The α particle as a canonically quantized multiskyrmion

A. Acus* and E. Norvaišas
VU Institute of Theoretical Physics and Astronomy, Goštauto 12, Vilnius 01108 and Department of Physics and Technology, Vilnius Pedagogical University, Lithuania

D. O. Riska†
Helsinki Institute of Physics, POB 64, 00014 University of Helsinki, Finland

(Received 6 January 2006; published 14 August 2006)

The rational map approximation to the solution to the SU(2) Skyrme model with baryon number $B = 4$ is canonically quantized. The quantization procedure leads to anomalous breaking of the chiral symmetry, and exponential fall-off of the energy density of the soliton at large distances. The model is extended to SU(2) representations of arbitrary dimension. These soliton solutions capture the double node feature of the empirical α particle charge form factor, but as expected lead to a too compact matter distribution. Comparison to phenomenology indicates a preference for the fundamental representation.

DOI: 10.1103/PhysRevC.74.025203 PACS number(s): 12.39.Dc, 21.10.Dr, 24.85.+p, 27.10.+h

I. INTRODUCTION

The chiral topological soliton model developed by Skyrme [1], which represents a dynamical realization of the large N limit of QCD, describes many of the key qualitative features of of baryons and nuclei [2–4]. The model describes baryons and nuclei as spatially extended topologically stable solitons of the chiral meson field. The soliton solutions of the equation of motion are characterized by the winding number or topological charge of the mapping $S^3 \to S^3$, which is interpreted as the baryon number B. Numerical study has shown that the shape of the ground state field configuration for nuclei with $B > 1$ has an intriguing geometrical structure [5]. For $B = 2$ the ground state solution is toroidal and for $B = 4$ the structure it is octahedral. Higher baryon number solutions are associated with more complicated symmetric polyhedral shapes. Such shapes also appear as variational solutions to interaction part of the nuclear Hamiltonian [6].

The rational map (RM) ansatz proposed for the SU(2) Skyrme model in Ref. [7] provides a remarkably accurate analytic approximation to the ground state solution of the model. This ansatz preserves the essential symmetries of the numerical solutions of the exact Skyrme model equations. The identification of the topological number with baryon number also leads to solitonic fullerene structures in light atomic nuclei [8]. The RM ansatz has been generalized to the SU(3) Skyrme model as well [9].

The rational map ansatz for the SU(2) skyrmion for $B = 2$, which represents the deuteron, has been canonically quantized in Ref. [10] for representations of arbitrary dimension of the Skyrme model Lagrangian. The canonically quantized deuteron solutions and their physical characteristics depend on the dimension of the representation in contrast to the semiclassically quantized solution.

The matter density of the canonically quantized skyrmion soliton falls off exponentially at long range in contrast to the power law fall-off of the classical soliton without a pion mass term [10,11,18,20]. In the case of the $B = 1$ skyrmion the inverse of the length scale of this exponential fall-off for corresponds to the pion mass, which arises because of the anomalous breaking of chiral symmetry by the canonical quantization procedure [11]. In the case of the α-particle it should correspond to $2\sqrt{mE_0}$, where M is the nucleon mass and E_0 is the binding energy [12]. Numerical calculation shows that the RM approximation leads to exponential falloff at a somewhat smaller rate than this. This feature may be traced to the fact that the Skyrme model represents a large N approximation to QCD, in which the kinetic energy term for the nucleons vanish. The ground state solution to the Skyrme model therefore corresponds to the variational solution to the interaction part of the nuclear Hamiltonian, as the kinetic energy terms vanish in the large N limit.

Below the static observables and the charge form factor of ^{4}He are calculated from the the quantum solution of the $B = 4$ skyrmion obtained with the rational map in SU(2) representations of arbitrary dimension. The calculated charge form factor has the same two-node structure as the experimental form factor, but the two zeros appear at smaller values of momentum transfer than in the empirical form factor. This shows that the ground state solution of the Skyrme model has an unrealistically compact structure, as expected. It is instructive to compare the results to those previously obtained with the product ansatz for the soliton field [13]. The product ansatz describes the asymptotic long range four-skyrmion structure of the solution, and leads to a charge form factor for ^{4}He, where nodes of the calculated form factor in contrast occur at too large values of momentum transfer. It is then natural to conjecture, that as the empirical form factor is bracketed by the form factor calculated with the too compact rational map approximation and with the too extended asymptotic product ansatz, a more realistic solution of the Skyrme model might provide an adequate description of the observed form factor.

*Email address: acus@itpa.lt, norvaisas@itpa.lt
†Email address: riska@pcu.helsinki.fi

The organization of this paper is the following. In Sec. II the RM ansatz for the classical soliton of octaedral symmetry is generalized to representations of arbitrary dimension. In Sec. III canonical quantization of the soliton is developed in the collective coordinate approach. The numerical results for the properties of the quantized solution are compared to the observables of ^{4}He in Sec. IV. Finally a summarizing discussion is given in Sec. V.

II. THE CLASSICAL SOLITON OF OCTAEDRAL SYMMETRY

The Skyrme model is a Lagrangian density for a unitary field $U(\mathbf{x}, \mathbf{t})$ that belongs to the representation of SU(2) group. In a general reducible representation $U(\mathbf{x}, t)$ may be expressed as a direct sum of Wigner's D matrices for irreducible representations as

$$U(\mathbf{x}, t) = \sum_j \oplus D^j(\boldsymbol{\alpha}(\mathbf{x}, t)). \tag{1}$$

The D^j matrices are functions of three unconstrained Euler angles $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$.

The chirally symmetric Lagrangian density of the Skyrme model has the form

$$\mathcal{L}(U(\mathbf{x}, t)) = -\frac{f_\pi^2}{4}\mathrm{Tr}\{R_\mu R^\mu\} + \frac{1}{32e^2}\mathrm{Tr}\{[R_\mu, R_\nu]^2\}. \tag{2}$$

Here the "right" current is defined as

$$R_\mu = (\partial_\mu U)U^\dagger, \tag{3}$$

and f_π (the pion decay constant) and e are parameters.

