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Power counting with one-pion exchange

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Techniques developed for handing inverse-power-law potentials in atomic physics are applied to the tensor one-pion exchange potential to determine the regions in which it can be treated perturbatively. In S, P, and D waves, the critical values of the relative momentum are less than or of the order of 400 MeV. The Wilsonian renormalization group (RG) is then used to determine the power counting for short-range interaction in the presence of this potential. In the P and D waves, where there are no low-energy bound or virtual states, these interactions have half-integer RG eigenvalues and are substantially promoted relative to naive expectations. These results are independent of whether the tensor force is attractive or repulsive. In the $^3S_1$ channel, the leading term is relevant, but it is demoted by half an order compared to the counting for the effective-range expansion with only a short-range potential. The tensor force can be treated perturbatively in those F waves and above that do not couple to P or D waves. The corresponding power counting is the usual one given by naive dimensional analysis.

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I. INTRODUCTION

Since Weinberg [1] first proposed that the ideas of chiral perturbation theory (ChPT) could be applied to nuclear forces, there has been a continuing debate over which parts of interaction can be treated perturbatively and which can, or indeed must, be treated nonperturbatively. This has led to two widely used schemes for constructing effective field theories (EFT's) to represent these forces.[1]

One is based on Weinberg's original suggestion [1] and has been widely applied by van Kolck and collaborators [3–5]. It will be referred to here as the WvK scheme. In it, one first expands potential using perturbative "Weinberg" power counting (like that in ChPT for mesons or single nucleons [6]). Then one constructs the scattering amplitude by iterating the lowest-order terms: the leading, energy-independent contact interaction and one-pion exchange (OPE).

The other scheme, developed by Kaplan, Savage, and Wise (KSW) [7], starts from a nontrivial fixed point of the renormalization group, which corresponds to a two-body system with an infinite scattering length. In the expansion around this point, all pion-exchange forces as well as momentum- or energy-dependent contact interactions are treated as perturbations.

At very low momenta, pion-range physics is not resolved, and nuclear forces can be described only in terms of contact interactions. In this regime the two schemes, WvK and KSW, are equivalent, since iterating the leading contact interaction leads to the same power counting [7–9], and they just reproduce the effective-range expansion [10,11]. However, at higher momenta, they treat OPE differently. There, problems with the perturbative KSW scheme have been identified by Fleming, Mehen, and Stewart [12]. In particular, they have shown that the expansion is only slowly convergent in the $^1S_0$ channel and, worse, it seems not to converge at all in the $^3S_1$ channel. This has led Beane *et al.* [13] to propose a hybrid approach, using KSW in the former and WvK in the latter.

[1]For reviews, see Refs. [2,3].

The strength of the OPE potential is given by the square of the pseudovector coupling which, to leading chiral order, is given by $f^2_{\pi NN} = g_A^2 m_\pi^2/(16\pi F_\pi^2)$. Empirical determinations lead to values for $f^2_{\pi NN}$ of about 0.075 [14]. If we factor the nucleon mass out of the Hamiltonian, we find that the OPE potential contains the scale

$$\lambda_\pi = \frac{16\pi F_\pi^2}{g_A^2 M_N} \simeq 290 \text{ MeV}, \tag{1}$$

which is constructed from the nucleon mass and the pion decay constant. Since both of these are high-energy scales in ChPT, one would naturally take this to be a high-energy quantity. If λ_π were much larger than m_π, OPE could then be treated as weak for scattering momenta of the order of m_π, and the KSW scheme would apply. This would be the case in a world with much smaller up- and down-quark masses, where one would be much closer to the chiral limit, and m_π would be smaller than f_π, not just smaller than $4\pi f_\pi$—the typical combination that appears in chiral expansions for processes involving at most one nucleon [6]. Unfortunately, in the real world λ_π is only about twice m_π, and so we do not have a good separation of scales. This is what underlies the difficulties in building a useful perturbative EFT with pion-exchange forces.

The first question is whether to iterate OPE or not. However, Nogga, Timmermans, and van Kolck [15] have shown numerically that iterated OPE cannot be consistently renormalized if the contact interactions are assigned the orders they would have in naive dimensional analysis. A similar observation is also made in footnote 5 of Ref. [16], but without examining in detail the consequences for power counting. Hence, if we choose to iterate OPE, we are forced to address a second question: what power counting should we use for the resulting contact interactions?

In this work, I address the first question by constructing exact solutions to the Schrödinger equation and examining them for nonanalytic dependence on the strength of the OPE potential. In the chiral limit, this potential has a tensor form proportional to $1/r^3$. Solutions can be obtained using

techniques that have been developed in atomic physics [17–19]. As in the case of the simple $1/r^3$ potential studied by Gao [19], the solutions in each partial wave become nonanalytic in the strength above some critical value for the dimensionless product of the momentum and coupling strength. This implies that the potential must be treated nonperturbatively in this region.

For the $1/r^3$ in uncoupled partial waves, the critical values have been determined by Gao [19]. Here I apply these results to the tensor potential in coupled waves. Since the strength of OPE is given in terms of $1/\lambda_\pi$, these results can be converted into a critical value for the relative momentum in each channel, above which OPE must be treated nonperturbatively. In channels involving waves with $l \leqslant 2$, these critical momenta are $\lesssim$400 MeV, implying that OPE needs to be iterated in them, in agreement with the observations in Ref. [12]. In contrast, the critical momenta in channels that involve only waves with $l \geqslant 3$ are well above 1 GeV, and so OPE can be treated perturbatively.

To answer the second question, I use the Wilsonian renormalization group (RG) [20] to determine the scale dependence of the interactions between two nucleons. In this approach, one imposes a floating cutoff Λ, lying between the low-energy scales of interest and high scales of the underlying interest. Demanding that the scattering amplitude be independent of the cutoff then leads to an RG equation for the effective short-range potential describing the physics that is not resolved at the scale Λ [21–23].

This cutoff could be imposed on a plane-wave basis, but in the presence of a known long-range potential, it turns out to be more convenient to work in the basis of distorted waves (DW's) of that potential [22,23]. For a long-range potential that is singular at the origin, the scaling behavior, and hence the power counting, is controlled by the power-law dependence of the DW's for small r. In the case of the $1/r^3$ tensor potential, I find that the short-range interactions have half-integer RG eigenvalues (anomalous dimensions) and so their scaling is quite different from that given by naive dimensional analysis. These eigenvalues are all positive if the scattering is weak, but they are smaller than they would be in the absence of the long-range potential. Compared to the usual power counting [1], these interactions are "promoted" to lower orders in the expansion in small scales and hence are more important for low-energy scattering than one would naively expect. This agrees with the numerical observations in Ref. [15]. Similar conclusions about the need for a modified power counting are drawn in Refs. [16,24,25], which use a related DW approach with a radial cutoff only. The RG analysis here provides the new power counting. It also shows that in waves where the tensor force can be treated perturbatively, the usual power counting still applies.

