Impact parameter dependence of heavy ion e^+e^- pair production to all orders in $Z\alpha$

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The heavy ion probability for continuum e^+e^- pair production has been calculated to all orders in $Z\alpha$ as a function of impact parameter. The formula resulting from an exact solution of the semiclassical Dirac equation in the ultrarelativistic limit is evaluated numerically. In a calculation of $\gamma = 100$ colliding Au ions, the probability of e^+e^- pair production is reduced from the perturbation theory result throughout the impact parameter range.

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I. INTRODUCTION

The impact parameter dependence of heavy ion e^+e^- pair production to all orders in $Z\alpha$ is of direct relevance to experiments carried out at the BNL Relativistic Heavy Ion Collider (RHIC) and possible at the CERN Large Hadron Collider (LHC). The usual method of scattering theory formally integrates over the impact parameter to turn the momentum transfer of the amplitude into that of the cross section via a δ function [1]. However, a standard method of triggering measured events at RHIC and LHC uses zero degree calorimeters, devices that respond to neutrons from Coulomb dissociation of the beam ions [2]. The physical process of a measured pair production event then also includes simultaneously the Coulomb dissociation of the ion nucleus. Assuming independence of the two processes (pair production and Coulomb dissociation), the probability of an event is then the product of the probability of pair production times the probability of Coulomb dissociation. Predicted cross sections are then appropriately constructed from the integral of the product of the pair production probability times the dissociation probability. I will confine myself in this paper to evaluating the impact parameter dependent probability for heavy ion production of e^+e^- pairs. Combining the probability of the particular e^+e^- phase space acceptance of physical detectors with the probability of Coulomb dissociation will be left for future treatment.

The impact parameter dependent calculation of heavy ion e^+e^- pair production to all orders in $Z\alpha$ described in this paper is based on techniques previously developed. That development began with the realization that in an appropriate gauge [3], the electromagnetic field of a relativistic heavy ion is to a very good approximation a δ function in the direction of motion of the heavy ion times the two-dimensional solution of Maxwell's equations in the transverse direction [4], and that an exact solution of the appropriate Dirac equation could be obtained for excitation of bound-electron positron pairs exhibiting a reduction from perturbation theory of a little less than 10% for Au+Au at RHIC [5].

An analytical solution of the Dirac equation was subsequently obtained [6–8] for the analogous case of continuum e^+e^- pair production induced by the corresponding countermoving δ-function potentials produced by heavy ions in an ultrarelativistic collider such as RHIC. It was noted at the time [7,8] that the exact solution appeared to agree with the perturbation theory result.

Several authors then argued [9–11] that a correct regularization of the exact Dirac equation amplitude should lead to a reduction of the total cross section for pair production from perturbation theory, the so-called Coulomb corrections. The first analysis was done in a Weizsacker-Williams approximation [9]. Subsequently, Lee and Milstein computed [10,11] the total cross section for e^+e^- pairs using approximations to the exact amplitude that led to a higher order correction to the well-known Landau-Lifshitz expression [12]. One might also have noted that it is well known that photoproduction of e^+e^- pairs on a heavy target shows a negative (Coulomb) correction proportional to Z^2 that is well described by the Bethe-Maximon theory [13]. In fact some years ago Bertulanni and Baur [14] applied the structure of the Bethe-Maximon theory to obtain an impact dependent probability for heavy ion reactions with a negative Coulomb correction. I will comment on their results and compare them with the results of the present work in Sec. V.

In a previous paper, I tried to explicate the Lee and Milstein approximate results and argued their qualitative correctness [15]. Subsequently, I undertook the full numerical calculation of electromagnetically induced ultrarelativistic heavy ion electron-positron pair production [16]. That evaluation of the "exact" semiclassical total cross section for e^+e^- production with gold or lead ions showed reductions from perturbation theory of 28% for the CERN Super Proton Synchrotron (SPS) case, 17% for RHIC, and 11% for LHC. For large Z, no final momentum region was found in which there was no reduction or an insignificant reduction of the exact cross section from the perturbative cross section.

In the present paper, I reformulate the approach used for my previous cross section calculations in order to calculate the corresponding impact parameter dependent probabilities.

II. IMPACT PARAMETER DEPENDENT PROBABILITIES

For production of continuum pairs in an ultrarelativistic heavy ion reaction, one may work in a frame of two countermoving heavy ions with the same relativistic γ, and the electromagnetic interaction arising from them goes to the limit of two δ-function potentials

$$V(\boldsymbol{\rho}, z, t) = \delta(z - t)(1 - \alpha_z)\Lambda^-(\boldsymbol{\rho}) + \delta(z + t)(1 + \alpha_z)\Lambda^+(\boldsymbol{\rho}), \quad (1)$$

where

$$\Lambda^{\pm}(\boldsymbol{\rho}) = -Z\alpha \ln \frac{(\boldsymbol{\rho} \pm \mathbf{b}/2)^2}{(b/2)^2}. \tag{2}$$