The rational map ansatz [7] is an approximation to the ground state solution of the Skyrme model with baryon number $B > 1$ takes the following form in a representation of arbitrary dimension:

$$U_R(\mathbf{r}) = \exp(2\mathrm{i}\,\hat{n}^a \hat{J}_{(a)} F(r)). \tag{4}$$

Here $\hat{J}_{(a)}$ are SU(2) generators in a given representation. The unit vector $\hat{\mathbf{n}}$ may be defined in terms of a rational complex function $R(z)$ as

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}\{2\Re(R), 2\Im(R), 1-|R|^2\}. \tag{5}$$

For baryon number $B = 4$ the function,

$$R(z) = \frac{z^4 + 2\sqrt{3}\mathrm{i}z^2 + 1}{z^4 - 2\sqrt{3}\mathrm{i}z^2 + 1}, \tag{6}$$

has been found to be a suitable choice [7]. Here $z = \tan(\theta/2)\mathrm{e}^{\mathrm{i}\varphi}$ is a complex coordinate that is parametrized by azimuthal and polar angles θ and φ. The circular components of the unit vector $\hat{\mathbf{n}}_R$ are

$$\begin{aligned}
\hat{n}_{+1} &= -\frac{1}{\sqrt{2}} + \frac{\sqrt{3}\sin^2\theta(\sqrt{3}\sin^2\theta - \mathrm{i}(1+\cos^2\theta)\cos 2\varphi)}{2\sqrt{2}(1-\sin^2\theta + \sin^4\theta(1-\sin^2\varphi+\sin^4\varphi))},\\
\hat{n}_0 &= \frac{\sqrt{3}\sin^2\theta\cos\theta\sin 2\varphi}{1-\sin^2\theta+\sin^4\theta(1-\sin^2\varphi+\sin^4\varphi)},\\
\hat{n}_{-1} &= \frac{1}{\sqrt{2}} + \frac{\sqrt{3}\sin^2\theta(-\sqrt{3}\sin^2\theta - \mathrm{i}(1+\cos^2\theta)\cos 2\varphi)}{2\sqrt{2}(1-\sin^2\theta+\sin^4\theta(1-\sin^2\varphi+\sin^4\varphi))}.
\end{aligned} \tag{7}$$

The rational map (6) has cubic symmetry. The orientation is fixed below so that the z-direction is that of the third component of the angular momentum.

Differentiation of $\hat{\mathbf{n}}$ yields the relation

$$\begin{aligned}
(-1)^s(\nabla_{-s} r\hat{n}_m)(\nabla_s r\hat{n}_{m'}) = \hat{n}_m\hat{n}_{m'} + \mathcal{I}((-1)^m\delta_{-m,m'}\\
- \hat{n}_m\hat{n}_{m'}),
\end{aligned} \tag{8}$$

which proves to be useful in the explicit calculation of Lagrangian density (2). Here ∇_s are the circular components of the nabla operator. The symbol $\mathcal{I}$ here denotes the function:

$$\mathcal{I} = \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2, \tag{9}$$

the explicit form of which is

$$\mathcal{I} = \frac{12\sin^2\theta(1-\sin^2\theta+\sin^4\theta\sin^2\varphi\cos^2\varphi)}{(1-\sin^2\theta+\sin^4\theta(1-\sin^2\varphi+\sin^4\varphi))^2}. \tag{10}$$

Integrals of powers of $\mathcal{I}$ over θ and ϕ can be regarded as Morse functions [7].

The baryonic charge density takes the following form in the irrep j:

$$\begin{aligned}
\mathcal{B}(r,\theta,\varphi) &= \varepsilon^{0k\ell m}\mathrm{Tr}R_k R_\ell R_m\\
&= -8j(j+1)(2j+1)\mathcal{I}\frac{F'(r)\sin^2 F}{r^2}.
\end{aligned} \tag{11}$$

Because of the presence of the $\mathcal{I}$ function in this expression, there is no need to modify usual boundary conditions $F(0) = \pi$; $F(\infty) = 0$ for the chiral angle. The baryon number therefore takes the standard expression:

$$B = \frac{1}{24N\pi^2}\int_0^\infty \mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \mathrm{d}\theta\,\mathcal{B}\,r^2\sin\theta, \tag{12}$$

with the normalization factor $N = \frac{2}{3}j(j+1)(2j+1)$, as expected [11]. The normalization factor is chosen to be unity in the fundamental representation of SU(2). The present choice of boundary conditions ensures that the integral of the $\mathcal{I}$ function is proportional to the baryon number:

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \mathrm{d}\theta\,\mathcal{I}\sin\theta = 4\pi B. \tag{13}$$

Substitution of the rational map ansatz (4) into the Lagrangian density (2) leads to the classical Skyrme model density:

$$\mathcal{L}_{\rm cl}(r,\theta,\varphi) = -N\left(f_\pi^2\left(\frac{F'^2(r)}{2}+\frac{\mathcal{I}\sin^2 F}{r^2}\right)+\frac{1}{e^2}\frac{\mathcal{I}\sin^2 F}{r^2}\left(F'^2(r)+\frac{\mathcal{I}\sin^2 F}{2r^2}\right)\right). \quad (14)$$

Note, that the symmetry of the Lagrangian density (14) in the θ,φ space is completely determined by the function $\mathcal{I}$ and its (more symmetric) powers.