II. SOLUTIONS IN THE CHIRAL LIMIT

The long-range OPE potential has a central piece

$$V_{\pi C}(r) = \frac{1}{3} f_{\pi NN}^2 \frac{e^{-m_\pi r}}{r} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \tag{2}$$

and a tensor piece

$$V_{\pi T}(r) = \frac{1}{3} \frac{f_{\pi NN}^2}{m_\pi^2} \left(3 + 3m_\pi r + m_\pi^2 r^2\right) \frac{e^{-m_\pi r}}{r^3} S_{12}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \tag{3}$$

where $S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$.

In the spin-singlet channels, only the central Yukawa potential $V_{\pi C}$ contributes. This has a $1/r$ singularity at the origin. Even when iterated, this is not sufficient to affect the power counting for the short-range interactions [22]. However, in the case of the spin-triplet channels, we have to deal with the tensor piece of OPE. This behaves like $1/r^3$ at short distances, so it is not obvious that it can ever be treated perturbatively. If we do iterate this interaction, the resulting nonperturbative short-distance physics can alter the power counting.

In fact, a $1/r^3$ singularity is sufficiently short ranged that waves with low momenta do not resolve the singularity, and their scattering can still be treated perturbatively. As discussed below, the critical value of the momentum above which nonperturbative behavior sets in is proportional to the scale λ_π in Eq. (1) and increases rapidly with the orbital angular momentum of the wave. This is because in higher partial waves, the centrifugal barrier "protects" low-energy waves from probing the singularity.

At long distances, the potential falls off exponentially as a result of the finite pion mass. This ensures that there are no nonanalytic terms in the scattering amplitude and so, at very low energies, the effective-range expansion [10,11] can be applied to it. However, it does not alter the singular behavior at short distances, so the effects of that can still be analyzed using the simpler form of the potential in the chiral limit ($m_\pi \to 0$).[2]

In the chiral limit the tensor interaction has the $1/r^3$ form

$$V_{\pi T}(r) = \frac{1}{M_N \lambda_\pi} \frac{1}{r^3} S_{12}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \tag{4}$$

This and other singular inverse-power-law potentials also arise for systems in atomic physics, where techniques for solving the corresponding Schrödinger equations have developed.[3]

In applications in both nuclear and atomic physics, these singular potentials should not be regarded as "fundamental". Instead they provide the long-range parts of effective interactions. At small separations, the finite sizes and structures of nucleons or atoms cannot be ignored, and other, short-range interactions are important. The solutions to the Schrödinger equation with one of these potentials can be used as a DW basis for analyzing the scale dependence of the associated short-range terms. Provided they are viewed in this way, as

[2] If λ_π had been much smaller than m_π then this would not be possible; the exponential falloff would mean that waves did not probe the singularity until their momenta were at least of order m_π. As discussed below, the critical momenta obtained in the chiral limit are greater than m_π in all channels except the ${}^3S_1$–${}^3D_1$. Even in that case, the critical momentum is of the order of $m_\pi/2$, so the conclusion that perturbation theory breaks down for momenta of the order of m_π should still be valid, although the precise value of the momentum at which this happens will be different.

[3] For a recent review, see Ref. [26]. A review of older approaches can be found in Ref. [27].

pieces of effective theories, even attractive singular potentials are meaningful (contrary to the comment at the end of Ref. [19]).

A. Uncoupled channels

The tensor interaction couples spin-triplet partial waves with $l = j \pm 1$, such as ${}^3S_1$ and ${}^3D_1$, but not those with $l = j$. I consider the latter first. For these, the tensor operator is just $S_{12} = 2$ [28], and the potential has a simple $1/r^3$ form. The solutions to the corresponding Schrödinger equation can be constructed as series expansions in Bessel functions using the method of Refs. [17–19].[4] Solutions for the pure $1/r^3$ potential have previously been obtained by Gao [19], but I recap some of the main features of the method here before applying it to the coupled waves.

The radial Schrödinger equation describing the relative motion of the two particles has the form

$$-\frac{1}{M_N}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+1)}{r^2}\right]\psi(r)+\frac{B_3}{r^3}\psi(r)=E\psi(r). \tag{5}$$

The strength of the potential is $B_3 = -6/(M_N\lambda_\pi)$ in the uncoupled isospin-singlet waves, such as ${}^3D_2$ and ${}^3G_4$, and $B_3 = 2/(M_N\lambda_\pi)$ in the isospin-triplet ones, such as ${}^3P_1$ and ${}^3F_3$. It is convenient to rewrite this equation in dimensionless form by introducing the coordinate $x = pr$ and the coupling $\kappa = pM_NB_3$, where $p = \sqrt{M_NE}$ is the on-shell relative momentum. It is also convenient to put the radial equation into a form similar to Bessel's equation by defining $\phi(x) = x^{1/2}\psi(x)$. After some rearrangement, the resulting equation is

$$\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}+x^2-\left(l+\frac{1}{2}\right)^2\right]\phi(x)=\frac{\kappa}{x}\phi(x). \tag{6}$$

In this form we see that the behavior of the solutions is controlled by the single combination of the energy and the coupling strength, κ.

This equation can be solved analytically with the aid of the methods in Refs. [17–19] by expanding $\phi(x)$ in terms of Bessel functions as

$$\phi(x)=\sum_{n=-\infty}^{\infty}a_nJ_{n+\nu}(x). \tag{7}$$

The shift in the order by ν is needed because the interaction on the right-hand side generates secular perturbations that must be resummed [17] (see also Ref. [34], Sec. 11.1). Substituting the expansion (7) into Eq. (6) leads to an infinite set of linear equations for the coefficients a_n:

$$\left[(n+\nu)^2-\left(l+\frac{1}{2}\right)^2\right]a_n=\frac{\kappa}{2(n+\nu+1)}a_{n+1}+\frac{\kappa}{2(n+\nu-1)}a_{n-1}, \tag{8}$$

for $-\infty < n < \infty$. By introducing $b_n = a_n/(n+\nu)$ and

$$f_l(n+\nu)=2(n+\nu)\left[(n+\nu)^2-\left(l+\tfrac{1}{2}\right)^2\right], \tag{9}$$

the linear equations can be put into the more symmetric form

$$\kappa b_{n-1}-f_l(n+\nu)b_n+\kappa b_{n+1}=0,\quad -\infty<n<\infty. \tag{10}$$

Following Refs. [18,19] (see also Ref. [29]), one can solve these equations to get a representation of the ratios of coefficients in terms of a continued fraction. For positive n, I define the ratios

$$R_n=\frac{b_n}{\kappa b_{n-1}}. \tag{11}$$

These satisfy the recurrence relation

$$R_n=\frac{1}{f_l(n+\nu)-\kappa^2R_{n+1}}. \tag{12}$$

In terms of these, the b_n can all be related to b_0 and expressed as

$$b_n=\left(\prod_{m=1}^{n}R_m\right)\kappa^nb_0. \tag{13}$$

In a similar way, for negative n, I define

$$\overline{R}_n=\frac{b_{-n}}{\kappa b_{-(n-1)}}, \tag{14}$$

and these satisfy

$$\overline{R}_n=\frac{1}{f_l(-n+\nu)-\kappa^2\overline{R}_{n+1}}. \tag{15}$$

The corresponding coefficients can be written as

$$b_{-n}=\left(\prod_{m=1}^{n}\overline{R}_m\right)\kappa^nb_0. \tag{16}$$

Using these results in Eq. (10), with $n = 0$, gives

$$f_l(\nu)-\kappa^2\left(R_1+\overline{R}_1\right)=0. \tag{17}$$

This is a nonlinear eigenvalue equation which determines the shift ν. Alternatively, if one is simply interested in the value of ν, one can look for the zeros of the infinite-dimensional Hill determinant of the coefficients in Eq. (10) (see Ref. [34], Sec. 7.5). If the set of equations is truncated to a finite number, this can be done straightforwardly with the aid of *Mathematica* [35]. Taking $|n| \lesssim 20$ is sufficient to determine the zeros to six significant figures, at least for small angular momenta ($l \lesssim 5$).