I calculate the number weighted probability (or number operator) $\langle N(b)\rangle$ for producing e^+e^- pairs at some impact parameter b

$$\langle N(b)\rangle = \sum_{n=1}^{\infty} n P_n(b) = \int \frac{m^2 d^3p d^3q}{(2\pi)^6 \epsilon_p \epsilon_q} |M(p,q)|^2, \tag{3}$$

with the previously derived exact semiclassical amplitude for electron-positron pair production [6–8] written in the notation of Lee and Milstein [10]

$$M(p,q) = \int \frac{d^2k}{(2\pi)^2} \exp[i\mathbf{k}\cdot\mathbf{b}] \mathcal{M}(\mathbf{k}) F_B(\mathbf{k}) \times F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}), \tag{4}$$

where p and q are the four-momenta of the produced electron and positron, respectively, $p_\pm = p_0 \pm p_z$, $q_\pm = q_0 \pm p_z$, $\gamma_\pm = \gamma_0 \pm \gamma_z$, $\boldsymbol{\alpha} = \gamma_0 \boldsymbol{\gamma}$, $\mathbf{k}$ is an intermediate transverse momentum transfer from the ion to be integrated over,

$$\mathcal{M}(\mathbf{k}) = \bar{u}(p) \frac{\alpha\cdot(\mathbf{k}-\mathbf{p}_\perp) + \gamma_0 m}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2 + i\epsilon} \gamma_- u(-q) + \bar{u}(p) \frac{-\alpha\cdot(\mathbf{k}-\mathbf{q}_\perp) + \gamma_0 m}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2 + i\epsilon} \gamma_+ u(-q), \tag{5}$$

and the effect of the potential [Eqs. (1) and (2)] is contained in integrals F_B and F_A over the transverse spatial coordinates [6–8],

$$F(\mathbf{k}) = \int d^2\rho \exp[-i\mathbf{k}\cdot\boldsymbol{\rho}] \{\exp[-iZ\alpha \ln \rho] - 1\}. \tag{6}$$

$F(\mathbf{k})$ has to be regularized or cut off at large ρ. How it is regularized is the key to understanding Coulomb corrections, as I will review in Sec. III.

In Sec. III, I will show how a properly regularized transverse potential in Eq. (6) necessitates a numerical integration over ρ to obtain $F(\mathbf{k})$. With the resultant $F(\mathbf{k})$, the integration over $\mathbf{k}$ in Eq. (4) is then not amenable to the usual Feynman integral techniques and must be carried out numerically. For numerical convenience, I have chosen to express $\mathbf{k}$ in Cartesian coordinates and rewrite Eq. (5) as

$$\mathcal{M}(\mathbf{k}) = \bar{u}(p) \frac{\alpha_x k_x + \alpha_y k_y - \boldsymbol{\alpha}\cdot\mathbf{p}_\perp + \gamma_0 m}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2 + i\epsilon} \gamma_- u(-q) + \bar{u}(p) \frac{-\alpha_x k_x - \alpha_y k_y + \boldsymbol{\alpha}\cdot\mathbf{q}_\perp + \gamma_0 m}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2 + i\epsilon} \gamma_+ u(-q). \tag{7}$$

The expression for the amplitude $M(p,q)$ in Eq.(4) then becomes

$$M(p,q) = \bar{u}(p)[I_{px}\alpha_x + I_{py}\alpha_y + (-\boldsymbol{\alpha}\cdot\mathbf{p}_\perp + \gamma_0 m) J_p] \times \gamma_- u(-q) + \bar{u}(p)[-I_{qx}\alpha_x - I_{qy}\alpha_y + (\boldsymbol{\alpha}\cdot\mathbf{q}_\perp + \gamma_0 m) J_q] \gamma_+ u(-q), \tag{8}$$

where, letting $\mathbf{b}$ define the x axis,

$$I_{px} = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) k_x dk_y}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2}, \tag{9}$$

$$I_{py} = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) k_y dk_y}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2}, \tag{10}$$

$$I_{qx} = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) k_x dk_y}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2}, \tag{11}$$

$$I_{qy} = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) k_y dk_y}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2}, \tag{12}$$

$$J_p = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) dk_y}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2}, \tag{13}$$

$$J_q = \frac{1}{(2\pi)^2} \int \exp[ik_x b]\, dk_x \times \int \frac{F_B(\mathbf{k}) F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}) dk_y}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2}. \tag{14}$$

After squaring, summing over spin states, and taking traces with the aid of the computer program FORM [17], I obtain the expression for the amplitude squared

$$\begin{aligned} |M(p,q)|^2 = {} & p_+q_-[(m^2 + p_\perp^2)|J_p|^2 + |I_{px}|^2 + |I_{py}|^2 \\ & - 2p_x \mathrm{Re}(J_p I_{px}^*) - 2p_y \mathrm{Re}(J_p I_{py}^*)] \\ & + p_-q_+[(m^2 + q_\perp^2)|J_q|^2 + |I_{qx}|^2 + |I_{qy}|^2 \\ & - 2q_x \mathrm{Re}(J_q I_{qx}^*) - 2q_y \mathrm{Re}(J_q I_{qy}^*)] \\ & + 2[(m^2 + p_\perp^2)(q_x \mathrm{Re}(J_p I_{qx}^*) \\ & + q_y \mathrm{Re}(J_p I_q y^*)) + (m^2 + q_\perp^2)(p_x \mathrm{Re}(J_q I_{px}^*) \\ & + p_y \mathrm{Re}(J_q I_{py}^*)) + (\mathbf{p}_\perp\cdot\mathbf{q}_\perp - m^2)(\mathrm{Re}(I_{px} I_{qx}^*) \\ & + \mathrm{Re}(I_{py} I_{qy}^*)) - (m^2 + p_\perp^2)(m^2 + q_\perp^2) \\ & \times \mathrm{Re}(J_p J_q^*) - (p_x q_y + p_y q_x)(\mathrm{Re}(I_{px} I_{qy}^*) \\ & + \mathrm{Re}(I_{py} I_{qx}^*)) - 2p_x q_x \mathrm{Re}(I_{px} I_{qx}^*) \\ & - 2p_y q_y \mathrm{Re}(I_{py} I_{qy}^*)]. \end{aligned} \tag{15}$$