It is useful to introduce dimensionless coordinates $\tilde{r} = ef_\pi r$. Variation of Lagrangian then yields the following differential equation for chiral angle:

$$F''(\tilde r)\Big(1+\frac{2B\sin^2 F(\tilde r)}{\tilde r^2}\Big)+\frac{2F'(\tilde r)}{\tilde r}+\frac{F'^2(\tilde r)B\sin 2F(\tilde r)}{\tilde r^2} - \frac{B\sin 2F(\tilde r)}{\tilde r^2}-\frac{I_2\sin^2 F(\tilde r)\sin 2F(\tilde r)}{\tilde r^4}=0. \quad (15)$$

Here we have used the abbreviation

$$I_2 = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\mathcal{I}^2\sin\theta. \quad (16)$$

In the limit $\tilde r\to\infty$, the Eq. (15) reduces to simple asymptotic form

$$F''(\tilde r)+\frac{2F'(\tilde r)}{\tilde r}-\frac{2BF(\tilde r)}{\tilde r^2}=0. \quad (17)$$

From this the asymptotic large distance solution, which satisfies physical boundary conditions, can easily be obtained as

$$F(\tilde r) = C_1\tilde r^{-\frac{1+\sqrt{1+8B}}{2}}. \quad (18)$$

Here C_1 a constant to be determined later by continuous joining of the numerical small distance solution onto the analytic asymptotic solution. For $B=4$, the power of $\tilde r$ in Eq. (18) is ≈ -3.37. Note that Eqs. (14)–(18) are valid for all B, provided that the corresponding function $\mathcal{I}$ is used.

III. CANONICAL QUANTIZATION IN THE COLLECTIVE COORDINATE APPROACH

The quantization of the Skyrme model in a general representation [11] can be carried out by means of collective rotational coordinates that separate the variables, which depend on the time and spatial coordinates [14]:

$$U(\mathbf{r},\mathbf{q}(t)) = A\left(\mathbf{q}(t)\right)U_R(\mathbf{r})A^\dagger\left(\mathbf{q}(t)\right). \quad (19)$$

Here the three real Euler angles $\mathbf{q}(t)=(q^1(t),q^2(t),q^3(t))$ are quantum variables. These are sufficient for the α particle ground state, for which $S=T=0$.

The canonical quantization with constraints procedure employed here was originally suggested by Dirac [15], and further developed in Refs. [16,17]. In this formalism the Skyrme Lagrangian (2) is considered quantum mechanically *ab initio* in contrast to the conventional semiclassical quantization of the Skyrmion as a rigid body. In the SU(2) case canonical quantization implies that the three independent generalized coordinates $\mathbf{q}(t)$ and the corresponding velocities $\dot{\mathbf{q}}(t)$ satisfy the following commutation relations [19]:

$$[\dot q^a, q^b] = -\mathrm{i}f^{ab}(\mathbf{q}). \quad (20)$$

Here $f^{ab}(\mathbf{q})$ are functions of generalized coordinates $\mathbf{q}$ only, the explicit forms of which are determined self-consistently upon imposition of the quantization condition. The tensor f^{ab} is symmetric with respect to interchange of the indices a and b by the relation $[q^a,q^b]=0$.

The commutation relation between a generalized velocity component $\dot q^a$ and an arbitrary function $G(\mathbf{q})$ is given by

$$[\dot q_a, G(\mathbf{q})] = -i\sum_r f^{ar}(\mathbf{q})\frac{\partial}{\partial q^r}G(\mathbf{q}). \quad (21)$$

Here Weyl ordering of the operators has been employed:

$$\partial_0 G(\mathbf{q}) = \frac{1}{2}\left\{\dot q^\alpha, \frac{\partial}{\partial q^\alpha}G(\mathbf{q})\right\}. \quad (22)$$

The curly brackets denote an anticommutator. With this choice of operator ordering no further ordering ambiguity appears.

To derive the Lagrangian the expression (19) is substituted into the Lagrangian density (2). Consider first the term that is quadratic in the generalized velocities. After integration over the spatial coordinates the Lagrangian takes the form

$$L(\mathbf{q},\dot{\mathbf{q}},F) = \frac{1}{N}\int \mathrm{d}^3\mathbf{r}\mathcal{L}(\mathbf{r},\mathbf{q}(t),F(r)) = \frac{1}{2}\dot q^\alpha g_{\alpha\alpha'}\dot q^{\alpha'}+\mathcal{O}(\dot q^0). \quad (23)$$

Here the momentum of inertia tensor is

$$g_{\alpha\alpha'} = C_\alpha^{\prime(b)}(\mathbf{q})E_{(b)(b')}C_{\alpha'}^{\prime(b')}(\mathbf{q}). \quad (24)$$

Here $E_{(b)(b')}$ is defined as

$$E_{(b)(b')} = -\tfrac{1}{2}(-1)^b a_b(F)\delta_{b,-b'} \quad \text{(no summation over } b). \quad (25)$$

Here $a_1 = a_{-1}$. The soliton momenta of inertia are given as

$$a_0(F) = \frac{\tilde a_0}{e^3 f_\pi} = 4\pi\int_0^\infty r^2\sin^2 F \times\left((1-N_2)\left(f_\pi^2+\frac{1}{e^2}F'^2\right)+\frac{2}{3}\frac{B}{e^2}\frac{\sin^2 F}{r^2}\right)\mathrm{d}r,$$
$$a_1(F) = \frac{\tilde a_1}{e^3 f_\pi} = 2\pi\int_0^\infty r^2\sin^2 F \times\left((1+N_2)\left(f_\pi^2+\frac{1}{e^2}F'^2\right)+\frac{4}{3}\frac{B}{e^2}\frac{\sin^2 F}{r^2}\right)\mathrm{d}r. \quad (26)$$

The symbol N_k in this expression denotes the angular integrals:

$$N_k = \frac{1}{4\pi}\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\sin\theta\hat n_0^k. \quad (27)$$

