Quadrupolar contact terms and hyperfine structure

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(Received 17 April 2006; published 4 August 2006)

In the interaction of two electric quadrupoles, there is at short range a contact term proportional to the second derivative of a δ function. This contact term contributes to the hyperfine splitting of bound states of two particles with spin one or higher — for example, the bound state of Ω^- and a nucleus with spin one. The contact hyperfine splitting occurs in states with orbital angular momentum unity (P wave), in contrast to the Fermi contact interaction, which is in S states. We find that these contact splittings will be observable with Ω^- atoms and help measure the quadrupole moment and charge radius of the hyperon.

DOI: 10.1103/PhysRevC.74.024001 PACS number(s): 25.80.Pw, 32.10.Fn, 13.40.Em, 41.20.−q

I. INTRODUCTION

It is well known in the context of atomic hyperfine interactions that the magnetic dipole-dipole interaction has a contact term, called the Fermi contact interaction [1]. A well-known example is the splitting of $F = 0$ from the $F = 1$ states of the hydrogen atom, responsible for the microwave radiation of interstellar hydrogen. [The quantum number F is defined in the text below Eq. (3.6)]. It is natural to ask whether such contact interactions occur in higher multipoles, for example, in the interaction of two electric quadrupoles. Only particles with angular momentum $J = 1$ or higher can have quadrupoles. We consider here as an example the bound state of an Ω^- hyperon ($J = 3/2$) to a nucleus of angular momentum one (say nitrogen-14), though there are other possibilities, such as a bound state of an antideuteron to a nucleus, and so on. We find in addition to the well-known long-range electrostatic interaction of two quadrupoles ($\sim 1/r^5$) there are short-range contributions some of which are proportional to the second derivative of a δ function such as $\Delta\delta(\mathbf{r})$. These contact terms give rise to splitting in the P wave ($L = 1$) states of the bound system of an Ω^- with a nucleus, or other systems with two quadrupoles. As far as we know this higher contact interaction has never been described, although it is a consequence of ordinary electromagnetism and quantum mechanics. We find that this splitting may be useful to determine the electric quadrupole moment of the Ω^-, which is not yet known experimentally.

In an ordinary atom, an electron is bound to the nucleus. The electron has spin, and associated with the spin there is a magnetic moment. This magnetic moment interacts with the magnetic fields in the atom and leads to splittings in the energy levels which are called fine structure- if the nucleus itself has no magnetic moment, for example a spinless nucleus such as helium-4. However, if the nucleus itself has spin and magnetic moment, the nuclear magnetic moment (which is much smaller than the electronic magnetic moment) gives rise to further splittings which are called hyperfine structure. Other contributions to atomic hyperfine structure arise from nuclear electric quadrupole moment (which is sensitive to electric field gradients at the nucleus). Higher nuclear moments tend to produce smaller effects. In an Ω^- atom where the Ω^- is bound to a nucleus, there are interesting effects coming from the magnetic moment of the Ω^- (which is known experimentally) and the electric quadrupole moment (which is not known experimentally but estimated theoretically). The fine structure in an Ω^- bound to a spinless nucleus was discussed by Goldhaber and Sternheimer [2]. In the scenario of Ref. [2] the electric quadrupole dominates over magnetic contribution to the fine structure because the electric quadrupole is assumed to be large. However, later estimates of the Ω^- quadrupole, using Quantum chromodynamics-inspired quark models [8] are about 100 times smaller than assumed in Ref. [2]. This leads to a completely different scenario, in which the fine structure is dominated by magnetic interactions, and the effect of the electric quadrupole is to be found in hyperfine interactions. This is the scenario we discuss here by taking a nucleus that has an electric quadrupole and a (small) magnetic dipole to bind the Ω^-. The hyperfine effects will now be smaller than fine structure but still large enough to be observed. As far as we know these hyperfine effects have not been discussed in the literature. The most interesting of these effects is in the interaction of two electric quadrupoles, where there is a contact term in the $l = 1$ (p-wave) states. This contact term is discussed in Sec. II and Appendixes A and B. Then in Secs. III and IV we discuss an application to Ω^- atoms. We conclude in Sec. V with a summary of the results.

II. CONTACT INTERACTION OF QUADRUPOLES

The long-range interaction of two quadrupoles $Q^{(1)}$ and $Q^{(2)}$ may be written in terms of Cartesian components in the form [4]:

$$\begin{aligned} H_{\text{tensor}} &= \frac{1}{48\pi} Q^{(1)}_{ij} \nabla_i \nabla_j \left(\frac{x_k x_l}{r^5} Q^{(2)}_{kl} \right) \\ &= \frac{1}{48\pi r^5} \left(2 Q^{(1)}_{ij} Q^{(2)}_{ij} - 20 Q^{(1)}_{ij} Q^{(2)}_{ik} n_j n_k \right. \\ &\quad \left. + 35 Q^{(1)}_{ij} Q^{(2)}_{kl} n_i n_j n_k n_l \right). \end{aligned} \tag{1}$$