The resulting eigenvalue equation, in either version, is an even function of κ; hence, the roots of the equation are the same for both repulsive and attractive potentials of the same strength. From the fact that $f_l(\nu)$ is an odd function, it follows that

$$\overline{R}_n(-\nu)=-R_n(\nu). \tag{18}$$

Hence, if ν is a solution of Eq. (17), so is ν. Also, the roots are periodic under addition of any integer to ν. From now on, I shall use ν to denote the root whose real part lies between l and $l+\frac{1}{2}$. This ensures that coefficient a_0 is large in the expansion of the corresponding solution, $\phi^{(+)}(x)$, at least for small κ. A second, independent solution, $\phi^{(-)}(x)$, is obtained

[4] Other closely related methods exist for solving equations with inverse-power-law potentials. These are based on Laurent [29–31] or Bessel-product expansions [32,33].

by replacing ν by $-\nu$. From the symmetry of R_n, it follows that the coefficients in the two solutions are related by

$$a_{-n}^{(-)} = (-1)^n a_n^{(+)}, \tag{19}$$

if we choose

$$a_0^{(-)} = a_0^{(+)} \equiv a_0. \tag{20}$$

The coefficient a_0 can be fixed by requiring that the solutions have the standard asymptotic normalization of a Bessel function for large r.

Solutions to these equations can be obtained using perturbation theory if κ is small, corresponding to a weak coupling or, equivalently, low energies. This is just the Born expansion, and it leads to expressions for the solutions as power series in κ. The advantage of the Bessel expansion [17–19] is that it determines the radius of convergence of this series, as well as providing analytic forms for the solutions in nonperturbative cases.

For $\kappa = 0$, the order of the Bessel function is $\nu = l + \frac{1}{2}$, and the solutions of the radial Schrödinger equation are just spherical Bessel functions $j_l(pr)$. Consider next very small values of κ. For these, we can approximate the ratios by their leading-order expressions from Eqs. (11) and (15):

$$R_n \simeq \frac{1}{f_l(n+\nu)}, \qquad \overline{R}_n \simeq \frac{1}{f_l(-n+\nu)}, \tag{21}$$

and the eigenvalue equation (17) becomes

$$f_l(\nu) - \kappa^2 \left[\frac{1}{f_l(1+\nu)} + \frac{1}{f_l(-1+\nu)} \right] = 0. \tag{22}$$

Since the solution ν lies very close to $l + \frac{1}{2}$ in this limit, we can write

$$\nu = l + \tfrac{1}{2} - \delta\nu. \tag{23}$$

In partial waves with $l \geqslant 1$, we can approximate $f_l(n+\nu)$ by

$$\begin{aligned} f_l(\nu) &\simeq -4\left(l+\tfrac{1}{2}\right)^2 \delta\nu, \\ f_l(n+\nu) &\simeq 4n\left(l+n+\tfrac{1}{2}\right)\left(l+\tfrac{1}{2}(n+1)\right). \end{aligned} \tag{24}$$

Using these in the eigenvalue equation we get, at order κ^2,

$$\delta\nu = \frac{3}{16} \frac{\kappa^2}{\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) l(l+1)}. \tag{25}$$

In the corresponding solutions to the Schrödinger equation, only the coefficent a_0 is large.

For s waves, we need to be more careful since $f_0(-1+\nu)$ is small in the weak-coupling limit:

$$f_0(-1+\nu) \simeq -\delta\nu. \tag{26}$$

TABLE I. Critical values of the dimensionless coupling κ for which eigenvalues ν form degenerate pairs. These agree with the values of ϵ_{sc} in Table I of [19], where $\epsilon_{sc} = \kappa_c^2/4$.

l	κ_c
0	0.318058
1	2.51811
2	8.33342
3	19.6983
4	38.6026
5	67.0469

Using this in the eigenvalue equation, we find that the leading shift $\delta\nu$ is of order κ and is given by[5]

$$\nu = \tfrac{1}{2} - |\kappa|. \tag{27}$$

In the corresponding wave function, a_{-1} is of the same order in κ as a_0.

For larger values of κ we need to iterate the recurrence relations for the R_n but, provided κ is small enough, we can still expand the results as power series in κ. Similarly the solutions to the eigenvalue equation and hence the wave functions themselves can be found perturbatively as Born expansions in powers of κ [17]. Since $\kappa \propto pB_3$, these expansions describe systems with either weak coupling or low energy.

As $|\kappa|$ increases, these series converge more slowly and eventually a perturbative treatment becomes impossible. A definite upper bound on the value of κ for which this occurs can be found by following the behavior of the eigenvalues of Eq. (17). As $|\kappa|$ increases, pairs of eigenvalues approach integer values from above and below until, at some critical value, they form degenerate pairs. Then, for larger values of $|\kappa|$, they move off into the complex plane [19]. The presence of a square-root branch point where this happens means that eigenvalue ν cannot be expanded in powers of κ above this critical value, κ_c. If the eigenvalue cannot be expanded in this way then neither can the solutions to the set of linear equations (10), and hence κ_c also provides an upper limit on the convergence of any perturbative expansion of the solutions.

The values of κ_c for low partial waves are listed in Table I. For large orbital angular momentum ($l \gtrsim 20$) κ_c grows roughly as $0.3l$.[3] This can be understood if nonperturbative behavior sets in when waves can penetrate the centrifugal barrier to radii where the $1/r^3$ and centrifugal potentials are roughly equal. To estimate where this occurs we can set

$$E \simeq \frac{l(l+1)}{M_N r^2} \simeq \frac{B_3}{r^3}. \tag{28}$$

[5]Compare Ref. [26] Sec. 2.5.2, noting that I have made a different choice from the multiple roots of the equation for ν. For s-wave scattering by a pure $1/r^3$ potential, this dependence of the order of the Bessel functions on κ leads to nonanalytic terms in the phase shift [19,36,37]. However, these are not present for OPE, even in the chiral limit, since the tensor potential vanishes in s waves.

This leads to

$$\kappa \simeq [l(l+1)]^{3/2}, \tag{29}$$

which is consistent with the observed growth with l.