There is an apparent numerical difficulty in evaluating Eqs. (9)–(14) due to the oscillating factor $\exp[ik_x b]$. Of course, in the $\mathbf{b} = 0$ limit, this factor is absent, and so we will first investigate this numerically more tractable case. The general case of nonzero $\mathbf{b}$ will then be addressed by a technique involving piecewise analytical integration.

The derived impact parameter dependent amplitude squared, $|M(p,q)|^2$, is not simply the square of the amplitude

for the excitation of a particular (correlated) electron-positron pair [18]. However, as has been discussed in the literature [19–26], the number operator for a particular (uncorrelated) electron or positron can be constructed by integrating over the positron or electron momenta, respectively, and likewise the number operator for total pair production. In previous articles [15,16], I have also discussed these matters in more detail.

For the present, I will limit myself to calculating the number operator $\langle N(b)\rangle$ for total pair production, and from it the total pair production cross section σ_T,

$$\langle N(b)\rangle = \int \frac{m^2 d^3p d^3q}{(2\pi)^6 \epsilon_p \epsilon_q} |M(p,q)|^2. \tag{16}$$

Note that σ_T corresponds to a peculiar type of inclusive cross section which we should call the "number weighted total cross section,"

$$\sigma_T = \int d^2b \langle N(b)\rangle = \int d^2b \sum_{n=1}^{\infty} n P_n(b), \tag{17}$$

in contrast to the usual definition of an inclusive total cross section σ_I for pair production,

$$\sigma_I = \int d^2b \sum_{n=1}^{\infty} P_n(b). \tag{18}$$

III. EIKONAL, EXACT, AND PERTURBATIVE CASES

If one merely regularizes the transverse integral Eq. (6) at large ρ, one obtains [7,8] apart from a trivial phase

$$F(\mathbf{k}) = \frac{4\pi\alpha Z}{k^{2-2i\alpha Z}}. \tag{19}$$

Then all the higher order $Z\alpha$ effects in $M(p,q)$ are contained only in the phase of the denominator of Eq. (19). It was noted that as $Z \to 0$ this photon propagator leads to the known perturbative result for $M(p,q)$ if $F(\mathbf{k})$ is modified to

$$F(\mathbf{k}) = \frac{4\pi\alpha Z}{(k^2+\omega^2/\gamma^2)^{1-i\alpha Z}}. \tag{20}$$

In this approach, a lower k cutoff at some ω/γ has to be put in by hand to obtain dependence on the beam energy and to agree with the known perturbative result in that limit.

Rather than put in the ω^2/γ^2 cutoff by hand, my physically motivated ansatz is to apply a spatial cutoff to the transverse potential $\chi(\boldsymbol{\rho})$ (which has been up to now set to $2Z\alpha \ln\rho$) in order to obtain an expression consistent with the perturbation theory formula [27,28] in the ultrarelativistic limit. In the Weizsacker-Williams or equivalent photon treatment of electromagnetic interactions, the effect of the potential is cut off at impact parameter $b \simeq \gamma/\omega$, where γ is the relativistic boost of the ion producing the photon and ω is the energy of the photon. If

$$\chi(\boldsymbol{\rho}) = \int_{-\infty}^{\infty} dz V\left(\sqrt{z^2+\rho^2}\right) \tag{21}$$

and $V(r)$ is cut off in such a physically motivated way, then [11]

$$V(r) = \frac{-Z\alpha \exp[-r\omega_{A,B}/\gamma]}{r}, \tag{22}$$

where

$$\omega_A = \frac{p_+ + q_+}{2}, \qquad \omega_B = \frac{p_- + q_-}{2}, \tag{23}$$

with ω_A the energy of the photon from ion A moving in the positive z direction and ω_B the energy of the photon from ion B moving in the negative z direction. Note that we work in a different gauge than that used to obtain the original perturbation theory formula, and thus our potential picture is somewhat different. The transverse potential will be smoothly cut off at a distance where the the longitudinal potential δ-function approximation is no longer valid.