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In Eq. (1) repeated indices are summed; in the first term the two quadrupoles are fully contracted and in the next terms $n_i = x_i/r$, etc. The interaction of two quadrupoles decreases very rapidly with distance— such as r^{-5}, as follows from dimensions— much faster than the fine structure coupling, which decreases only such as r^{-3}. As a result the quadrupolar coupling is more important for smaller orbits, at low quantum numbers. Therefore we focus on low-lying P states of the atom. It can be checked that the tensor quadrupolar interaction [Eq. (1)] is ambiguous in P states of the atom; the angular averaging vanishes because the angular dependence of the interaction (when expressed in spherical harmonics) involves $Y_{4,m}(\theta, \phi)$, where the angles θ, ϕ are those of the vector connecting the two quadrupoles [4]. The expectation value of a spherical harmonic $Y_{4,m}$ vanishes in a P-wave state. The radial integral, however, diverges. This ambiguity is similar to the case of the dipole-dipole interaction in S waves, where the contact interaction solves the ambiguity. Similarly here, instead of the tensor interaction of two quadrupoles we have a contact term, involving the derivative of a δ function, which generalizes the contact interaction where there is only a δ function. The quadrupolar contact coupling is a scalar interaction coupling the two quadrupole tensors and second derivatives of a δ function (as required by dimensional considerations) to a scalar. The precise formula for the quadrupolar contact term is given below:

$$H_{\text{cont}} = -\tfrac{1}{36}\left[\tfrac{4}{7}\left(\nabla_j\nabla_k - \tfrac{1}{3}\nabla^2\delta_{jk}\right)\delta^{(3)}(\mathbf{r}) + \tfrac{2}{15}\delta_{jk}\nabla^2\delta^{(3)}(\mathbf{r})\right] Q^{(1)}_{ij} Q^{(2)}_{ik}. \quad (2)$$

Equation (2) is derived in Appendix A. In this formula $Q^{(1),(2)}$ are the quadrupole moments of the hyperon and the nucleus, and $\nabla^2 = \nabla_i\nabla_i$ is the Laplace operator. This contact term gives a finite contribution in a P-wave state where the two spatial derivatives are taken up by the wave function squared.

III. STRUCTURE OF Ω ATOM

Consider an Ω^- interacting with nucleus with charge Ze. In the first approximation this is Coulomb system with spectrum

$$E_{n,l} = -\frac{Z^2\alpha^2}{2n^2}m, \quad (3)$$

where n and l are the principal and angular momentum quantum numbers, m is the reduced mass

$$m = \frac{m_\Omega}{1 + m_\Omega/M},$$

m_Ω is Ω mass, and M is the mass of the nucleus. The corresponding Rydberg constant for heavy nucleus is

$$R_\Omega = R_e\frac{m_\Omega}{m_e} = 44\,\text{keV}. \quad (4)$$

The interaction of charge with magnetic and quadrupole moments and of the multipole moments with each other is smaller than Coulomb interaction of charges and can be considered as perturbation to Coulomb levels Eq. (3). In the case of ordinary atoms perturbations split Coulomb levels into two main structures: fine structure (because of the interaction of the magnetic moment of the electron with the angular momentum of the atom) and then fine structure levels into hyperfine structure (because of the interaction of the electron magnetic moment with the nuclear magnetic moment). In the case of the Ω^- atom the splitting of the Coulomb levels can be more complicated because Ω^- has spin $S = 3/2$ and magnetic and quadrupole interactions can have rather different scales of strength. It is not a big deal to write equations for splitting for any possible case. In this article we restrict ourselves to the case most similar to the ordinary atoms, i.e., we suppose that there is fine structure because of the interaction of the *magnetic* moment of Ω^- with orbital angular momentum $\mathbf{L}$. That means in particular that the magnetic moment of the nucleus is much smaller than the magnetic moment of Ω^- and the corresponding interaction of quadrupole moments with $\mathbf{L}$ is also much smaller.

For fine structure levels we have one "good" quantum number: the total angular momentum $\mathbf{J}$ that is the sum of orbital angular momentum $\mathbf{L}$ and spin $\mathbf{S}$ of Ω^-:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

Each Coulomb level $E_{n,l}$ splits in general into four levels with different total angular momentum $j = l + 3/2$, $j = l + 1/2$, and $j = l - 1/2$, and $j = l - 3/2$. The magnetic interaction for Ω^- atoms is similar to the case of ordinary atoms

$$V_{\text{mag}} = -\frac{Z\alpha}{2m^2r^3}(g-1)\mathbf{L}\cdot\mathbf{S}, \quad (5)$$

where the g factor relates the magnetic moment to the spin $\mu_\Omega = \text{gS}\,(e/2m_\omega)$.

The average value of $1/r^3$ has to be calculated with unperturbed Coulomb wave functions

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{(Z\alpha m)^3}{n^3 l(l+1/2)(l+1)}. \quad (6)$$

Thus the fine splitting is equal to

$$\Delta E(n,l,j) = \frac{(Z\alpha)^4}{n^3}m(g-1)\frac{j(j+1) - l(l+1) - 15/4}{2l(2l+1)(l+1)}. \quad (7)$$

Magnetic moment of Ω^- is known experimentally $\mu_\Omega = -2.02 \pm 0.05\ \mu_N$ [6]. Thus substituting these factors into V_{mag} we get for fine splitting of p-wave levels into three states with total angular moment $j = 5/2, 3/2, 1/2$:

$$\Delta E(n,l,j) = \frac{Z^4}{12n^3}\left[j(j+1) - \frac{23}{4}\right](6.64\ \text{ev}). \quad (8)$$

This formula gives the scale and exact form of the fine splitting.