This dependence of κ_c on the orbital angular momentum demonstrates that it is short-distance physics that leads to the breakdown of perturbation theory: the wave function has to penetrate the centrifugal barrier before it can "see" the singular core. The radius at which this happens is, from Eq. (28), of the order of

$$r_0 \simeq \frac{\beta_3}{l(l+1)}, \tag{30}$$

where

$$\beta_3 = M_N B_3 \tag{31}$$

is the length scale associated with the strength of the potential. These estimates of the critical value of κ and the associated radius fail for small angular momenta. In particular, κ_c is nonzero for $l = 0$, because the $1/r^3$ potential is short ranged in the sense that, apart from a single logarithmic term, its low-energy scattering amplitude can be expanded in powers of the energy [19,36,37] (unlike the Coulomb or $1/r^2$ potentials). Hence, a minimum momentum is required before its singular nature can be resolved.

In the realistic case of finite pion mass, the long-range tail of potential is replaced by an exponential falloff. Since the breakdown of perturbation theory is a short-range effect, it should not be altered qualitatively by a nonzero m_π, especially in high partial waves. Provided the radius r_0 is much smaller than $1/m_\pi$, the finite mass will not even change the critical value of κ significantly. In lower waves, where r_0 is comparable to or less than $1/m_\pi$, momenta of the order of m_π are needed before the singularity is resolved and nonperturbative behavior sets in. Hence, in channels where the momentum corresponding to κ_c is less than m_π, such nonperturbative effects are still expected for momenta of order m_π, but not for much lower momenta.

Because the forms of the wave functions at short distances will be an important ingredient in the RG analysis below, I will outline their basic features here. More details of the solutions can be found in Ref. [19] for the case of repulsive $1/r^3$ potentials. For small r, we can use the WKB approximation to find their forms. In the case of a repulsive potential, this shows that the solutions have exponential dependence on $\sqrt{\kappa/x}$, or equivalently $\sqrt{\beta_3/r}$. In dimensionless form, the small-x solutions are

$$\phi^{(\pm)}(x) \sim A^{(\pm)}(\kappa) x^{1/4} \exp\left[2\sqrt{\frac{\kappa}{x}}\right] + B^{(\pm)}(\kappa) x^{1/4} \exp\left[-2\sqrt{\frac{\kappa}{x}}\right]. \tag{32}$$

The determination of the coefficient A of the dominant piece of a solution requires careful asymptotic analysis of the series (7). The important terms at small x are those for large negative values of n. These can be summed using Laplace's method (see Ref. [34], Sec. 6.7). Under the analytic continuation $x \to e^{i2\pi} x$, the dominant and subdominant pieces of the solution exchange roles (an example of Stokes's phenomenon—see Ref. [34], Sec. 3.7). This shows that their coefficients are related by

$$B^{(\pm)} = -ie^{\pm i2\pi\nu} A^{(\pm)}. \tag{33}$$

Having found the small-x forms of the two independent solutions $\phi^{(\pm)}(x)$, we can then build the regular solution (which behaves like $x^{1/4} \exp[-2\sqrt{\kappa/x}]$) as a linear combination of them. These regular solutions form a complete, orthogonal set of basis functions for the RG analysis of the short-range physics.

In the case of an attractive inverse-cube potential, we need to be more careful. Since, as already noted, ν and the ratios of determinants are independent of the sign of the potential, solutions for this case can be obtained by replacing κ by $-\kappa$ in the recursion relations for the ratios of coefficients a_n. Alternatively one can make an analytic continuation $x \to e^{i\pi} x$ of the solutions already found for the repulsive case. The latter method shows that the solutions have the small-x forms

$$\phi^{(\pm)}(x) \sim C^{(\pm)}(\kappa) x^{1/4} \cos\left[2\sqrt{\frac{\kappa}{x}} + \left(\pm\nu - \frac{1}{4}\right)\pi\right], \tag{34}$$

where I have now defined the dimensionless coupling to be positive-definite: $\kappa = pM_N|B_3|$. The continuation also shows that the $C(\kappa)$ are related by

$$C^{(\pm)} = 2A^{(\pm)}, \tag{35}$$

to the coefficients $A(\kappa)$ of the dominant pieces of solutions with the same ν for a repulsive potential of the same strength.

Both of the solutions for an attractive potential display oscillatory behavior as $x \to 0$, and so any linear combination of them is an equally good solution. For the RG analysis, we need a well-defined set of orthogonal basis functions. As in the case of an attractive inverse-square potential,[6] this can be obtained if we choose a self-adjoint extension of the original Hamiltonian. In practice this means fixing the phase of these short-distance oscillations [25,38]. This phase should be independent of energy to form an extension whose eigenfunctions are orthogonal. In essence one short-distance parameter, the leading energy-independent term of the effective potential, has been used to provide a well-defined set of DW's of the long-range potential. Clearly this leads to a redundancy in the parametrization: a different choice of extension can be compensated by changing the leading term in the short-range potential [23]. However, any energy or momentum dependence due to short-range physics can be described entirely by higher-order terms in that potential. Note that here I am requiring orthogonality simply in order to generate a suitable DW basis for studying the effects of the short-range interactions. This is in contrast to the approach developed in Refs. [16,24,25], which imposes an orthogonality condition on the full wave functions, thus leading to very strong constraints on the short-distance interactions.

The forms of the short-distance wave functions for attractive and repulsive potentials look very different, depending

[6] See Refs. [23,39] and references therein for more discussion of this potential.

sinusoidal or exponentially on $\sqrt{\beta_3/r}$. However, we shall see that it is their power-law radial dependence that controls RG flow of short-distance interactions, and this is the same for both cases. Although the wave functions for the attractive potential oscillate at short distances, these oscillations depend on a scale, β_3, unlike the analogous ones found for the inverse-square potential. The scale-free oscillations found there and in the corresponding three-body systems lead to limit cycles in the RG flow [23,38–43], but here the scale dependence of the $1/r^3$ potential means that we should not expect to find similar limit cycles.

B. Coupled channels

The solution of the Schrödinger equation for the coupled spin-triplet channels proceeds along very similar lines. Using the matrix elements of S_{12} in the two-component basis of waves with $l = j \pm 1$ [28], the chiral limit of the tensor potential can be written in the form

$$\mathbf{V}_{\pi T}(r) = \frac{1}{2j+1}\begin{pmatrix} -2(j-1) & 6\sqrt{j(j+1)} \\ 6\sqrt{j(j+1)} & -2(j+2) \end{pmatrix}\frac{B_T}{r^3}, \quad (36)$$

where $B_T = -3/(M_N\lambda_\pi)$ for isospin-singlet waves, with $l = j \pm 1$ even; and $B_T = 1/(M_N\lambda_\pi)$ for isospin triplets, with $l = j \pm 1$ odd. Rescaling the equation as above, it can be written in a dimensionless form similar to Eq. (6):

$$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + x^2\right]\boldsymbol{\phi} - \left(\mathbf{L}_j + \frac{1}{2}\mathbf{1}\right)^2\boldsymbol{\phi} = \frac{1}{x}\mathbf{K}_j\boldsymbol{\phi}, \quad (37)$$

where the 2×2 matrices are

$$\begin{aligned} \mathbf{L}_j &= \begin{pmatrix} j-1 & 0 \\ 0 & j+1 \end{pmatrix}, \\ \mathbf{K}_j &= \frac{\kappa_T}{2j+1}\begin{pmatrix} -2(j-1) & 6\sqrt{j(j+1)} \\ 6\sqrt{j(j+1)} & -2(j+2) \end{pmatrix}, \end{aligned} \quad (38)$$

and the dimensionless combination of momentum and coupling strength is

$$\begin{aligned} \kappa_T &= pM_N B_T \\ &= \begin{cases} -3p/\lambda_\pi, & \text{isospin singlet,} \\ +p/\lambda_\pi, & \text{isospin triplet.} \end{cases} \end{aligned} \quad (39)$$