The integral Eq. (21) can be carried out to obtain

$$\chi(\rho) = -2Z\alpha K_0(\rho\omega_{A,B}/\gamma), \tag{24}$$

and Eq. (6) is modified to

$$F_{A,B}(\mathbf{k}) = 2\pi \int d\rho\rho J_0(k\rho) \times \{\exp[2iZ_{A,B}\alpha K_0(\rho\omega_{A,B}/\gamma)] - 1\}. \tag{25}$$

$F_A(\mathbf{k})$ and $F_B(\mathbf{k})$ are functions of virtual photon ω_A and ω_B, respectively. The modified Bessel function

$$K_0(\rho\omega/\gamma) = -\ln(\rho) - \ln(\omega/2\gamma) - \gamma_e \tag{26}$$

for $\rho \ll \gamma/\omega$ (γ_e is the Euler constant), and cuts off exponentially at $\rho \approx \gamma/\omega$. This is the physical cutoff to the transverse potential. One may define $\xi = k\rho$ and rewrite Eq. (25) in terms of a normalized integral $I_{A,B}(\gamma k/\omega)$

$$F_{A,B}(\mathbf{k}) = \frac{4\pi i Z_{A,B}\alpha}{k^2} I_{A,B}(\gamma k/\omega), \tag{27}$$

where

$$I_{A,B}(\gamma k/\omega) = \frac{1}{2iZ_{A,B}\alpha} \int d\xi\xi J_0(\xi) \times \{\exp[2iZ_{A,B}\alpha K_0(\xi\omega/\gamma k)] - 1\}. \tag{28}$$

Note that $F_{A,B}$ is equal to $4\pi Z_{A,B}/k^2$ times a function of $(\gamma k/\omega)$. The form of the integral, Eq. (6) without the cutoff [here with added arbitrary phase constants consistent with $K_0(\xi\omega/\gamma k)$ for small ξ, Eq. (26)], was solved in closed form [7,8]

$$I^E_{A,B}(\gamma k/\omega) = -i\left(\frac{\exp[\gamma_e]\omega}{\gamma k}\right)^{-2i\alpha Z} \frac{\Gamma(-i\alpha Z)}{\Gamma(i\alpha Z)} \times \frac{1}{(1+\omega^2/k^2\gamma^2)^{1-i\alpha Z}}. \tag{29}$$

I will refer to this form as the eikonal. In the limit as $Z \to 0$, $I^E_{A,B}(\gamma k/\omega)$ goes to

$$I^0_{A,B}(\gamma k/\omega) = \frac{-i}{1+\omega^2/k^2\gamma^2}, \tag{30}$$

leading to the familiar perturbation theory form

$$F^0_{A,B}(\mathbf{k}) = \frac{4\pi i Z_{A,B}\alpha}{k^2+\omega^2/\gamma^2}. \tag{31}$$

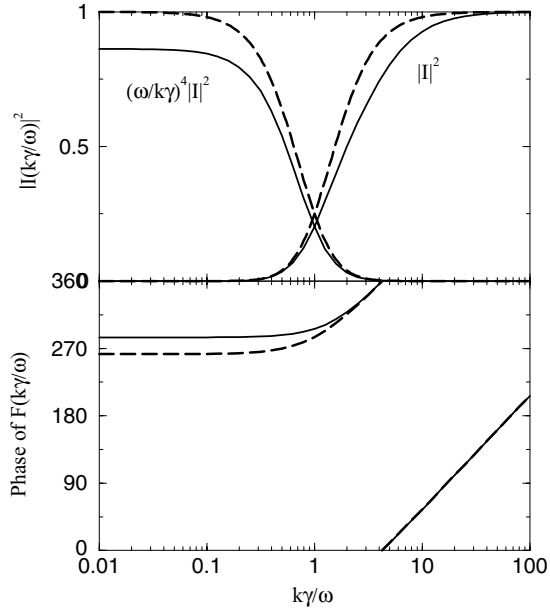

FIG. 1. Top: Relative magnitude squared of virtual photon source; solid line, exact; long dashed line, eikonal or perturbative (see text). Bottom: Corresponding phase of virtual photon source; the perturbative phase (not shown) does not vary with $k\gamma/\omega$.

The top panel of Fig. 1 displays the results of the numerical calculation of $|I(k\gamma/\omega)|^2$ for $Z = 79$ and in the perturbative limit. The curves that go to zero on the left are $|I(k\gamma/\omega)|^2$. The curves that go to zero on the right are $|I(k\gamma/\omega)|^2$ multiplied by $(\omega/k\gamma)^4$ to exhibit the reduction at high Z on the left of the curve. $(\omega/k\gamma)^4|I(k\gamma/\omega)|^2$ goes to unity as $k\gamma/\omega$ goes to zero in the perturbative case; it goes to a reduced constant value as $k\gamma/\omega$ goes to zero for $Z = 79$. The bottom panel exhibits the phase of the full expression $F(\mathbf{k})$. The perturbative phase (not shown) is constant. Both exact and eikonal phases are identical at high $k\gamma/\omega$ and diverge by 22° for this $Z = 79$ case as they approach constants at low $k\gamma/\omega$.