We are interested in the hyperfine splitting of these levels by interaction of the Ω quadrupole moment with the nucleus. In this case only the total angular moment of the system $\mathbf{F} = \mathbf{J} + \mathbf{I}$ is a good quantum number. (Here $\mathbf{I}$ is a nuclear spin). The splitting because of the long-range interaction was considered in Ref. [2]. But interaction of quadrupole moment with charged system has also local terms proportional to the δ function and to it derivatives. Thus there should be additional hyperfine structure for states with small angular momentum,

i.e., additional small shifts of s and p states. We are interested in hyperfine structure because of these contact terms.

There are well-known local terms, such as the Fermi contact interaction of two magnetic moments $\mu^{(1),(2)}$

$$U(x) = -\tfrac{2}{3}\left(\mu_i^{(1)}\mu_i^{(2)}\right)\delta^{(3)}(\mathbf{r}). \tag{9}$$

Fermi term contributes to the s wave only and it does not compete with the hyperfine structure because of quadrupoles in p waves.

Interaction of charge $e_{1,2}$ with electromagnetic radius $r_{2,1}$ and radius-radius interaction are also local

$$U(x) = -e_1e_2\left[\tfrac{1}{6}\left(r_1^2 + r_2^2\right)\delta^{(3)}(\mathbf{r}) + \tfrac{1}{36}r_1^2r_2^2\nabla^2\delta^{(3)}(\mathbf{r})\right]. \tag{10}$$

First term in Eq. (10) contributes to s-wave states shift and the second one to s- and p-wave states. But both of these terms do not depend on the spin. Thus they contribute to shift but not to splitting of states with different J and/or F.

Contact terms from Eqs. (9) and (10) are well known and their derivation can be found in text books on classical electrodynamics (see, e.g., Ref. [5]). Similar local terms exist also in the interaction of quadrupole moments, which we cannot find in the literature. There are two such terms (for derivation see Appendix A). Interaction of quadrupole moment Q_{ij} with electromagnetic radius r

$$V(x) = -\tfrac{1}{36}\left(e_1r_1^2Q_{ij}^{(2)} + e_2r_2^2Q_{ij}^{(1)}\right)\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}) \tag{11}$$

and quadrupole-quadrupole interaction

$$V(x) = -\tfrac{1}{63}Q_{ij}^{(1)}Q_{ik}^{(2)}\left(\nabla_j\nabla_k - \tfrac{1}{10}\nabla^2\delta_{jk}\right)\delta^{(3)}(\mathbf{r}). \tag{12}$$

These contact terms contribute to splitting of s- and p-wave states.

Notice that *long-range* interaction of two quadrupoles does not contribute to splitting of s and p states. Any matrix element of this $1 = 4$ interaction vanishes identically for s and p states.

We are interested in determining the value of Ω quadrupole moment from the spectrum of Ω atom. Thus we concentrate on hyperfine structure of p states.

There are three sources of splitting for p states as it is seen from Eqs. (11) and (12).

First there is the Ω quadrupole-nuclear radius interaction. It looks like

$$V(x) = \frac{\mathrm{Ze}}{36}r_{(Z)}^2Q_{ij}^{(\Omega)}\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}). \tag{13}$$

The quadrupole moment of Ω is related to the spin operator $\mathbf{S}$ of Ω

$$\begin{aligned} Q_{ij}^{\Omega} &= \frac{3Q^{(\Omega)}}{2S(2S-1)}\left[S_iS_j + S_jS_i - \frac{2}{3}S(S+1)\delta_{ij}\right] \\ &= \frac{Q^{(\Omega)}}{2}\left(S_iS_j + S_jS_i - \frac{5}{2}\delta_{ij}\right). \end{aligned} \tag{14}$$

Thus the contact term Eq. (13) does not depend on the value of spin $\mathbf{I}$ of nucleus. As a result the corresponding shift of p level can depend only on the value of $\mathbf{J} = \mathbf{L} + \mathbf{S}$, i.e., this contact term contributes into fine splitting of Coulomb p levels, not into hyperfine ones. According to our assumption fine structure is mainly determined by interaction of Ω magnetic moment with orbit. Thus interaction Eq. (13) slightly change this fine shift of levels with different angular momentum j but does not split them. Here and below we use lowercase letters for the quantum numbers j, f (formerly J, F) to distinguish them from J_i, F_k, which are operators.

There is a similar term in interaction of nuclear quadrupole moment $Q_{ik}^{(Z)}$ with electromagnetic radius of Ω

$$V(\mathbf{r}) = \frac{(-e)}{36}r_{(\Omega)}^2Q_{ij}^{(Z)}\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}) \tag{15}$$

with

$$Q_{ij}^{(Z)} = \frac{3Q^{(Z)}}{2I(2I-1)}\left[I_iI_j + I_jI_i - \frac{2}{3}I(I+1)\delta_{ij}\right]. \tag{16}$$