The solutions to these equations can be expanded in Bessel functions as

$$\boldsymbol{\phi}(x) = \sum_{n=-\infty}^{\infty} \mathbf{a}_n J_{n+\nu}(x), \quad (40)$$

where $\mathbf{a}_n$ are two-component vectors. Substituting this into Eq. (37) leads to an infinite set of linear equations. As in the uncoupled case, these can be put into a symmetric form by defining $\mathbf{b}_n = \mathbf{a}_n/(n+\nu)$ and

$$\mathbf{F}(n+\nu) = \begin{pmatrix} f_{j-1}(n+\nu) & 0 \\ 0 & f_{j+1}(n+\nu) \end{pmatrix}. \quad (41)$$

The resulting equations are then

$$\mathbf{K}\mathbf{b}_{n-1} - \mathbf{F}(n+\nu)\mathbf{b}_n + \mathbf{K}\mathbf{b}_{n+1} = 0, \quad -\infty < n < \infty. \quad (42)$$

If we write

$$\mathbf{b}_n = \mathbf{R}_n\mathbf{K}\mathbf{b}_{n-1}, \quad (43)$$

then the $\mathbf{R}_n$ satisfy the recurrence relation

$$\mathbf{R}_n = \left[\mathbf{F}(n+\nu) - \mathbf{K}\mathbf{R}_{n-1}\mathbf{K}\right]^{-1}. \quad (44)$$

In terms of these matrices, the coefficients $\mathbf{b}_n$ for $n \geqslant 1$ are

$$\mathbf{b}_n = \left(\prod_{m=1}^{\infty}\mathbf{R}_m\mathbf{K}\right)\mathbf{b}_0, \quad (45)$$

where the matrix product should be read as starting with $m = 1$ at the right. Similarly we can write the coefficients $\mathbf{b}_{-n}$ as

$$\mathbf{b}_{-n} = \left(\prod_{m=1}^{\infty}\overline{\mathbf{R}}_m\mathbf{K}\right)\mathbf{b}_0, \quad (46)$$

where the $\overline{\mathbf{R}}_n$ are given by

$$\overline{\mathbf{R}}_n = \left[\mathbf{F}(-n+\nu) - \mathbf{K}\overline{\mathbf{R}}_{n-1}\mathbf{K}\right]^{-1}. \quad (47)$$

Consistency of the equation for $n = 0$ requires that ν satisfy

$$\det\left[\mathbf{F}(\nu) - \mathbf{K}\left(\mathbf{R}_1 + \overline{\mathbf{R}}_1\right)\mathbf{K}\right] = 0. \quad (48)$$

This is equivalent to the Hill determinant corresponding to Eq. (42). As for the simple inverse-cube potential, the roots of this equation are symmetric in ν and periodic.

The 2×2 nature of the problem means that this eigenvalue equation has two roots ν_1 and ν_2 with real parts between l and $l + \frac{1}{2}$. The solutions to the Schrödinger equation (37) for these roots will be denoted by $\boldsymbol{\phi}^{(1,+)}(x)$ and $\boldsymbol{\phi}^{(2,+)}(x)$, respectively. There are also two other independent solutions $\boldsymbol{\phi}^{(1,-)}(x)$ and $\boldsymbol{\phi}^{(2,-)}(x)$, corresponding to the roots $-\nu_1$ and $-\nu_2$. The starting coefficients $\mathbf{b}_0^{(i,\pm)}$ must be eigenvectors of the $n = 0$ equation

$$\left[\mathbf{F} - \mathbf{K}(\mathbf{R}_1 + \overline{\mathbf{R}}_1)\mathbf{K}\right]_{\nu=\nu_i}\mathbf{b}_0^{(i,\pm)} = 0. \quad (49)$$

Since $\mathbf{F}(n+\nu)$ is an odd function, the matrices for positive and negative roots are related by

$$\overline{\mathbf{R}}_n(-\nu) = -\mathbf{R}_n(\nu). \quad (50)$$

If we choose the starting coefficients such that

$$\mathbf{a}_0^{(i,-)} = \mathbf{a}_0^{(i,+)} \equiv \mathbf{a}_0^{(i)}, \quad (51)$$

then the coefficients in the pairs of solutions are related by

$$\mathbf{a}_{-n}^{(i,-)} = (-1)^n\mathbf{a}_n^{(i,+)}. \quad (52)$$

The $\mathbf{a}_0^{(i)}$ can again be fixed using the asymptotic normalization of the solutions.

As for the case of uncoupled channels, the eigenvalues ν move into the complex plane for large enough values of κ_T, and the solutions can no longer be expanded perturbatively. The critical values for which this happens are listed in Table II. The coupled nature of the equations means that there are in general two of these values. The one exception is $j = 0$,

TABLE II. Critical values of the dimensionless coupling κ_T for which eigenvalues ν form degenerate pairs.

j	l	κ_{c1}	κ_{c2}
0	1	0.629528	
1	0,2	0.683495	2.48290
2	1,3	1.61857	6.91983
3	2,4	3.95647	14.3624
4	3,5	8.02206	23.2001

where only $l = 1$ is possible, and the critical value of κ_T is just a quarter of the one in Table I for a P wave.

The same methods outlined above can be used to find the coefficient vectors $\mathbf{a}_n$ and hence to construct solutions to the radial Schrödinger equation (37). Again, the full forms of these are not needed here, just their short-distance behaviors. These can be obtained by applying the WKB method to the eigenchannels of the potential. Since the eigenvalues of $\mathbf{K}_j$ are $+2\kappa_T$ and $-4\kappa_T$, one of these channels is repulsive and one attractive. For small x, all the solutions are mixtures of exponential and sinusoidal pieces, with the form

$$\begin{aligned}\boldsymbol{\phi}(x) \sim{}& \mathbf{e}_+ A(\kappa_T) x^{1/4} \exp\left[2\sqrt{\frac{\kappa_+}{x}}\right] \\ &+ \mathbf{e}_+ B(\kappa_T) x^{1/4} \exp\left[-2\sqrt{\frac{\kappa_+}{x}}\right] \\ &+ \mathbf{e}_- C(\kappa_T) x^{1/4} \cos\left[2\sqrt{\frac{\kappa_-}{x}} + \left(\pm\nu - \frac{1}{4}\right)\pi\right], \end{aligned} \quad (53)$$

where κ_+ and $-\kappa_-$ are the eigenvalues of $\mathbf{K}_j$,

$$\kappa_+ = \begin{cases} 12p/\lambda_\pi, \\ 2p/\lambda_\pi, \end{cases} \qquad \kappa_- = \begin{cases} 6p/\lambda_\pi, & \text{isospin singlet}, \\ 4p/\lambda_\pi, & \text{isospin triplet}, \end{cases} \quad (54)$$

and $\mathbf{e}_\pm$ are the corresponding eigenvectors

$$\begin{aligned} \mathbf{e}_+ &= \frac{1}{\sqrt{2j+1}} \begin{pmatrix} \sqrt{j+1} \\ \sqrt{j} \end{pmatrix}, \\ \mathbf{e}_- &= \frac{1}{\sqrt{2j+1}} \begin{pmatrix} \sqrt{j} \\ -\sqrt{j+1} \end{pmatrix}, \end{aligned} \quad (55)$$

for an isospin triplet channel, and *vice versa* for a singlet. As in the single-channel case, analytic continuation in x can be used to relate $B(\kappa_T)$ to $A(\kappa_T)$, and to express $C(\kappa_T)$ in terms of the $A(\kappa_T)$ of the solution for a repulsive potential.