IV. THE b = 0 LIMIT

Since the present calculations, based on the ultrarelativistic limit, employ a somewhat different approach than that taken in previous perturbative calculations but should agree with them in the perturbative limit, I have deemed it prudent to do some comparison of numerical results. As a first step, I computed the $\mathbf{b} = 0$ limit, both as a test of the method and general structure of the computer code as well as for physical interest. Figure 2 shows a pertubation theory calculation of the energy spectrum of produced positrons at $\mathbf{b} = 0$ compared with a previous perturbation theory calculation of Hencken, Trautmann, and Baur [29]. The same dipole form factor has been utilized for the ions with $\Lambda = 83$ MeV, and the curves have been divided by $(Z\alpha)^4$. The heavy ions are from Au+Au at RHIC, and the energy of the electrons has been integrated over. Except at the lowest energies, the agreement is good, and the integrated probability of the present calculation $P^0(0) = 1.64$ is in good agreement with the previous calculation $P^0(0) = 1.6$. The integrated numbers have not been divided by $(Z\alpha)^4$ as the displayed curves were.

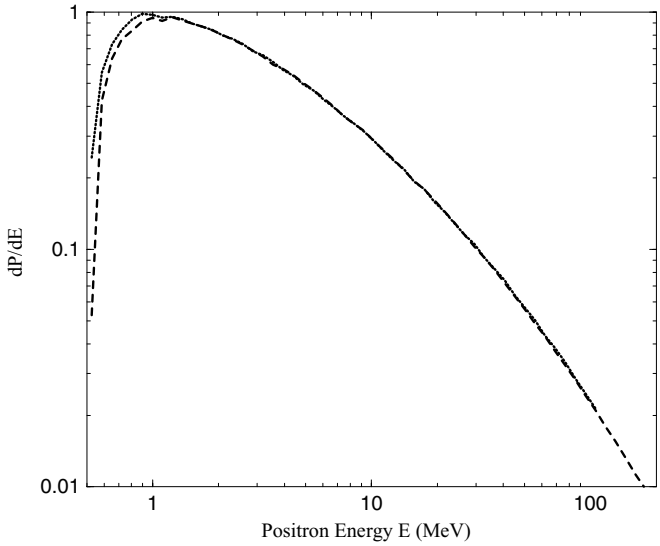

FIG. 2. Total probability as a function of energy of the positron. The curves have been divided by $(Z\alpha)^4$. Short dashes, present perturbation theory; dotted line, previous perturbation theory calculation [29].

Figure 3 shows the effect of higher order contributions. My perturbation result of Fig. 2 is repeated for reference. The solid line is the result of the exact calculation. It is clearly reduced from perturbation theory for the lowest to highest values of the energy range; the exact energy integrated $P(0) = .94 = .57P^0(0)$. Only slightly larger in magnitude is the result of an eikonal calculation where the energy integrated $P^E(0) = 1.03 = .63P^0(0)$.

The overall reduction from perturbation theory is almost as large in the eikonal case as it is in the exact calculation, and the two curves in Fig. 3 do not diverge much at any energy value.

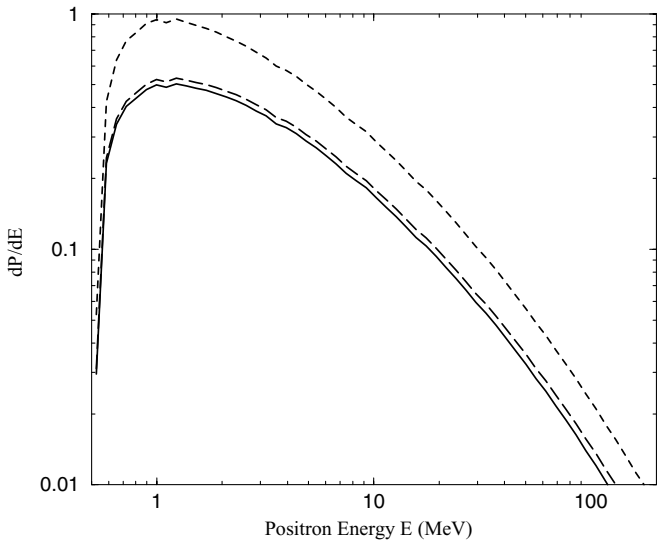

FIG. 3. Probabilities as in Fig. 2. Short dashes, present perturbation theory; solid line, exact; long dashes, eikonal.

This would suggest that similar effective physics is involved in the two calculations. Recall from Fig. 1 the reduction in magnitude of the virtual photon source for the exact calculation as compared to the eikonal (which is identical in magnitude to perturbation theory) for $k\gamma/\omega < 50$. For $k > 100\omega/\gamma$, the magnitude of $F(\mathbf{k})$ goes over into the eikonal or perturbative result, while the phase of $F(\mathbf{k})$ is identical to the eikonal. It is in this large k region, where $F(\mathbf{k})$ becomes identical the eikonal in both magnitude and phase, that most of the contribution for $\mathbf{b} = 0$ comes. Test calculations show that 80% of the contribution to the total probability at $\mathbf{b} = 0$ comes from $k > 100\,\omega/\gamma$, and that less than a tenth of a percent comes from $k < 10\,\omega/\gamma$. It is not the region of reduced magnitude that dominates for the exact case $\mathbf{b} = 0$, but the region of rotating phase, and that rotating phase is what reduces the cross section from perturbation theory. The eikonal $F(\mathbf{k})$ is a fair approximation to the exact at $\mathbf{b} = 0$, but as we will see in Sec. V this is not true for larger $\mathbf{b}$.