For baryon number $B=1$ and $B=2$ the integrals may be evaluated in closed form to yield $N_2(\text{nucleon})=\frac{1}{3}$; $N_4(\text{nucleon})=\frac{1}{5}$ and $N_2(\text{deuteron})=-1+\frac{\pi}{2}$, $N_4(\text{deuteron})=-1/3+\frac{\pi}{4}$. For $B=4$ the numerical values of the corresponding integrals are $N_2\approx 0.218897$ and $N_4\approx 0.118382$. The other integrals, which explicitly enters calculation of the inertia tensor (25), may be evaluated analytically by the following expression:

$$\int\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2\left(\frac{1-|R|^2}{1+|R|^2}\right)^m\frac{2\mathrm{i}\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}$$
$$=\int_0^\pi\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\sin\theta\mathcal{I}\hat{n}_0^m$$
$$=2\pi B\frac{(-1)^m+1}{m+1},\quad m\in\text{Rationals};\quad m\geqslant 0. \tag{28}$$

The validity of expression has been verified numerically for a number of randomly chosen rational maps with different baryon numbers B to a very high degree of precision. There is good reason to conjecture that the integrals are topologically conserved quantities valid for all rational maps. Note that the relation (13) is a particular case ($m=0$) of Eq. (28). Here the function $\mathcal{I}$ plays an intriguing role as an "integrating" factor.

The coefficients $C_\alpha^{\prime(b)}$ and their inverses $C_{(b)}^{\prime\alpha}$ are functions of the dynamical variables, which appear in the differentiation of the Wigner D matrices:

$$\frac{\partial}{\partial\alpha^k}D_{mn}^j(\boldsymbol{\alpha})=-\frac{1}{\sqrt{2}}C_k^{\prime(a)}(\boldsymbol{\alpha})D_{mm'}^j(\boldsymbol{\alpha})\langle jm'|J_{(a)}|jn\rangle. \tag{29}$$

The conventional quantum mechanical commutation relations $[p_\alpha,q^\beta]=-\mathrm{i}\delta_{\alpha\beta}$ for the momenta $p_\alpha=\frac{\partial L}{\partial\dot{q}^\alpha}=\frac{1}{2}\{\dot{q}^\beta,g_{\alpha\beta}\}$ then leads to the following expression for the tensor $f^{\alpha\beta}$ [Eq. (20)]:

$$f^{\alpha\beta}(\mathbf{q})=g_{\alpha\beta}^{-1}(\mathbf{q}). \tag{30}$$

It is convenient to introduce the following angular momentum operators on the hypersphere S^3 (the manifold of the SU(2) group):

$$\hat{J}'_{(a)}=-\frac{\mathrm{i}}{\sqrt{2}}\{p_\alpha,C_{(a)}^{\prime\alpha}\}. \tag{31}$$

It is readily verified that the operator $\hat{J}'_a$ is a $D^j(\mathbf{q})$ "right rotation" generator that has the well defined actions

$$\hat{J}'^2\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle=\ell(\ell+1)\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle;$$
$$\hat{J}_0'^2\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle=m_t^2\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle; \tag{32}$$

on the normalized state vectors with fixed spin and isospin ℓ:

$$\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle=\frac{\sqrt{2\ell+1}}{4\pi}D_{m_s,m_t}^\ell(\mathbf{q})\,|0\rangle\,. \tag{33}$$

The explicit form of the function $f^{ab}(\mathbf{q})$, in turn, leads to to an explicit expression of the Skyrme model Lagrangian density (2) in the collective coordinate approach. Lengthy manipulation and use of computer algebra [21] yields the result

$$\mathcal{L}_{\mathrm{qt}}(r)=-N\left(f_\pi^2\left\{\frac{F'^2}{2}+\frac{\mathcal{I}\sin^2F}{r^2}-\frac{\sin^2F}{8}\left[\left(\frac{1}{a_0}+\frac{3}{a_1}\right)\mathbb{C}\right.\right.\right.$$
$$-\frac{2}{a_1}\left(\frac{1}{a_0}+\frac{1}{a_1}\right)+\frac{(2j-1)(2j+3)\sin^2F}{5}$$
$$\left.\left.\times\left(3\,\mathbb{C}^2-\frac{4}{a_1}\mathbb{C}+\frac{4}{a_1^2}\right)\right]\right\}$$
$$+\frac{1}{e^2}\left[\frac{\mathcal{I}\sin^2F}{r^2}\left(F'^2+\frac{\mathcal{I}\sin^2F}{2r^2}\right)\right.$$
$$-\frac{\sin^2F}{8}\left(\frac{\mathcal{I}\sin^2F}{r^2}\left[\left(\frac{1}{a_0}+\frac{1}{a_1}\right)\mathbb{C}-\frac{2}{a_0a_1}\right]\right.$$
$$+F'^2\left[\left(\frac{1}{a_0}+\frac{3}{a_1}\right)\mathbb{C}-\frac{2}{a_1}\left(\frac{1}{a_0}+\frac{1}{a_1}\right)\right]$$
$$+\frac{(2j-1)(2j+3)}{5}\left\{-\frac{\mathcal{I}\sin^2F}{r^2}\left[3\,\mathbb{C}^2\right.\right.$$
$$\left.-2\left(\frac{2}{a_0}+\frac{5}{a_1}\right)\mathbb{C}+\frac{2}{a_1}\left(\frac{4}{a_0}+\frac{3}{a_1}\right)\right]$$
$$\left.\left.\left.\left.+F'^2\left(3\mathbb{C}^2-\frac{4}{a_1}\mathbb{C}+\frac{4}{a_1^2}-2\mathbb{C}^2\sin^2F\right)\right\}\right)\right]\right). \tag{34}$$

Here the following notation has been introduced:

$$\mathbb{C}=\frac{1}{a_0}+\frac{1}{a_1}-\left(\frac{1}{a_0}-\frac{1}{a_1}\right)\hat{n}_0^2. \tag{35}$$