This interaction leads to hyperfine splitting of p levels indeed. The mathematics that helps to calculate matrix elements of tensor operators between wave functions from multiplets with different j and f is very similar to the case of ordinary atomic physics [3]. The result of this calculation is

$$\Delta E^{(1)}(n, l = 1, j, f) = \tfrac{e}{6}Q^{(Z)}r_{(\Omega)}^2|f(0)|^2X^{(1)}(j)T(f, j), \tag{17}$$

where $f(0)$ is the derivative of Coulomb $l = 1$ wave function at origin (see Ref. [3])

$$|f(0)|^2 = \frac{n^2-1}{3\pi n^5}(mZ\alpha)^5, \tag{18}$$

and $T(f, j)$ and $X(j)$ are spin factors

$$\begin{aligned} T(f, j) &= I_iI_j\left\{J_iJ_j + J_jJ_i - \tfrac{2}{3}j(j+1)\delta_{ij}\right\} \\ &= 2(\mathbf{J}\cdot\mathbf{I})^2 + (\mathbf{J}\cdot\mathbf{I}) - \tfrac{2}{3}I(I+1)j(j+1), \end{aligned} \tag{19}$$

and

$$X^{(1)}(j) = \frac{3[j(j+1) - 77/12]}{8j(j+1)}. \tag{20}$$

Here

$$J_iI_i = \mathbf{J}\cdot\mathbf{I} = \tfrac{1}{2}[f(f+1) - I(I+1) - j(j+1)].$$

For $I = 1$ nucleus and for p states the only interesting states are those with $j = 3/2$ and $j = 5/2$. For this cases $X^{(1)}(j = 3/2) = -4/15$ and $X^{(1)}(j = 5/2) = 1/10$. For $j = 1/2$, $T(f, j)$ vanishes, as it should, because one cannot have a quadrupole at $j = 1/2$.

In this article we are interested mainly in Ω *quadrupole - nuclear quadrupole* contact interaction

$$V(x) = -\tfrac{1}{63}Q_{ij}^{(\Omega)}Q_{ik}^{(Z)}\left(\nabla_j\nabla_k - \tfrac{1}{10}\nabla^2\delta_{jk}\right)\delta(\mathbf{r}). \tag{21}$$

Matrix elements of tensor operators for this interactions can be calculated in the similar way. The result is

$$\Delta^{(2)}E(n, l = 1, j, f) = -\tfrac{1}{21}Q^{(Z)}Q^{(\Omega)}|f(0)|^2X^{(2)}(j)T(f, j), \tag{22}$$

with spin factor

$$X^{(2)}(j) = -\frac{1}{16}\left[j(j+1) - \frac{69}{10} - \frac{49\cdot 31}{80j(j+1)}\right]. \tag{23}$$

For $I = 1$ nucleus and for p states the only interest is in $j = 3/2$ and $j = 5/2$, as before. For this cases $X^{(2)}(j = 3/2) = 77/150$ and $X^{(2)}(j = 5/2) = 1/50$.

For the derivation of Eqs. (20) and (23) see the next section.

IV. CALCULATION OF SPIN FACTORS

The calculation of matrix elements of tensor operators between states that correspond to irreducible representation of rotation group is well known (see, e.g., Ref. [3]). In this section we present some intermediate formulas that help to get Eqs. (20) and (23).

Matrix elements of any tensor in the subspace of multiplets with given value of $\mathbf{J}$ can be represented as a matrix elements of appropriate combination of operators $\hat{\mathbf{J}}$.

We start with the state with given $\mathbf{L}$, namely with subspace of p states. In the special basis where $l = 1$ wave function is represented as a vector function $\mathbf{\Psi}(x) = \mathbf{x} f(r)$ $(r = |\mathbf{x}|)$ one can check that

$$\nabla_i \nabla_j \delta^{(3)}(\mathbf{r}) = |f(0)|^2 (\hat{L}^2 \delta_{ij} - \hat{L}_i \hat{L}_j - \hat{L}_j \hat{L}_i).$$

Thus interaction of nuclear quadrupole moment $Q_{ik}^{(Z)}$ with the e.m. radius of Ω can be rewritten as

$$V(x) = \frac{er^2_{(\Omega)}}{36} Q_{ij}^{(Z)} |f(0)|^2 (\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i). \tag{24}$$

(The term $\sim\delta_{ij}$ does not contribute to the interaction because the tensor $Q_{ij}^{(Z)}$ is traceless).

The second step is to average this operator over states with a given $\mathbf{J}$. At this step any tensor can be constructed from operators $\hat{J}_i$ so that

$$\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i - 2/3 \hat{L}^2 \delta_{ij} = X^{(1)}(j) \cdot (\hat{J}_i \hat{J}_j + \hat{J}_j \hat{J}_i - 2/3 \hat{J}^2 \delta_{ij}), \tag{25}$$

where $X^{(1)}(j)$ is an unknown constant. To calculate this constant we use the following relations

$$\begin{aligned} T_0 &= \hat{J}_i (\hat{J}_i \hat{J}_j + \hat{J}_j \hat{J}_i - \tfrac{2}{3} \hat{J}^2 \delta_{ij}) \\ \hat{J}_j &= 4/3 j(j+1)(j-1/2)(j+3/2); \end{aligned} \tag{26}$$

$$\begin{aligned} T_1 &= \hat{J}_i (\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i - \tfrac{2}{3} \hat{L}^2 \delta_{ij}) \hat{J}_j \\ &= 2(\mathbf{J} \cdot \mathbf{L})^2 - (\mathbf{J} \cdot \mathbf{L}) - \tfrac{2}{3} l(l+1) j(j+1) \\ &= \tfrac{1}{2}(j - 1/2)(j + 3/2)[j(j+1) - 77/12], \end{aligned} \tag{27}$$

where in the last line we take $l = 1$. Thus for constant $X^{(1)}(j)$ one gets the relation

$$X^{(1)} = T_1 / T_0,$$

that is equivalent to Eq. (20) from the previous section.