It is worth emphasizing that calculations labeled perturbative, exact, and eikonal differ only in the expressions used for $F_{A,B}(\mathbf{k})$. The analytical expression Eq. (31) is used for perturbative calculations, and Eqs. (27) and (29) for the eikonal calculations. The exact calculations use the expressions in Eqs. (27) and (28) , which must be evaluated numerically, but only once for each $Z_{A,B}$ of interest.

V. b DEPENDENT PROBABILITIES AND THE TOTAL e^+e^- CROSS SECTION: Au + Au AT RHIC

With the addition of impact parameter dependence, we are faced with a nine-dimensional integral for the total cross section as compared with the seven-dimensional integral in the previous section, or the seven-dimensional integral in the representation of cross sections without impact parameter [16]. Although the usual method of evaluation, e.g., in perturbation theory, is via Monte Carlo, I have chosen to do the multidimensional integral directly on meshes uniform on a logarithmic scale in each radial momentum dimension and on a logarithmic scale in impact parameter. It was computationally tedious but possible to carry out the calculations to some rough but significant accuracy without using Monte Carlo because the integrands are very smooth and they smoothly go to zero at both the high and low ends of the momentum ranges and the impact parameter range.

Apart from the fact that impact parameter dependence adds two more dimensions to calculations (the magnitude of $\mathbf{b}$ and the angle of the outgoing pair relative to $\mathbf{b}$), evaluation of the square of the pair production amplitude, Eq. (15), is considerably more difficult for nonzero impact parameter b due to the factor $\exp[ik_x b]$ in Eqs. (9)–(14). For increasing values of $\mathbf{b}$, the integral over k_x oscillates rapidly and presents a challenge to numerical integration. Furthermore, while the oscillation is on a linear scale in k_x, experience has shown that the natural scale for integration over momenta in this electromagnetic problem is logarithmic; we want to choose numerical mesh points on a logarithmic scale. I have chosen to work in Cartesian coordinates because after a numerical integration over the comparative smooth variation in k_y I am able to make use of a relatively simple piecewise analytical method to integrate over k_x. I do the integration over k_x in the manner described in the Appendix.

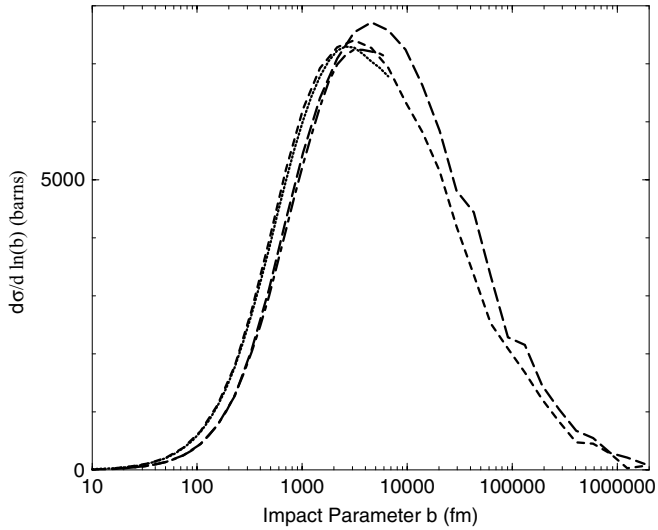

FIG. 4. Impact parameter dependence of contribution to total cross section. Dashed line, perturbation theory; long dashed line, eikonal. Comparable results are shown derived from the calculations of Hencken, Trautmann, and Baur [30]: Dotted line, perturbation theory; dot-dashed line, eikonal.

Calculations were again carried out for the RHIC case of Au + Au at $\gamma = 100$. For the perturbation theory and eikonal probabilities, I was able to do a rough check of my computer code by comparing my results with another set of published calculations [30]. Figure 4 shows the comparison of cross section contributions derived from the present calculations with those derived from the $\mathbf{b}$ dependent probabilities of Hencken, Trautmann, and Baur. The logarithmic derivative was chosen so that contributions to the integrated cross section go as the area under the curves on this natural log scale plot. No form factor was utilized in either calculation as its effect should be small for total cross section probabilities at these impact parameters. Agreement is reasonably good, considering the large grids used in the present calculations. Note that for both sets of calculations, the eikonal solution is less than perturbation theory for low impact parameters, and then it crosses over to be greater than perturbation theory at about $b = 3000$ fm and larger. For small impact parameters, there is fair agreement, but there is an incipient divergence between the curves for the present calculations and those of Hencken, Trautmann, and Baur where the latter cut off. However, one of the authors had warned me not to trust their results far out in b too much because the Fourier transform used was not as good there [31].

