The product

$$(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2) = \gamma_x\gamma_y\gamma_z\frac{1}{2E_1}[E_1A_1 + p_{1z}\gamma_z\gamma_0A_1 + m\gamma_0A_1 + p_{1\perp}\gamma_y\gamma_0B_1]\gamma_x\gamma_y\gamma_z\frac{1}{2E_2} \times [E_2A_2 + p_{2z}\gamma_z\gamma_0A_2 + m\gamma_0A_2 + p_{2\perp}\gamma_y\gamma_0B_2]. \quad \text{(D4)}$$

Now

$$\gamma_x\gamma_y\gamma_z\frac{1}{2E_1}[E_1A_1 + p_{1z}A_1\gamma_z + mA_1 - p_{1\perp}B_1\gamma_y] = \frac{1}{2E_1}[E_1\gamma_x\gamma_y\gamma_zA_1 + p_{1z}\gamma_x\gamma_yA_1 + \gamma_x\gamma_y\gamma_z mA_1 - p_{1\perp}\gamma_x\gamma_y\gamma_zB_1\gamma_y]. \quad \text{(D5)}$$

Hence

$$(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2) = \frac{1}{4E_1E_2}[E_1\gamma_x\gamma_y\gamma_zA_1 + p_{1z}\gamma_x\gamma_yA_1 + \gamma_x\gamma_y\gamma_z mA_1 - p_{1\perp}\gamma_x\gamma_y\gamma_zB_1\gamma_y] \times [E_2\gamma_x\gamma_y\gamma_zA_2 + p_{2z}\gamma_x\gamma_yA_2 + \gamma_x\gamma_y\gamma_z mA_2 - p_{2\perp}\gamma_x\gamma_y\gamma_zB_2\gamma_y]. \quad \text{(D6)}$$

Now to obtain the Tr $[(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2)]$, we use the results

$$\gamma_x\gamma_y\gamma_z\gamma_x\gamma_y\gamma_z = -\gamma_x\gamma_y\gamma_z\gamma_x\gamma_z\gamma_y = \gamma_x\gamma_y\gamma_x\gamma_y = -1, \quad \gamma_x\gamma_y\gamma_x\gamma_y = -1, \quad \text{(D7)}$$

$$\gamma_x\gamma_y\gamma_z\gamma_0\gamma_x\gamma_y\gamma_z\gamma_0 = +1, \quad \gamma_x\gamma_y\gamma_0\gamma_x\gamma_y\gamma_0 = -1. \quad \text{(D8)}$$

Then we finally get

$$\mathrm{Tr}\,[(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2)] = \frac{1}{4E_1E_2}[-4E_1E_2 - 4p_{1z}p_{2z} - 4m^2 - 4mE_2 - 4mE_1] = -\left[1 + \frac{p_{1z}p_{2z}}{E_1E_2} + \frac{m^2}{E_1E_2} + m\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\right]. \quad \text{(D9)}$$

The integration over p_{1z} and p_{2z} are done in the same manner as for Tr $[(\beta\rho_1)(\beta\rho_2)]$.

[1] R. C. Duncan and C. Thompson, Astrophys. J. Lett. **392**, L9 (1992); C. Thompson and R. C. Duncan, Astrophys. J. **408**, 194 (1993); Mon. Not. Roy. Astr. Soc. **275**, 255 (1995); Astrophys. J. **473**, 322 (1996).

[2] P. M. Woods *et al.*, Astrophys. J. Lett. **519**, L139 (1999); C. Kouveliotou *et al.*, Nature **393**, 235 (1999).

[3] K. Hurley *et al.*, Astrophys. J. Lett. **510**, L111 (1999).

[4] S. Mereghetti and L. Stella, Astrophys. J. Lett. **442**, L17 (1999); J. van Paradihs, R. E. Taam, and E. P. J. van den Heuvel, Astron. Astrophys. **299**, L41 (1995); S. Mereghetti, astro-ph/9911252; see also A. Reisenegger, astro-ph/0103010.

[5] D. Bandyopadhyay, S. Chakrabarty, P. Dey, and S. Pal, Phys. Rev. D **58**, 121301 (1998).

[6] S. Chakrabarty, D. Bandyopadhyay, and S. Pal, Phys. Rev. Lett. **78**, 2898 (1997); D. Bandyopadhyay, S. Chakrabarty, and S. Pal, *ibid.* **79**, 2176 (1997).

[7] C. Y. Cardall, M. Prakash, and J. M. Lattimer, Astrophys. J. **554**, 322 (2001), and references therein; E. Roulet, astro-ph/9711206; L. B. Leinson and A. Pérez, astro-ph/9711216; D. G. Yakovlev and A. D. Kaminkar, in *The Equation of States in Astrophysics*, edited by G. Chabrier and E. Schatzman (Cambridge University, Cambridge, 1994), p. 214.

[8] S. Chakrabarty and P. K. Sahu, Phys. Rev. D **53**, 4687 (1996).

[9] S. Chakrabarty, Phys. Rev. D **51**, 4591 (1995); **54**, 1306 (1996).

[10] T. Ghosh and S. Chakrabarty, Phys. Rev. D **63**, 043006 (2001); Int. J. Mod. Phys. D **10**, 89 (2001).

[11] V. G. Bezchastrov and P. Haensel, Phys. Rev. D **54**, 3706 (1996).

[12] S. Mandal and S. Chakrabarty (two papers on weak and electromagnetic processes to be published).
[13] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D **52**, 4747 (1995).
[14] D. M. Gitman, S. D. Odintsov, and Yu. I. Shil'nov, Phys. Rev. D **54**, 2968 (1996).
[15] D. S. Lee, C. N. Leung, and Y. J. Ng, Phys. Rev. D **55**, 6504 (1997).
[16] V. P. Gusynin and I. A. Shovkovy, Phys. Rev. D **56**, 5251 (1995).
[17] D. S. Lee, C. N. Leung, and Y. J. Ng, Phys. Rev. D **57**, 5224 (1998).
[18] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. **83**, 1291 (1999).
[19] E. V. Gorbar, Phys. Lett. **B491**, 305 (2000).
[20] T. Inagaki, T. Muta, and S. D. Odintsov, Prog. Theor. Phys. **127**, 93 (1997); T. Inagaki, S. D. Odintsov, and Yu.I. Shil'nov, Int. J. Mod. Phys. A **14**, 481 (1999); T. Inagaki, D. Kimura, and T. Murata, Prog. Theor. Phys. **111**, 371 (2004).
[21] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Lett. **B349**, 477 (1995).
[22] S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992).
[23] C. N. Leung and S.-Y. Wang, hep-ph/0503298, Nucl. Phys. **B747**, 266 (2006).
[24] Shuzi Li, R. S. Bhalerao, and R. K. Bhaduri, Int. J. Mod. Phys. A **6**, 501 (1991).
[25] J. da Providência, M. C. Ruivo, and C. A. de Sousa, Phys. Rev. D **36**, 1882 (1987).
[26] S. Ghosh, S. Mandal, and S. Chakrabarty, Ann. Phys. (NY) **312**, 398 (2004).
[27] S. Mandal, R. Saha, S. Ghosh, and S. Chakrabarty, Phys. Rev. C **74**, 015801 (2006).