Quadrupole-quadrupole interaction for the states with given $\mathbf{L}$ can be rewritten as

$$V(x) = -\tfrac{1}{63} |f(0)|^2 Q_{ij}^{(\Omega)} Q_{ik}^{(Z)} \left\{ \tfrac{9}{10} \hat{L}^2 \delta_{jk} - \hat{L}_j \hat{L}_k - \hat{L}_k \hat{L}_j \right\}, \tag{28}$$

where the quadrupole tensors are

$$Q_{ij}^{(Z)} = \frac{3Q^{(Z)}}{2I(2I-1)} \left[I_i I_j + I_j I_i - \frac{2}{3} I(I+1) \delta_{ij} \right] \tag{29}$$

and

$$Q_{ij}^{(\Omega)} = \frac{3Q^{(\Omega)}}{2S(2S-1)} \left[S_i S_j + S_j S_i - \frac{2}{3} S(S+1) \delta_{ij} \right]. \tag{30}$$

For multiplet with given total angular moment $\mathbf{J}$

$$V(x) = -\tfrac{1}{63} |f(0)|^2 Q_{ij}^{(Z)} \left\langle Q_{ik}^{(\Omega)} \left(\tfrac{9}{10} \hat{L}^2 \delta_{jk} - \hat{L}_j \hat{L}_k - \hat{L}_k \hat{L}_j \right) \right\rangle_J, \tag{31}$$

where $< .. >_J$ is average over multiplet with given $\mathbf{J}$ and

$$\left\langle Q_{ik}^{(\Omega)} \left(\tfrac{9}{10} \hat{L}^2 \delta_{jk} - \hat{L}_j \hat{L}_k - \hat{L}_k \hat{L}_j \right) \right\rangle_J = X^{(2)}(j) \left(\hat{J}_i \hat{J}_j + \hat{J}_j \hat{J}_i - \tfrac{2}{3} \hat{J}^2 \delta_{ij} \right). \tag{32}$$

To calculate $X^{(2)}(j)$ we need new relations

$$\begin{aligned} T_2 &= \hat{J}_i \left(\hat{S}_i \hat{S}_j + \hat{S}_j \hat{S}_i - \tfrac{2}{3} \hat{S}^2 \delta_{ij} \right) \hat{J}_j \\ &= \tfrac{1}{2}(j - 1/2)(j + 3/2)[j(j+1) - 7/4], \end{aligned} \tag{33}$$

$$\begin{aligned} T_3 &= \hat{J}_i \left(\hat{S}_i \hat{L}_j + \hat{S}_j \hat{L}_i - \tfrac{2}{3} \mathbf{S} \cdot \mathbf{L} \delta_{ij} \right) \hat{J}_j \\ &= \tfrac{1}{6}(j - 1/2)(j + 3/2)[j(j+1) + 49/4]. \end{aligned} \tag{34}$$

After some algebra one obtains Eq. (23) for $X^{(2)}(j)$.

V. NUMERICAL ESTIMATES

Thus for hyperfine splitting we obtain

$$\Delta E(n, l = 1, j, f) = \tfrac{1}{6} Q^{(Z)} |f(0)|^2 T(f, j) \left[X^{(1)}(j) \, \mathrm{er}^2_{\Omega} - \tfrac{2}{7} X^{(2)}(j) Q^{(\Omega)} \right], \tag{35}$$

with

$$|f(0)|^2 = \frac{n^2 - 1}{3\pi n^5} (mZ\alpha)^5, \tag{36}$$

$$\begin{aligned} T(f = 1/2) &= 5, \\ T(f = 3/2) &= -4, \\ T(f = 5/2) &= 1 \end{aligned} \tag{37}$$

for $j = 3/2$ and

$$\begin{aligned} T(f = 3/2) &= 28/3, \\ T(f = 5/2) &= -32/3, \\ T(f = 7/2) &= 10/3 \end{aligned} \tag{38}$$

for $j = 5/2$. One can check that hyperfine splitting satisfies the well-known sum rules

$$2T(1/2) + 4T(3/2) + 6T(5/2) = 0 \tag{39}$$

and

$$4T(3/2) + 6T(5/2) + 8T(7/2) = 0. \tag{40}$$

Unknown quantities in Eq. (35) are electric radius er^2_{Ω} and quadrupole moments $Q^{(Z)}$ and $Q^{(\Omega)}$. If we take $\mathrm{er}^2_{\Omega} \sim 10^{-2}\mathrm{F}^2$ and $Q^{(\Omega)} \sim 10^{-2}\mathrm{F}^2$ we obtain units for $Z = 7$ hyperfine splitting of the order of keV. We believe that such splitting can be measured experimentally.