The expression (34) does not contain the operator component. Integration of the latter (operator component) yields matrix elements, which depend on spin and isospin ℓ:

$$\left\langle\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right|\int_0^\infty\mathrm{d}r\int_0^\pi\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\left(f_\pi^2+\frac{1}{e^2}\left[\frac{\mathcal{I}}{r^2}\sin^2F\right.\right.$$
$$\left.\left.+F'^2(r)\right]\right)\left(\frac{1}{a_1^2}\hat{\mathbf{J}}'^2(q)+\left(\frac{1}{a_0^2}-\frac{1}{a_1^2}\right)\hat{J}_0'^2(q)\right.$$
$$\left.-\left[\frac{1}{a_1}(\hat{\mathbf{J}}'(q)\cdot\hat{\mathbf{n}})+\left(\frac{1}{a_0}-\frac{1}{a_1}\right)\hat{J}_0'(q)\hat{n}_0\right]^2\right)\left|\begin{matrix}\ell\\ m_s,m_t\end{matrix}\right\rangle$$
$$=m_t^2\left(\frac{1}{a_0}-\frac{1}{a_1}\right)+\frac{\ell(\ell+1)}{a_1}. \tag{36}$$

This expression vanishes in the case of ^{4}He for which $m_t=\ell=0$.

Integration and subsequent variation of Lagrangian density (34) then leads to the following integrodifferential equation for the quantum chiral angle in the dimensionless coordinate $\tilde{r}=ef_\pi r$:

$$F''(\tilde{r})\left(4\tilde{r}^2+8B\sin^2F(\tilde{r})+e^4\tilde{r}^2\sin^2F(\tilde{r})\left[4\tilde{\mu}^2\right.\right.$$
$$\left.\left.+\frac{(2j-1)(2j+3)}{5}(\mathbb{A}+2\mathbb{B}+(\mathbb{A}+\mathbb{B})\cos2F(\tilde{r}))\right]\right)$$
$$+F'^2(\tilde{r})\left(4B\sin2F(\tilde{r})+e^4\tilde{r}^2\sin2F(\tilde{r})\left[2\tilde{\mu}^2\right.\right.$$
$$\left.\left.+\frac{(2j-1)(2j+3)}{10}(\mathbb{B}+2(\mathbb{A}+\mathbb{B})\cos2F(\tilde{r}))\right]\right)$$
$$+\tilde{r}F'(\tilde{r})\left(8+e^4\sin^2F(\tilde{r})\left[8\tilde{\mu}^2+\frac{2(2j-1)(2j+3)}{5}\right.\right.$$

$$\times(\mathbb{A}+2\mathbb{B}+(\mathbb{A}+\mathbb{B})\cos 2F(\tilde r))\Big]\Bigg)$$
$$-\sin 2F(\tilde r)\Bigg(4B+\frac{4I_2\sin^2F(\tilde r)}{\tilde r^2}$$
$$+e^4\tilde r^2\left(2\tilde\mu^2+\frac{(2j-1)(2j+3)}{5}(2\mathbb{A}+3\mathbb{B})\sin^2F(\tilde r)\right)$$
$$+2e^4B\sin^2F(\tilde r)\Bigg\{2\tilde\mu_0^2+\frac{1}{3\tilde a_1^2}+\frac{2}{3\tilde a_0\tilde a_1}$$
$$+\frac{(2j-1)(2j+3)}{5}\Bigg(-\frac{8}{15\tilde a_0^2}+\frac{6}{15\tilde a_0\tilde a_1}-\frac{13}{15\tilde a_1^2}$$
$$+\frac{4\pi(-1+3N_2)}{9}\left(\frac{1}{\tilde a_0}-\frac{1}{\tilde a_1}\right)\Bigg[\left(\frac{3}{\tilde a_0^2}+\frac{2}{\tilde a_0\tilde a_1}+\frac{1}{\tilde a_1^2}\right)$$
$$\times\int\tilde r^2\sin^2 2F(\tilde r)\mathrm{d}\tilde r$$
$$+8\left(\frac{1}{\tilde a_0^2}+\frac{1}{\tilde a_0\tilde a_1}+\frac{1}{\tilde a_1^2}\right)\int\tilde r^2\sin^4F(\tilde r)F'^2(\tilde r)\mathrm{d}\tilde r\Bigg]\Bigg)\Bigg\}\Bigg). \tag{37}$$

Here

$$\mathbb{A}=-\frac{4}{\tilde a_1^2}+\frac{4}{\tilde a_1}(-1+N_2)\left(\frac{1}{\tilde a_0}-\frac{1}{\tilde a_1}\right), \tag{38}$$

$$\mathbb{B}=(-1+2N_2-N_4)\left(\frac{1}{\tilde a_0}-\frac{1}{\tilde a_1}\right)^2. \tag{39}$$

Above $\tilde\mu^2$ denotes the following integral:

$$\frac{4\tilde\mu^2}{e^4}=\frac{(-1+4m_t)(-1+N_2)}{\tilde a_0^2}-\frac{\tilde a_0(1+N_2)}{\tilde a_1^3}$$
$$+\frac{2(1+(1+N_2)(1+m_t-\ell(\ell+1)))}{\tilde a_1^2}$$
$$+\frac{8\pi B}{3\tilde a_1}\left(\frac{2(-1+N_2)}{\tilde a_0^2}-\frac{1+N_2}{\tilde a_0\tilde a_1}-\frac{1+N_2}{\tilde a_1^2}\right)$$
$$\times\int\sin^4F(\tilde r)\mathrm{d}\tilde r+\frac{(2j-1)(2j+3)}{5}$$
$$\times\Bigg(\frac{3(-1+N_2)}{\tilde a_0^2}\left(N_4-5-\frac{2\tilde a_1(-1+N_4)}{\tilde a_0}\right)$$
$$+\frac{2(1+2N_2-3N_4)}{\tilde a_0\tilde a_1}$$
$$+\frac{1+N_2}{2\tilde a_1^2}\left(3N_4+9+\frac{\tilde a_0(1+3N_4)}{\tilde a_1}\right)$$
$$+16\pi\Bigg(\frac{-1+N_2}{\tilde a_0^2}\left(\frac{1-2N_2+N_4}{\tilde a_0}-\frac{-1+N_4}{\tilde a_1}\right)$$
$$+\frac{1+N_2}{2\tilde a_1^2}\left(\frac{-1+N_4}{\tilde a_0}-\frac{1+2N_2+N_4}{\tilde a_1}\right)\Bigg)$$
$$\times\int\tilde r^2\sin^4F(\tilde r)F'^2(\tilde r)\mathrm{d}\tilde r$$
$$+\frac{8\pi B}{15}\Bigg(\frac{-1+N_2}{\tilde a_0^2}\left(\frac{-1+45N_4}{\tilde a_0}-\frac{-31+45N_4}{\tilde a_1}\right)$$
$$+\frac{1+N_2}{2\tilde a_1^2}\left(\frac{-31+45N_4}{\tilde a_0}-\frac{29+45N_4}{\tilde a_1}\right)\Bigg)$$
$$\times\int\sin^4F(\tilde r)\mathrm{d}\tilde r+2\pi\Bigg(\frac{-1+N_2}{\tilde a_0^2}\Bigg(\frac{3(1-2N_2+N_4)}{\tilde a_0}$$
$$+\frac{1+2N_2-3N_4}{\tilde a_1}\Bigg)-\frac{1+N_2}{2\tilde a_1^2}\Bigg(\frac{1+2N_2-3N_4}{\tilde a_0}$$
$$+\frac{3+2N_2+3N_4}{\tilde a_1}\Bigg)\Bigg)\int\tilde r^2\sin^2 2F(\tilde r)\mathrm{d}\tilde r\Bigg). \tag{40}$$

The symbol $\tilde\mu_0^2$ represents the special case of $\tilde\mu^2$ integral, when $N_2=\frac{1}{3}$.

At large distances Eq. (37) reduces to the asymptotic form

$$\tilde r^2F''(\tilde r)+2\tilde rF'(\tilde r)-(2B+\tilde\mu^2\tilde r^2)F(\tilde r)=0. \tag{41}$$

From this asymptotic equation it follows that the quantity $\tilde\mu$ describes the fall-off rate of the chiral angle at large distances:

$$F(\tilde r)=C_1\mathrm{e}^{-\tilde\mu\tilde r}\left(\frac{\tilde\mu}{\tilde r}+\frac{B}{\tilde r^2}\right). \tag{42}$$

The related quantity $\mu=ef_\pi\tilde\mu$ describes the asymptotic fall-off $\exp(-2\mu r)$ of the soliton mass density for the dimensional coordinate r. Note the appearance of $\tilde\mu^2$ in all the higher derivative terms in Eq. (37), which is a nontrivial result. Eqs. (26), (28), (34), (37), and (40) are conjectured to be valid for all rational maps $R(z)$.

Because of the isospin of ^{4}He is zero, the charge distribution is proportional to the baryon density (11). The charge form factor then is the usual Fourier transform:

$$F_c(q)=\frac{1}{2}\int\mathrm{d}^3\,j_0(qr)\mathcal{B}(r,\theta,\varphi), \tag{43}$$

where j_0 denotes the spherical Bessel function of zero order.

IV. NUMERICAL RESULTS

The RM ansatz represents an approximation, which gives energies that fall above the numerically computed ground state energy by only a few percent [8]. Calculation of the static properties and the charge form factor of ^{4}He from the RM with $B=4$ should therefore be expected to give a good approximation to those for the exact ground state solution. In the present numerical calculation the parameters f_π end e in the Skyrme model Lagrangian were determined so as to reproduce the calculated static observables of the nucleons in the different representations j considered in Ref. [11].

The quantum integrodifferential equation (37) was solved numerically by the shooting method. In the initialization of the algorithm trial values for all the integrals $(a_0, a_1, \mu^2, \ldots)$ that appear in the equation had to be specified. For these estimates were obtained by employment of the semiclassical chiral angle of the $B=1$ skyrmion. Shooting from the point $\tilde r_{\max}$, [where $F(\tilde r)$ assumed to be of the form Eq. (42)] to

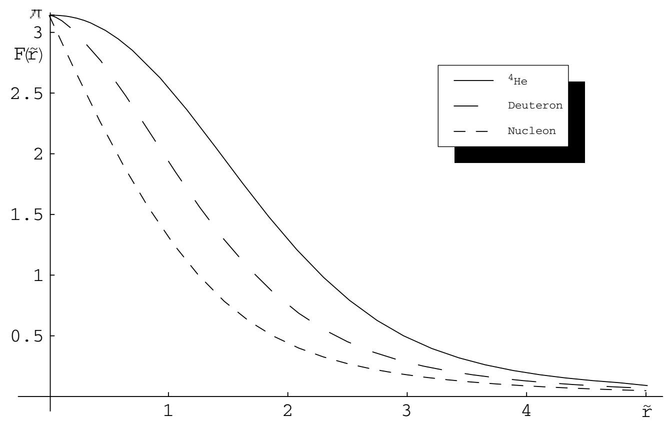

FIG. 1. ^{4}He (solid), deuteron (long-dashed) and nucleon (short-dashed) chiral angle profile functions in SU(2) representation $\frac{1}{2}$.

the point $\tilde{r}_{\rm min}$ (here $F(\tilde{r}) = F(\tilde{r}_{\rm min}) - (\tilde{r}_{\rm min} - \tilde{r})F'(\tilde{r}_{\rm min})$) and varying the only unknown constant C_1 in (42) leads to a continuous function C^2 that satisfies the required boundary conditions $F(0) = \pi$ and $F(\infty) = 0$.