A basis set of physical, orthogonal solutions can be formed by taking two linear combinations of these four independent solutions at each energy. These must be regular, with no admixture of the divergent $x^{1/4}\exp[2\sqrt{\kappa_+/x}]$ piece in the repulsive channel, as discussed in Refs. [16,25] for the ${}^3S_1$–${}^3D_1$ waves. As for the attractive inverse-cube potential, one energy-independent parameter is also needed to fix the phase of the oscillations in the attractive channel [13]. Although these solutions contain both oscillatory and exponentially decreasing pieces at small x, what matters in the RG analysis will be their power-law behavior, which is the same as for the uncoupled $1/r^3$ case above.

TABLE III. Critical values of the relative momentum at which pairs of eigenvalues become degenerate and hence the tensor potential cannot be treated perturbatively.

Channel	p_c (Mev)
${}^3S_1$–${}^3D_1$	66
${}^3P_0$	182
${}^3P_1$	365
${}^3P_2$–${}^3F_2$	470
${}^3D_2$	403
${}^3D_3$–${}^3G_3$	382
${}^3F_3$	2860
${}^3F_4$–${}^3H_4$	2330
${}^3G_4$	1870

C. Critical momenta

For the one-pion-exchange potential of interest here, the strengths of the tensor interaction in the various channels are all fixed in terms of the scale λ_π introduced in Eq. (1). The critical values of the dimensionless coupling can then be converted into critical values of the relative momentum for each scattering channel. Taking $\lambda_\pi = 290$ MeV leads to the critical momenta listed in Table III. These are the maximum values of the momenta for which one could attempt to construct a perturbative expansion of the solutions. In practice, one would expect such an expansion to be sufficiently convergent to be useful only for momenta well below these values.

From the table we see that in two channels, ${}^3S_1$–${}^3D_1$ and ${}^3P_0$, the critical values are less than or of the order of m_π. These are obtained from the chiral limit of the OPE; the values for finite m_π will be somewhat higher. However, they will still be of the order of m_π since, as discussed above, the exponential falloff cannot make the radius at which the waves probe the singular core much smaller than $1/m_\pi$. It is thus not surprising that Fleming, Mehen, and Stewart [12] found that the perturbative KSW approach fails for these cases. In the other P and D waves, the values are low enough, ~400 MeV, to suggest that these should also be treated nonperturbatively at the energies of interest for nuclear physics. Again, this is in accord with the findings of Ref. [9].

Between the D and F waves, the critical momenta jump by a factor of about 6. Two effects contribute to this: the critical dimensionless couplings are about twice as large for the F waves, and their physical couplings are three times smaller since the F waves are isospin triplets. Hence for the F waves and above that do not couple to P or D waves, the breakdown scales are well above 1 GeV. In these cases it should be possible to treat OPE perturbatively. A similar conclusion is reached in Ref. [15], but on the basis of very different arguments which rely heavily on keeping m_π finite.

III. RENORMALIZATION GROUP ANALYSIS

The long-range pion-exchange physics in low-energy nuclear EFT's can be calculated from ChPT [1–3]. The short-

range physics is then parametrized in terms of contact interactions. If the EFT is to have any predictive power, we must be able to organize these interactions systematically according to some power counting. In weakly interacting systems, one can do this by simply counting powers of the low-energy scales, generically denoted here by Q. This naive dimensional analysis leads to the power counting used in ChPT for mesons and single nucleons [6]. In contrast, nonperturbative effects can introduce new low-energy scales and generate anomalous dimensions for terms in the short-distance potential. As a result, these terms may not scale as predicted by naive dimensional analysis, so the power counting can be quite different.

The RG provides a general and powerful tool for analyzing scale dependences, particularly in nonperturbative systems. In the context of few-body systems, it is convenient to express this in the form of a differential equation describing the "flow" of the short-range potential as the cutoff is varied [21–23]. This equation can be constructed as follows. First, we apply a floating cutoff at some scale Λ that lies between the low-energy scales of interest and the scale of the underlying physics Λ_0. This assumes that these scales are well separated; if they are not, the expansion in powers of Q/Λ_0 will not converge and we shall not be able to construct a useful EFT. Second, we demand that physical observables be independent of the cutoff Λ, since its value is arbitrary. As a result, the couplings in our effective potential must depend on Λ. This dependence ensures, in particular, that the couplings cancel any parts of loop integrals that diverge for large Λ. Finally, we rescale the theory by expressing all dimensioned quantities in units of Λ. Powers of Λ can then be used to determine the net powers of low-energy scales in the terms in the potential.

Having constructed the RG equation for the rescaled potential, we can look for fixed points. These are Λ-independent solutions that can form the end points of the RG flow. They describe scale-free systems. Perturbations around a fixed point can be expanded in eigenfunctions of the linearized RG equation that scale with definite powers of Λ, given by the eigenvalues of this equation. These perturbations can be classified according to their eigenvalues as relevant, irrelevant, or marginal.

Marginal terms ("renormalizable" ones in field-theoretic language) have no power-law dependence on the cutoff after rescaling, although in general they can depend logarithmically on it. Like the fixed-point potential, these terms are important at all scales. They are the leading-order terms in the WvK scheme, of order $1/Q$ in low-energy scales. Since the loop integrals in the Lippmann-Schwinger equation are of order Q, all iterations of these terms are of the same order, and hence they need to be treated nonperturbatively. Irrelevant (or "nonrenormalizable") terms vanish as positive powers of Λ as $\Lambda \to 0$. These are higher-order terms which become weak at low energies and so can be treated perturbatively. Lastly, relevant (or "super-renormalizable") terms grow as negative powers of Λ. These are unimportant at high energies but become increasingly important at low energies, ultimately changing the nature of the low-energy EFT. If such terms are present, a fixed point is unstable and, for low-enough values of Λ, the theory will ultimately flow to a different point. An example of this is provided by the scattering-length term in the pionless EFT [21].

The RG approach developed in Ref. [22] assumes that a two-body potential consists of a known long-range piece V_L and a short piece V_S which parametrizes the physics that lies outside the scope of our effective theory. This method starts by using the "two-potential trick" [11] to define a T matrix describing scattering between distorted waves (DW's) of the long-range potential. The Hilbert space is then reduced by imposing a cutoff on the basis of DW's at momentum Λ. For this cutoff to lead to a well-defined space, the DW's should form a complete orthogonal set of basis functions. This requires that the long-range potential, and any parameter needed to form a self-adjoint extension of it, should be independent of energy.