The main background that interferes with hyperfine splitting is strong interaction of Ω with nucleus.[1] Little is known about strong interactions of Ω. To obtain an estimate for the shift and width of Coulomb levels in Ω atoms because of strong interactions we looked at p levels of antiprotonic atoms [9]. According to these data for antiprotonic hydrogen atom $\Gamma(2p) \sim 10^{-1}$ eV and for antiprotonic deuterium atoms $\Gamma(2p) \sim 5 \times 10^{-1}$ eV. For heavier atoms widths of the states grows rapidly. Thus $\Gamma(2p) \sim 660$ eV for 6 Li.[2]

There is good reason to assume that Ω interacts with a nucleus less than an antiproton. This is because of the absence of annihilation diagrams. Thus there is a good chance that strong interaction does not screen hyperfine structure of Ω atoms because of contact interactions. It is a challenge for experimentalist to measure quadrupole moment of Ω.

VI. SUMMARY AND CONCLUSIONS

We have discussed the hyperfine spectrum of a bound system of Ω^- and a nucleus of angular momentum one, in a scenario that corresponds to a small electric quadrupole moment for the Ω^-, of the order of 10^{-2} F^2. We find that there is a quadrupolar contact term that contributes to the *orbital* angular momentum one states. Our discussion suggests that contact terms are a universal property of the interaction between multipoles and not restricted only to the interaction of two dipoles—electric or magnetic. However, multipoles higher than quadrupoles require two particles of spin 3/2, and the number of such systems is quite small. We assume that the nucleus that binds the Ω^- has only an electric quadrupole moment and no magnetic dipole, an oversimplified case, which is approximated by the nucleus Nitrogen-14. We do not imply that this is the optimal case to determine the quadrupole moment of the Ω^-. It remains to be found by experiment if the scenario we discuss is in fact useful.

ACKNOWLEDGMENTS

This research was supported by the NSERC and partly by RFBR grant 05-02-17203. One of the authors (V.N.) is indebted to the Physics Department at the University of Guelph and to Perimeter Institute for hospitality.

APPENDIX A: ELECTROMAGNETIC INTERACTION OF TWO PARTICLES

The interaction of two multipole moments at long distances is well known and can be found in textbooks on classical electrodynamics. In this article we are mainly interested in the contact interactions of the moments. These terms are less familiar. We derive them using a Feynman diagram technique in nonrelativistic notations.

Consider the scattering of two particles with charges (e_1, e_2), dipole moments $(\mathbf{d}_1, \mathbf{d}_2)$, etc. The scattering amplitude $T(q)$ is given by one photon exchange diagram and is equal to

$$T(q) = -\frac{1}{q^2}[\rho^{(1)}(q)\rho^{(2)}(-q) - \mathbf{j}^{(1)}(q)\mathbf{j}^{(2)}(-q)], \qquad \text{(A1)}$$

where $\rho(q)$ and $\mathbf{j}(q)$ are the matrix elements of the density and of the current of the particle. In the nonrelativistic approximation $T(q) = -U(q)$, where $U(q)$ is the Fourier transform of the potential energy. This gives a systematic way to calculate potential energy between two particles with different multipoles moments.

To perform the calculation we need first of all of the expressions for the matrix element of the current operator $j(q) =< p_2|\hat{j}|p_1 >$ for a particle with electric dipole moment $\mathbf{d}$, with magnetic moment $\boldsymbol{\mu}$, with quadrupole moment Q_{ij}, etc. The expression for the matrix element of electromagnetic current $j_\mu = (\rho, \mathbf{j})$ is the following (for derivation see Appendix B):

$$\rho(q) = j_0(q) \simeq e\left(1 - \frac{1}{6}r^2q^2\right) - id_iq_i - \frac{1}{6}Q_{ij}q_iq_j + \cdots,$$

$$j_l(q) \simeq \frac{e}{2}(v_1 + v_2)_l - ie_{ikl}\mu_iq_k + \cdots, \qquad \text{(A2)}$$

where $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$ is momentum transfer and $\mathbf{v} = \mathbf{p}/m$ is velocity of a particle. We introduced form factor $F(q^2) \simeq 1 - (r^2q^2)/6$ to take into account the distribution of charge. Here r^2 is the electromagnetic radius of the distribution. The expression for the current density operator $\hat{\mathbf{j}}$ for a particle with charge e and magnetic moment μ moving in a magnetic field is well known and can be found in courses of quantum mechanics (see, e.g., Ref. [3]). For the derivation of other terms in Eq. (A2) see Appendix B.

Consider now how this procedure works in the case of **charge-charge** interactions. The scattering amplitude is

$$T(q) = -U(q) = -\frac{e_1e_2}{q^2}\left[1 - \frac{1}{6}q^2\left(r_1^2 + r_2^2\right) + \frac{1}{36}r_1^2r_2^2q^4\right].$$

In coordinate representation $1/q^2$ corresponds to the $1/4\pi r$, the constant in q space corresponds to the δ function in x space, and q^2 corresponds to the second derivative of the δ function:

$$(1/q^2) \Longrightarrow \frac{1}{4\pi r};$$
$$1 \Longrightarrow \delta^{(3)}(\mathbf{r});$$
$$(q_iq_j) \Longrightarrow -\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}).$$

As a result $U(x)$ is given by

$$U(x) = \frac{e_1e_2}{4\pi r} - \frac{e_1e_2}{6}\left(r_1^2 + r_2^2\right)\delta^{(3)}(\mathbf{r}) - \frac{e_1e_2}{36}r_1^2r_2^2\nabla^2\delta^{(3)}(\mathbf{r}). \qquad \text{(A3)}$$

We get long-range Coulomb term and *contact* interactions of charge with charge radius and of two charge radii.