In Fig. 5 one can observe the comparison of the exact calculation with perturbation theory and the eikonal. For all impact parameters, the probability of pair production is smaller in the exact calculation than it is in perturbation theory. The small impact parameter agreement between the exact calculation and the eikonal extends the mechanism discussed in the $\mathbf{b} = 0$ case, the reduction from pertubation theory due to the similar rotating phase of $F(k)$ for dominant

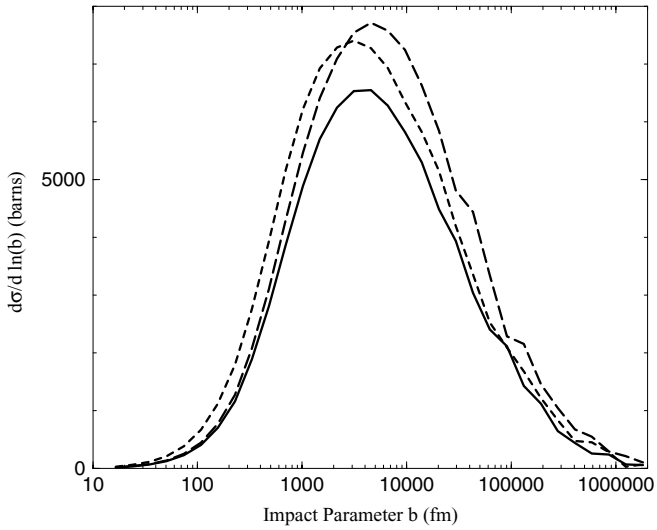

FIG. 5. Impact parameter dependence of contribution to total cross section. Dashed line, perturbation theory; long dashed line, eikonal; solid line, exact.

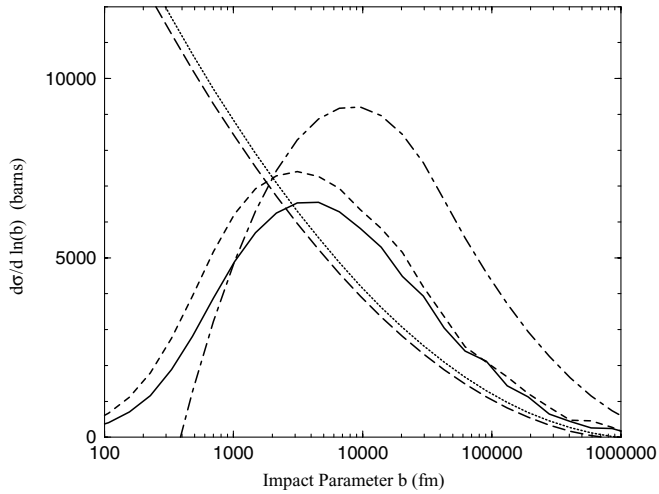

FIG. 6. Impact parameter dependence of contribution to total cross section. Present calculation: Dashed line, perturbation theory; solid line, exact. Comparable results are shown derived from the calculations of Bertulani and Baur [14]: Dotted line, perturbation theory; long dashed line, Coulomb corrected. Comparable perturbative results derived from the formula of Lee, Milstein, and Serbo [32] are represented by the dot-dashed line.

large values of **k** seen in Fig. 1. For larger impact parameters providing the dominant contribution to the cross section, a significant contribution comes in the region $k < 10\,\omega/\gamma$, where the magnitude of $F(\mathbf{k})$ is reduced from perturbation theory and the eikonal. This demonstrates how a correct physical cutoff process preserves the Coulomb corrections at high **b** and thus leads to Coulomb corrections in the total cross section.

The calculations can be integrated over impact parameter to compare with my previous calculations [15] that were not done in the impact parameter representation. That the present integrated perturbation theory calculation of 34.6 kb (kilobarns) is in such good agreement with my previous calculation of 34.6 kb must be considered fortuitous, given the relatively crude nature of the impact parameter calculation. The present integrated exact calculation of 29.4 kb is in fair agreement with my previous 28.6 kb. The integrated eikonal calculation of 35.5 kb should in principle be identical to the 34.6 kb perturbation theory calculation, but like the present exact calculation, it is slightly larger than my previously published non-impact-parameter calculation.

At this point, it is interesting to also compare the present results with two previous approximate analytical expressions for impact parameter dependence. In their classic article, Bertulani and Baur [14] presented an impact parameter formula that included a Coulomb correction. Figure 6 shows a comparison of the cross section contribution derived from their probability formula with the present results. Bertulani and Baur integrated over their stated range of validity, from $1/m_e$ to $\sim\gamma^2/m_e$, to obtain a Landau-Lifshitz perturbative result with a Coulomb correction. For symmetric heavy ions, the Coulomb correction that they obtained was of the same form as the result later obtained by Ivanov *et al.* [9] and by Lee and Milstein [10,11]. However, since Bertulani and Baur had postulated averaging the result of a Coulomb correction for the target with one for the projectile, they obtained half of the more correct later result based on adding the lowest order Coulomb correction for both target and projectile. The overall shapes of the Bertulani and Baur results differ from the present results, but their Coulomb correction reduces the probability throughout the impact parameter range.

Shortly after obtaining their above-mentioned Coulomb correction results for the total cross section [9–11], Lee, Milstein, and Serbo presented an analytical form for the the impact parameter dependence [32] that differed substantially from that of Bertulani and Baur, especially in the intermediate range from $1/m_e$ to γ/m_e. Figure 6 also displays the impact parameter contribution to the cross section obtained from their pertubative expression for the probability. The shape of the curve seems qualitatively more in agreement with present results. As in the case of Bertulani and Baur, the treatment is considered valid for values of impact parameter larger than $1/m_e$ (386 fm), and in both cases integration of the contribution from their expression is in agreement with the leading $\ln^3(\gamma^2)$ term of Landau and Lifshitz [12]. Lee, Milstein, and Serbo did not present a specific form for the impact parameter dependence of the Coulomb correction, but as noted above, their previous work indicates a negative Coulomb correction on average twice that of Bertulani and Baur over the indicated parameter range.