Typically $\tilde{r}_{\rm max} \approx 6$ and $\tilde{r}_{\rm min} \approx 10^{-2}$ (the equation has a singularity at the origin). The chiral angle function found by this method is then used to recalculate all required integrals and procedure is iterated until the integrals converge to a stable value. The convergence proved to be rapid, and faster than in the case of the nucleon. Every iteration increases the absolute integral precision approximately by one decimal point. Thus typically 10–15 iterations are enough to achieve an accurate solution for the chiral angle.

The canonical quantization of the $B = 1$ skyrmion, which describes the nucleon, was presented in Ref. [11]. Variation of the quantized energy functional revealed the existence of stable solutions for the nucleon. In that work the parameters were determined by the isoscalar radius (0.72 fm) and mass (939 MeV) of the nucleon. The same parameter values for f_π and e in the Skyrme model Lagrangian were employed here for the solution of the $B = 4$ soliton, which describes the ^{4}He nucleus in different representations. Figure 1 shows the chiral angle profile functions for different baryon numbers 1, 2 and 4 B. Here the rational map ansätze were used in the case of $B = 2$ and $B = 4$. It is notable that the exponential falloff rate of the chiral angle becomes slower and smoother with increasing baryon number.

The calculated values of the static observables of ^{4}He are listed in Table I. The best agreement between the calculated and the empirical values for the charge radius $\langle r_E^2\rangle^{1/2}$ and the corresponding binding energy E_0 values is found for the reducible representation $1 \oplus \frac{1}{2} \oplus \frac{1}{2}$ as in the case of the nucleon [11]. For the higher irreps no binding is found at all with these parameter values.

While the finite pion mass is conventionally introduced by adding an explicitly chiral symmetry breaking pion mass term to the Lagrangian density of the model [14], the canonical quantization procedure by itself gives rise to a finite pion mass. This realizes Skyrme's original conjecture that "This (chiral) symmetry is, however, destroyed by the boundary condition ($U(\infty) = 1$), and we believe that the mass (of pion) may arise as a self consistent quantal effect" [25].

TABLE I. The predicted static ^{4}He nuclei observables in different representations with fixed empirical values for the nucleon isoscalar radius 0.72 fm and nucleon mass $m_N = 939$ MeV [11]. The momenta of inertia, $\tilde{a}_i$, are in units of $1/(e^3 f_\pi)$.

j	1/2	1	3/2	$1 \oplus \frac{1}{2} \oplus \frac{1}{2}$	Exp.
f_π	59.8	58.5	57.7	58.8	93 MeV
e	4.46	4.15	3.86	4.24	
m	3585	3759	3975	3701	3728.55 MeV
μ	33.1	45.2	50.4	41.8	229 MeV
$\langle r_E^2\rangle^{1/2}$	1.39	1.52	1.65	1.49	1.676 fm
E_0	−171	+3	+219	−55	−28.11 MeV
$\tilde{a}_0$	157.1	154.6	152.9	155.2	
$\tilde{a}_1$	130.1	128.1	126.8	128.6	

The "quantal effect" (the exponential falloff rate of the mass density of ^{4}He, $e^{-2\mu r}$) which we find in Eq. (42) is, however, much smaller than the value that is obtained for a four-nucleon system with the empirical binding energy: $\mu = \sqrt{mE_0}$, where m denotes nucleon mass. The reason for this is that the rational map ansatz gives an approximation to the ground state solution, which does not contain the vibrational modes. This conclusion is also supported by comparison to the semiclassical approximation to the $B = 4$ skyrmion given in Ref. [22], which did take into account the vibrational modes, and obtained both a smaller binding energy (79 MeV) and concomitantly a larger radius (1.50 fm). Alternatively it may be viewed as natural consequence of the implied large N limit of the model, in which there is no kinetic energy contribution from the constituent nucleons.

The nonrelativistic charge form factors (see Fig. 2) which are calculated from fixed empirical values of nucleon [11] have the same qualitative features as the empirical form factor values taken from Refs. [23,24], with two nodes. The best agreement with experimental data is found for the fundamental representation $j = \frac{1}{2}$.

V. DISCUSSION

The main result derived above is the demonstration of the utility of the rational map approximation for the $B = 4$ skyrmion, which allows the complete canonical quantization of the soliton to be carried out in closed form in a way similar to, even though calculationally more cumbersome than, that used for the hedgehog solution for the $B = 1$ skyrmion.

From the phenomenological perspective the main result is however the explicit demonstration that the empirical charge form factor of the α-particle is bracketed between the form factor derived here by the rational map ansatz, which approximates the ground state, and the form factor that is obtained with the product ansatz [13], and which asymptotically approaches the configuration of four separated $B = 1$ skyrmions. This then suggests that there exists a smooth path between these two limiting configurations, and that a physically more realistic solution may eventually be found on

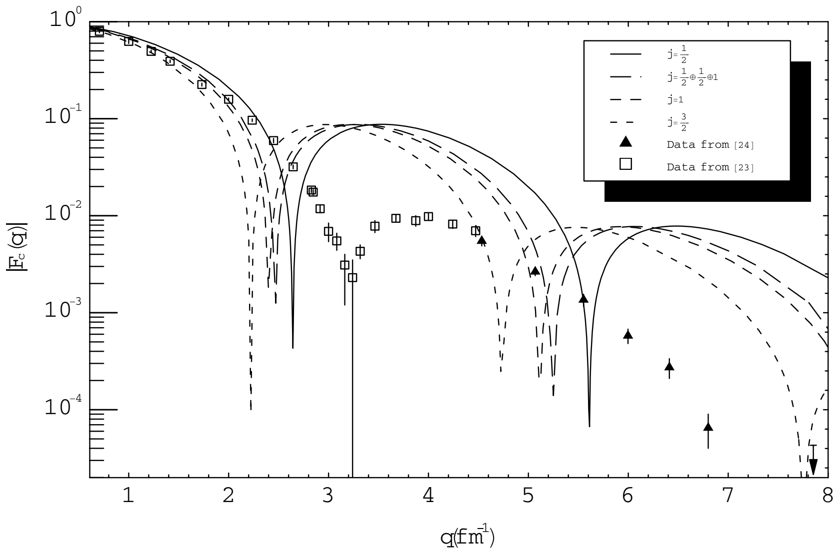

FIG. 2. Comparison of ^{4}He electric form factors in different representations of SU(2) with experimental data [23,24]. The form factors are calculated with parameters that yield the experimental nucleon mass $m_N = 939$ MeV and radius $r = 0.72$ fm [11].

this path. That is yet another example of the remarkably wide field of baryonic phenomenology, for which the Skyrme model provides a qualitative description.