Consider **dipole-dipole** interaction. In momentum space

$$U(q) = \frac{(\mathbf{d}^{(1)}\mathbf{q})(\mathbf{d}^{(2)}\mathbf{q})}{q^2}.$$

[1] We are grateful to M. Pospelov, who brought our attention to this subject.

[2] We are grateful to L. Bogdanova, B. Kerbikov, A. Kudryavtsev, D. Gotta, and D. C. Bailey for discussion of the current experimental results.

The product $q_i q_k$ can be decomposed into two irreducible tensors so that

$$\frac{q_i q_k}{q^2} = \left(\frac{q_i q_k}{q^2} - \frac{1}{3}\delta_{ik}\right) + \frac{1}{3}\delta_{ik} \tag{A4}$$

Irreducible tensors in momentum space correspond to irreducible tensors in coordinate space. Thus the Fourier transform of $(q_i q_k/q^2 - 1/3\delta_{ik})$ is traceless and the symmetric second-rank tensor in coordinate space is made from the second derivatives of $1/(4\pi r)$, i.e.,

$$\frac{q_i q_k}{q^2} - \frac{1}{3}\delta_{ik} \Longleftrightarrow \frac{-3n_i n_k + \delta_{ik}}{4\pi r^3},$$

where $\mathbf{n} = \mathbf{r}/r$. Using irreducible tensors we automatically subtract contact terms $\sim\delta(x)$ from the long-range potential. In spherical coordinates, the irreducible tensor of the second rank is proportional to the second spherical harmonic $Y_{2,m}(\theta, \phi)$.

The second term $\sim\delta_{ik}$ in Eq. (A4) is a constant in q space. In coordinate space gives the δ function, i.e.,

$$1 \Longleftrightarrow \delta^{(3)}(\mathbf{r})$$

In this way we obtain the well-known long-range potential for two electric dipoles in the p wave and the pointlike interaction of electric dipoles in the s wave

$$U(x) = \frac{(\mathbf{d}_1\mathbf{d}_2) - 3(\mathbf{d}_1\mathbf{n})(\mathbf{d}_2\mathbf{n})}{4\pi r^3} + \frac{1}{3}(\mathbf{d}_1\mathbf{d}_2)\delta^{(3)}(\mathbf{r}). \tag{A5}$$

Magnetic moment-magnetic moment interaction is very similar to electric moments interaction. The only difference is that $\boldsymbol{\mu}$ contributes to $\mathbf{j}$ (not to ρ) and $\boldsymbol{\mu}$ is an axial vector so that instead of dot products we have cross products of vectors:

$$U(q) = -\frac{1}{q^2}\left(e_{ikl}q_k\mu_l^{(1)}\right)\left(e_{imn}q_m\mu_n^{(2)}\right). \tag{A6}$$

Transforming this expression to coordinate space we find

$$U(x) = \frac{\mu_i^{(1)}\mu_i^{(2)} - 3\left(\mu_i^{(1)}n_i\right)\left(\mu_k^{(2)}n_k\right)}{4\pi r^3} - \frac{2}{3}\left(\mu_i^{(1)}\mu_i^{(2)}\right)\delta^{(3)}(\mathbf{r}). \tag{A7}$$

The long-range interaction is known from magnetostatics. The result for the contact term was found originally by Fermi and independently by Casimir in the framework of the relativistic Dirac equation. Later it was derived using the nonrelativistic Breit equation in Ref. [7]. Elementary derivation of contact terms in magnetostatics is also known (see Ref. [5]).

We are now ready to derive contact terms for quadrupole interaction.

Quadrupole-electric radius interaction in momentum space looks as

$$V(q) = \tfrac{1}{36}\left(e_1 r_1^2 Q_{ij}^{(2)} + e_2 r_2^2 Q_{ij}^{(1)}\right)q_i q_j. \tag{A8}$$

In coordinate space it gives

$$V(x) = -\tfrac{1}{36}\left(e_1 r_1^2 Q_{ij}^{(2)} + e_2 r_2^2 Q_{ij}^{(1)}\right)\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}). \tag{A9}$$

This interaction vanishes for the s wave but is nonzero for the p wave.

Quadrupole-quadrupole interaction in momentum space has the following form

$$V(q) = \frac{q_i q_j q_k q_l}{36q^2} Q_{ij}^{(1)} Q_{kl}^{(2)}, \tag{A10}$$

where we have to express the product of the four-momentum factors in terms of irreducible tensors. Different tensors represent interactions in d, p, and s waves.

We construct irreducible tensor $T_{ijkl}^{(4)}$ as a sum of all possible symmetric terms:

$$T_{ijkl}^{(4)} = q_i q_j q_k q_l + Aq^2(\delta_{ij}q_k q_l + \delta_{kl}q_i q_j + \delta_{ik}q_j q_l + \delta_{jl}q_i q_k + \delta_{il}q_j q_k + \delta_{jk}q_i q_l) + Bq^4(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The irreducible tensor $T_{ijkl}^{(4)}$ has to be traceless. The constraint $T_{iikl}^{(4)} = 0$ yields $A = -(1/7)$ and $B = (1/35)$.