Demanding that the (fully off-shell) T matrix be independent of the cutoff Λ leads to the differential equation

$$\frac{\partial V_S}{\partial \Lambda} = -V_S \frac{\partial G_L}{\partial \Lambda} V_S, \tag{56}$$

where G_L is the DW Green's function for the long-range potential. To treat cases where the DW's vanish or diverge at the origin as a result of nonperturbative effects of V_L, the short-range potential is taken to have the δ-shell form

$$V_S(p, \lambda, \Lambda, R; r) = V_S(p, \lambda, \Lambda, R) \frac{\delta(r - R)}{4\pi R^2}, \tag{57}$$

where λ denotes a generic low-energy scale associated with V_L. The radius R provides a second regulator here, which can be thought of as a "factorization scale" separating the long-range physics which lies within the domain of our EFT from the unknown nonperturbative physics at shorter distances. This second scale is introduced because it leads to an RG equation from which the scaling of the potential can be deduced in a particularly transparent way. In approaches based on a simple momentum cutoff, as in Ref. [15], or a coordinate-space regulator, as in Refs. [16,24,25], a single scale plays both roles of renormalization and factorization.[7] For present purposes, R should be chosen to be small enough such that the wave functions have reached a common, energy-independent form.

With this δ-shell form for V_S, Eq. (56) becomes

$$\frac{\partial V_S}{\partial \Lambda} = -\frac{M_N}{2\pi^2} |\psi_L(\Lambda, R)|^2 \frac{\Lambda^2}{p^2 - \Lambda^2} V_S^2(p, \lambda, \Lambda, R), \tag{58}$$

where $\psi_L(p, r)$ are the DW's for V_L and I have assumed that it does not produce any bound states. Taking the "factorization" radius R to be small enough that it lies in the asymptotic region, the DW's can all be written in the form

$$\psi_L(p, R) \sim \mathcal{N}(\lambda/p)(pR)^{(\sigma-1)/2} F(\lambda R). \tag{59}$$

This is a slight generalization of the cases considered in Ref. [22] to potentials that generate a non-power-law dependence on R in this region, described by the function $F(\lambda R)$.

[7] This is analogous to the choice that is often made in studies of the QCD evolution of structure functions.

Following the general method of Ref. [22], I introduce a rescaled on-shell momentum, $\hat{p} = p/\Lambda$. Other low-energy variables are treated similarly, and I then define the rescaled potential as

$$\hat{V}_S(\hat{p}, \hat{\lambda}, \Lambda) = \frac{M_N \Lambda}{2\pi^2}(\Lambda R)^{\sigma-1}|F(\Lambda\hat{\lambda}R)|^2 V_S(\Lambda\hat{p}, \Lambda\hat{\lambda}, \Lambda, R). \tag{60}$$

Inserting (60) and (59) into the differential equation (58) gives the RG equation for $\hat{V}_S$:

$$\Lambda\frac{\partial\hat{V}_S}{\partial\Lambda} = \hat{p}\frac{\partial\hat{V}_S}{\partial\hat{p}} + \hat{\lambda}\frac{\partial\hat{V}_S}{\partial\hat{\lambda}} + \sigma\hat{V}_S + \frac{|\mathcal{N}(\hat{\lambda})|^2}{1-\hat{p}^2}\hat{V}_S^2. \tag{61}$$

Since the function $|F(\Lambda\hat{\lambda}R)|^2$ depends on the product $\Lambda\hat{\lambda}$, its derivatives cancel in this equation. The evolution of $\hat{V}_S$ thus depends only on the power-law part of the DW's.

Note that in Eq. (60) I have implicitly demanded V_S depend on R in such a way that the rescaled potential is independent of R. This implies that $|\psi_L(p, R)|^2 V_S$ is also independent of R, and hence it ensures that scattering observables do not depend on this arbitrary radius. This is very similar to the philosophy adopted by Valderrama and Arriola [16,24,25], except that there the coordinate-space regulator is the only one. However, the complicated dependence of $F(\lambda R)$ on R means that the scaling behavior of the terms in the potential cannot easily be determined from this condition. In the present approach, the radial regulator separates off the regime of nonperturbative short-distance physics, while the momentum cutoff Λ is used to analyze the scale dependence. For this to work, R must be in the region where the DW's $\psi_L(p, R)$ have reached their energy-independent asymptotic form, at least for $p \leqslant \Lambda$.

The starting points for analyzing the scale dependence of the potential are the fixed points of the RG, solutions to Eq. (61) that are independent of Λ. Expanding $\hat{V}_S$ around one of these points and keeping only linear terms in the RG equation, we get an equation whose eigenfunctions scale with definite powers of Λ. Since the rescaling means that the power of Λ counts the net power of all low-energy scales in each term, these eigenvalues determine the power counting for the terms in the potential. For example, in the case of a pure short-range potential [21], the power counting for the expansion around the trivial fixed point is just the one originally proposed by Weinberg [1]. There is also a nontrivial fixed point, which describes a system with an infinite scattering length. The expansion around this point can be organized according to the power counting developed in Refs. [7–9]. The terms in this are in one-to-one correspondence with the terms in the effective-range expansion [10,11].

The form of the RG equation above can easily be used to analyze scale dependences in the vicinity of the trivial fixed point, $\hat{V}_S = 0$. To find and study nontrivial fixed points, it is more convenient to convert it into a linear equation for $1/\hat{V}_S$ [22]:

$$\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{1}{\hat{V}_S}\right) = \hat{p}\frac{\partial}{\partial\hat{p}}\left(\frac{1}{\hat{V}_S}\right) + \hat{\lambda}\frac{\partial}{\partial\hat{\lambda}}\left(\frac{1}{\hat{V}_S}\right) - \sigma\frac{1}{\hat{V}_S} - \frac{|\mathcal{N}(\hat{\lambda})|^2}{1-\hat{p}^2}. \tag{62}$$

The detailed forms of the fixed-point solutions to this equation are not necessary to determine the power countings for perturbations around them. If needed, they can be found by applying the methods of Refs. [22,23,44]. These rely on the fact that the basic loop integral from the Lippmann-Schwinger equation satisfies the Λ-independent version of Eq. (62). This integral contains a piece that is a nonanalytic function of $\hat{p}/\hat{\lambda}$. Cancelling off this piece then leaves a well-behaved solution, $\hat{V}_{S0}$, which is analytic in the low-energy variables, $\hat{p}$ and $\hat{\lambda}$.

So far I have discussed only energy-dependent perturbations. More generally, as an off-shell quantity, the short-range effective potential can also depend on momenta. This is equivalent to including terms with spatial derivatives in V_S. For the expansion of a pure short-range potential around the trivial fixed point, the power counting is the same for both energy- and momentum-dependent terms. Hence, one can "use the equation of motion," to exchange one dependence for the other [45].