VI. SUMMARY AND DISCUSSION

Calculated exact total probabilities for heavy ion e^+e^- pair production exhibit a reduction from the probabilities calculated in perturbation theory throughout the full range of impact parameters.

In principle, the impact parameter approach to calculating exact e^+e^- pair production probabilities is suited for combining with an impact parameter dependent mutual Coulomb dissociation calculation [2] to be able to compare with, e.g., data such as those obtained with the solenoidal tracker at RHIC (STAR) setup [33]. In this paper, I have presented calculations only for the total e^+e^- pair production probabilities. Still to be done is a sufficiently accurate calculation of the high transverse momentum slice of data seen by STAR to be combined with a Coulomb dissociation calculation for the zero degree calorimeter acceptance. Also, since the present approach is strictly speaking valid only when either the positrons or electrons have been integrated over, and in the STAR case both electron and positron are constrained to be in the high momentum slice, the present approach to e^+e^- production is not exactly valid for the STAR case. At present, the best one can do is observe that the present method is valid for both uncorrelated positrons and electrons of all momenta, and ignore the correlations. The effect of correlations averages to zero, but some estimate of individual magnitudes would be useful.

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APPENDIX: PIECEWISE ANALYTICAL FOURIER INTEGRATION

Consider the integral over k_x (I will drop the subscript x for now). It contains a smoothly varying part which I will call $f(k)$ times the rapidly varying coefficient $\exp[ikb]$, that is,

$$I = \int_{-\infty}^{\infty} f(k) \exp[ikb] dk. \tag{A1}$$

One can use integration by parts to transform I to

$$I = \int_{-\infty}^{\infty} \frac{(1 - \exp[ikb])}{ib} \frac{df(k)}{dk} dk, \tag{A2}$$

where the term 1 is inserted for numerical convenience. One now rewrites I in a form suggesting piecewise analytical integration from mesh point to mesh point

$$I = \sum_{k_{i\min}}^{k_{i\max}} \int_{k_i}^{k_{i+1}} \frac{(1 - \exp[ikb])}{ib} \frac{df(k)}{dk} dk. \tag{A3}$$

Taking the lowest order (constant) approximation to the derivative over the interval, one has

$$\begin{aligned} I &= \sum_{k_{i\min}}^{k_{i\max}} \left(\frac{f(k_{i+1}) - f(k_i)}{k_{i+1} - k_i} \right) \int_{k_i}^{k_{i+1}} \frac{(1 - \exp[ikb])}{ib} dk \\ &= \sum_{k_{i\min}}^{k_{i\max}} \left(\frac{f(k_{i+1}) - f(k_i)}{k_{i+1} - k_i} \right) \\ &\quad \times \left(\frac{k_{i+1} - k_i}{ib} + \frac{\exp[ik_{i+1}b] - \exp[ik_i b]}{b^2} \right). \end{aligned} \tag{A4}$$

This is the lowest order expression for piecewise analytical integration.

One can obtain the piecewise analytical expression good to the next leading order by taking three point Lagrange interpolation for $f(k)$ between k_i and k_{i+1} and then taking the derivative. One obtains

$$\begin{aligned} I = \sum_{k_{i\min}}^{k_{i\max}} \frac{1}{ib} \Bigg(& A(k_{i+1} - k_i) + \frac{B}{2}\left(k_{i+1}^2 - k_i^2\right) \\ & - \exp[ik_{i+1}b] \left(\frac{A + B\,k_{i+1}}{ib} + \frac{B}{b^2} \right) \\ & + \exp[ik_i b] \left(\frac{A + B\,k_i}{ib} + \frac{B}{b^2} \right) \Bigg), \end{aligned} \tag{A5}$$

where

$$\begin{aligned} A = -\Bigg(& \frac{f(k_i)\,(k_{i+1} + k_{i+2})}{(k_i - k_{i+1})(k_i - k_{i+2})} + \frac{f(k_{i+1})\,(k_i + k_{i+2})}{(k_{i+1} - k_i)\,(k_{i+1} - k_{i+2})} \\ & + \frac{f(k_{i+2})\,(k_i + k_{i+1})}{(k_{i+2} - k_i)\,(k_{i+2} - k_{i+1})} \Bigg), \end{aligned} \tag{A6}$$

and

$$\begin{aligned} B = 2\Bigg(& \frac{f(k_i)}{(k_i - k_{i+1})\,(k_i - k_{i+2})} + \frac{f(k_{i+1})}{(k_{i+1} - k_i)\,(k_{i+1} - k_{i+2})} \\ & + \frac{f(k_{i+2})}{(k_{i+2} - k_i)\,(k_{i+2} - k_{i+1})} \Bigg). \end{aligned} \tag{A7}$$

The expression Eqs. (A5)–(A7) exhibits numerical difficulties for small impact parameters. Therefore, for impact parameters in the range below the electron Compton wavelength (about 386 fm), I have utilized Eq. (A4); and I have utilized Eqs. (A5)–(A7) for higher impact parameter values.

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