Thus Eq. (A10) can be rewritten in following form:

$$V(q) = \frac{1}{36q^2}\left(Q_{ij}^{(1)}Q_{kl}^{(2)}T_{ijkl}^{(4)}\right) + \frac{1}{63}\left(Q_{ij}^{(1)}Q_{ik}^{(2)}q_j q_k - \frac{q^2}{10}Q_{ij}^{(1)}Q_{ij}^{(2)}\right). \tag{A11}$$

In coordinate space irreducible tensor $T_{ijkl}^{(4)}$ is proportional to fourth-harmonic $Y_{4,m}(\theta, \phi)$. Thus this term in Eq. (A11) represents the well-known long-range interaction of two quadrupoles in the d wave [see Eq. (1)].

When we go to coordinate space

$$1/q^2 \Longrightarrow \frac{1}{4\pi r}$$
$$q_i q_j \Longrightarrow -\nabla_i\nabla_j\delta^{(3)}(\mathbf{r}).$$

In this way differentiating $1/r$ we obtain the standard formula for quadrupole-quadrupole interactions at long distances. The second term in Eq. (A11) is the quadratic polynomial in momentum space. In coordinate space it corresponds to the second derivative of the δ function

$$V_{\text{cont}}(x) = -\tfrac{1}{63}Q_{ij}^{(1)}Q_{ik}^{(2)}\left(\nabla_j\nabla_k - \tfrac{1}{10}\nabla^2\delta_{jk}\right)\delta^{(3)}(\mathbf{r}), \tag{A12}$$

which is equivalent to Eq. (2) used in the text. This result is new.

APPENDIX B: ELECTROMAGNETIC VERTEX

One can find the standard derivation for the current density operator $\hat{\mathbf{j}}$ for a particle with charge e and magnetic moment $\boldsymbol{\mu}$ moving in a magnetic field in quantum mechanics textbooks (see, e.g., Ref. [3]). The matrix element of this operator between two states with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ is

$$\langle p_2|\hat{j}_i|p_1\rangle = \frac{e}{2}(v_1 + v_2)_i + i e_{ikl}q_k\mu_l, \tag{B1}$$

where $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$ is the momentum transfer and $\mathbf{v} = \mathbf{p}/m$ is the velocity of a particle. For the charge density $\rho = j_0$ the

matrix element is

$$\langle p_2|\hat{\rho}|p_1\rangle = eF(q^2) \simeq e\left(1 - \tfrac{1}{6}r^2q^2\right). \tag{B2}$$

We need a similar expression for the electromagnetic current of a particle with electric dipole moment **d** and with quadruple moment Q_{ij} (and with magnetic moment as well). One can derive this expression classically without any reference to quantum mechanics.

This is a two-step procedure (see also Ref. [10]). First, note that the interaction of the particle with electromagnetic field $A_\mu = (\phi, \mathbf{A})$ is described by currents

$$H_{\rm int} = \rho\phi - \mathbf{jA}. \tag{B3}$$

This is the definition of the current.

Second, note that in the static field the interaction energy of a particle is given by the multipole expansion. By definition of the multipole moments **d**, $\boldsymbol{\mu}$, Q_{ij}, etc., we have

$$\begin{aligned} H_{\rm int} &\simeq e\phi(0) - d_i E_i - \frac{1}{6}Q_{ij}\frac{\partial E_j}{\partial x_i} - \mu_i H_i \\ &= \left(e + d_i\nabla_i + \frac{1}{6}Q_{ij}\nabla_i\nabla_j\right) \\ &\quad \times \phi(x)|_{x=0} - e_{ikl}\mu_i\nabla_k A_l(x)|_{x=0}, \end{aligned} \tag{B4}$$

where $E_i(x) = -\nabla_i\phi(x)$ and $H_i = e_{ikl}\nabla_k A_l(x)$ are electric and magnetic fields, respectively.

Comparing Eqs. (B3) and (B4) we find the contribution of electric dipole, magnetic dipole, and quadrupole moments into an electromagnetic current.

In coordinate space it is equal to the sum of differential operators

$$\begin{aligned} \hat{j}_0 &= e + d_i\nabla_i + \tfrac{1}{6}Q_{ij}\nabla_i\nabla_j + \cdots, \\ \hat{j}_l &= e_{ikl}\mu_i\nabla_k. \end{aligned} \tag{B5}$$

In momentum space

$$j(q) = \int j(x)\exp(-i\mathbf{qx})d^3x$$

it can be written as

$$\begin{aligned} j_0 &= e - i\,d_i q_i - \tfrac{1}{6}Q_{ij}q_iq_j + \cdots, \\ j_l &= -i\,e_{ikl}\mu_i q_k + \cdots. \end{aligned} \tag{B6}$$

Fourier transform of the classical current corresponds to the matrix element of the quantum operator $j(q) = \langle p_2|\hat{j}|p_1\rangle$.

Thus we obtain the complete representation of the matrix element of the current operator in momentum space

$$\begin{aligned} j_0(q) &\simeq e\left(1 - \tfrac{1}{6}r^2q^2\right) - i\,d_iq_i - \tfrac{1}{6}Q_{ij}q_iq_j + \cdots, \\ j_l(q) &\simeq \tfrac{e}{2}(v_1 + v_2)_l - i\,e_{ikl}\mu_i q_k + \cdots. \end{aligned} \tag{B7}$$

In the language of Feynman diagrams Eq. (B7) corresponds to the vertex operator that describes interaction of the particle with the photon.

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