In the case of the $B = 4$ skyrmion it was found that the calculated observables in the fundamental representation lead to a better qualitative agreement with the empirical values than those obtained in representations of larger dimension. In the case of the $B = 1$ and $B = 2$ skyrmions there is no such clear preference for the fundamental representation [10].

The quantization of the deuteron (the $B = 2$ skyrmion) is of particular interest due to the different values of spin $S = 1$ and isospin $T = 0$. This implies that quantization with three quantum variables as in Eq. (19) is not sufficient. In Ref. [10] six independent degrees of freedom—ie right and left chiral transformations were therefore employed. This allowed the construction of quantum states with different values of spin and isospin. Such a quantization of classical states with predefined symmetry, $\hat{\mathbf{n}}$ should be applicable to a wide class of nuclei.

As noted above the canonical quantization procedure generates a pion mass term as originally conjectured by Skyrme [25]. In work based on the conventional semiclassical quantization the pion mass term has in contrast to be introduced by way of an explicit chiral symmetry breaking term. In that method the requirement of rotational stability requires a value for the pion mass that is considerably larger than the empirical value [26]. With such large values for the pion mass the chiral symmetry breaking term leads to spatial configurations for the ground state solution of the Skyrme model with baryon number larger than 4, which differ significantly from those obtained in the chiral limit [27]. It should be worthwhile to explore how the features implied by the overly large pion mass term in the semiclassical approximation are be modified once the mass term, which arises dynamically in consistent canonical quantization procedure are taken into account. The canonical quantization procedure here applied cannot, however, be directly applied to this question as it keeps the angular dependence of the ansatz fixed.

ACKNOWLEDGMENTS

E.N. acknowledges the hospitality of the Helsinki Institute of Physics during course of this work. Research supported in part by the Academy of Finland grant number 54038 and Lithuanian State Science and Studies Foundation grant number T-06034.

[1] T. H. R. Skyrme, Proc. R. Soc. London A **260**, 127 (1961).
[2] E. M. Nyman and D. O. Riska, Rept. Prog. Theor. Phys. **53**, 1137 (1990).
[3] V. G. Makhankov, Y. P. Rubakov, and V. I. Sanyuk, *The Skyrme Model: Fundamentals, Methods and Applications* (Springer Verlag, Berlin 1993), p. 265.
[4] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, 2004), p. 493.
[5] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **79**, 363 (1997).
[6] J. L. Forest, V. R. Pandharipande, S. C. Pieper, R. B. Wiringa, R. Schiavilla, and A. Arriaga, Phys. Rev. C **54**, 646 (1996).
[7] C. J. Houghton, N. S. Manton, and P. M. Sutcliffe, Nucl. Phys. **B510**, 507 (1998).
[8] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **86**, 3989 (2001); Rev. Math. Phys. **14**, 29 (2002).
[9] V. B. Kopeliovich, B. E. Stern, and W. J. Zakrzevski, Phys. Lett. **B492**, 39 (2000).
[10] A. Acus, J. Matuzas, E. Norvaišas, and D. O. Riska, Phys. Scr. **69**, 260 (2004).
[11] A. Acus, E. Norvaišas, and D. O. Riska, Phys. Rev. C **57**, 2597 (1998).
[12] J. L. Friar *et al.*, Phys. Lett. **B161**, 241 (1985).
[13] S. Boffi, O. Nicrosini, and M. Radici, Nucl. Phys. **A490**, 585 (1988).
[14] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[15] P. Dirac, Lectures on Quantum Mechanics, Yeshiva University, New York, 1964.
[16] T. Kimura, T. Chtani, and R. Sugano, Prog. Theor. Phys. **48**, 1395 (1972).
[17] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin, 1990), p. 291.
[18] A. Acus, E. Norvaišas, and D. O. Riska, Nucl. Phys. **A614**, 361 (1997).
[19] K. Fujii, K. Sato, N. Toyota, and A. Kobushkin, Phys. Rev. Lett. **58**, 651 (1987); K. Fujii, A. Kobushkin, K. Sato, and N. Toyota, Phys. Rev. D **35**, 1896 (1987).
[20] E. Norvaišas and D. O. Riska, Phys. Scr. **50**, 634 (1994); A. Acus, E. Norvaišas, and D. O. Riska, Nucl. Phys. **A614**, 361 (1997).
[21] Wolfram Research, Inc., *Mathematica*, Version 5.0, Champaign, IL, 2003.
[22] T. S. Walhout, Nucl. Phys. **A547**, 423 (1992).
[23] R. F. Frosch, J. S. McCarthy, R. E. Rand, and M. R. Yearian, Phys. Rev. **160**, 874 (1967).
[24] R. G. Arnold, B. T. Chertok, S. Rock, W. P. Schütz, Z. M. Szalata, D. Day, J. S. McCarthy, F. Martin, B. A. Mecking, I. Sick, and G. Tamas, Phys. Rev. Lett. **40**, 1429 (1978).
[25] T. H. R. Skyrme, Nucl. Phys. **31**, 556 (1962).
[26] R. A. Battye, S. Krusch, and P. M. Sutcliffe, Phys. Lett. **B626**, 120 (2005).
[27] R. A. Battye and P. M. Sutcliffe, Phys. Rev. C **73**, 055205 (2006).