The expansion around a nontrivial fixed point is more complicated. There, the equation of motion involves the fixed-point potential (which can include both long- and short-range pieces) and so purely momentum-dependent terms are not eigenfunctions of the linearized RG equation, in contrast to the energy-dependent ones discussed above. The coefficients of the latter terms are directly related to on-shell scattering observables through an effective-range expansion, as discussed in Refs. [21,22]. There are also terms with mixed momentum and energy dependences that scale with definite powers of Λ. However, these affect only the off-shell behavior of the scattering amplitude. Furthermore, they are of higher order than the corresponding energy-dependent terms [21]. Hence, if one expands a momentum-dependent term in eigenfunctions of the RG, the scaling with Λ of the dominant piece is governed by the eigenvalue of the corresponding energy-dependent term. This is why the analyses of Refs. [7–9], using momentum-dependent potentials, arrive at the same power counting as Ref. [21], which uses energy-dependent ones.

A. Central OPE

The first step in constructing any EFT is to identify all the important low-momentum scales. To illustrate the choices involved in the nucleon-nucleon system, I consider first the central piece of the OPE potential and summarize the relevant results from Ref. [22]. For scattering at energies of $\sim$100 MeV, the relative momentum and the pion mass obviously form two of these scales. For the central Yukawa potential, we can construct the scale $\alpha_\pi = m_\pi^2/\lambda_\pi \simeq 70$ MeV. In strict chiral power counting, this would be of order Q^2, since it contains two powers of m_π. However, if we choose to treat λ_π as an additional low-energy scale, then α_π is promoted to order Q.

To see the consequences of this in the context of the RG for the $^1S_0$ channel, we can multiply the effective Hamiltonian by M_N and define the dimensionless potential

$$\hat{V}_{\pi C}(r) = \frac{M_N V_{\pi C}(r)}{\Lambda^2}. \tag{63}$$

Then we need to express all low-energy scales in units of Λ. If we regard λ_π as a high-energy scale, then we define a rescaled on-shell momentum by $\hat{p} = p/\Lambda$, and similarly $\hat{m}_\pi = m_\pi/\Lambda$ and $\hat{r} = \Lambda r$.[8] The resulting rescaled potential

$$\hat{V}_{\pi C}(\hat{r}) = -\Lambda \frac{\hat{m}_\pi^2}{\lambda_\pi} \frac{e^{-\hat{m}_\pi \hat{r}}}{\hat{r}}, \tag{64}$$

is of order Λ, and hence is an irrelevant perturbation in the RG sense that it vanishes as $\Lambda \to 0$. This is the choice made in the KSW scheme. In contrast, if we treat λ_π as a low-energy scale and express it in units of Λ, writing $\lambda_\pi = \Lambda \hat{\lambda}_\pi$, then the rescaled potential is independent of Λ and so forms part of any fixed point of the RG. In Weinberg's power counting [1], this choice means that the potential (in momentum space) is of order $1/Q$.[9] It must thus be iterated to all orders when solving the Schrödinger equation, along with the leading contact interaction, and so this choice corresponds to the WvK scheme.

In the ${}^1S_0$ channel, the DW's of the Yukawa potential tend to constants as $R \to 0$ and so have the short-distance form

$$\psi_L(p, R) \sim \mathcal{N}(\alpha_\pi/p, m_\pi/p). \tag{65}$$

This corresponds to setting $\sigma = 1$ and $F(\lambda R) = 1$ in Eq. (59) above. The resulting RG equation for $\hat{V}_S$ is given by Eq. (61), with $\sigma = 1$ and three low-energy scales, $\hat{p}$, $\hat{\alpha}_\pi$, and $\hat{m}_\pi$.

The expansion of the potential around the trivial fixed point has the form

$$\hat{V}_S = \sum_{k,m,n} C_{kmn} \Lambda^\rho \hat{m}_\pi^{2k} \hat{\alpha}_\pi^m \hat{p}^{2n}. \tag{66}$$

The RG eigenvalues of these terms are $\rho = 2k + m + 2n + 1$ where k, m, and n are nonnegative integers. The corresponding power counting assigns them orders Q^d, where $d = \rho - 1$ [21,22]. As in the case of a pure short-range potential, their eigenvalues start at $\rho = 1$ and so they are all irrelevant. This would be the appropriate power counting if all short-range interactions in the ${}^1S_0$ channel were weak.

In the RG framework, the WvK treatment of this channel corresponds to an expansion of the short-distance potential around the nontrivial fixed point $\hat{V}_{S0}$. This can be written in the form

$$\frac{1}{\hat{V}_S} = \frac{1}{\hat{V}_{S0}} - \sum_{k,m,n} C_{kmn} \Lambda^\rho \hat{m}_\pi^{2k} \hat{\alpha}_\pi^m \hat{p}^{2n}. \tag{67}$$

The RG eigenvalues of these terms are $\rho = 2k + m + 2n - 1$, where k, m, and n are nonnegative integers. This is similar to the expansion around the nontrivial fixed point for a pure short-range potential [7,21] in that the leading perturbation is a relevant one, with eigenvalue $\rho = -1$. The terms are in one-to-one correspondence with the terms in a DW or "modified" effective-range expansion [10,47–49]. In it, all rapid energy dependence associated with the low-energy scales of OPE is factored out, to leave an amplitude whose energy dependence is controlled only by scales from the short-range physics.

In this channel, we therefore have a choice between the two schemes, both of which lead to consistent expansions of the low-energy physics. The KSW scheme suffers from poor convergence because of the smallness of λ_π; while the WvK one, by treating λ_π as a low-energy scale, converges better but lacks a clear connection with ChPT. The most immediate signal of the latter problem is the contact interaction needed to renormalize the logarithmic divergence produced by the $1/r$ singularity of the potential. This is proportional to α_π, which contains two powers of m_π in the chiral expansion but which must be treated as a single small scale in the the WvK scheme [22]. This is discussed by Beane *et al.* [13], who conclude that the KSW scheme should be used in the ${}^1S_0$ channel. However, it is worth stressing that the problem is a lack of consistency with the chiral power counting for other effective operators, not an internal inconsistency in the DW effective-range expansion.

B. Tensor OPE

The nonperturbative nature of the tensor piece of OPE at short distances means that the RG analysis is rather different than that for the central piece just discussed. In Sec. II, we saw that the wave functions in all partial waves (and for both attractive and repulsive potentials) have the same power-law dependence on $x = pr$ at short distances. If we choose R to be small enough that it lies in the asymptotic WKB region, then the uncoupled DW's can all be written in the form

$$\psi_L(p, R) \sim \mathcal{N}(\lambda_\pi/p)(pR)^{-1/4} F(\lambda_\pi R), \tag{68}$$

where $F(\lambda_\pi R)$ is the non-power-law part of the wave function. In the isospin-triplet waves, where the tensor potential is repulsive, this function is

$$F(\lambda_\pi R) = \exp\left[-2\sqrt{\frac{2}{\lambda_\pi R}}\right]. \tag{69}$$

In the isospin-singlet channels, we have

$$F(\lambda_\pi R) = \cos\left[2\sqrt{\frac{6}{\lambda_\pi R}} + \gamma\right], \tag{70}$$

where γ is an energy-independent phase. Here, for simplicity, I will continue to consider the chiral limit of the tensor OPE potential. Allowing for a finite m_π will not alter the form of the short-distance wave functions, but it will introduce a dependence on m_π/p into the normalization constants $\mathcal{N}$.

For the coupled channels, we have two independent solutions which satisfy the two boundary conditions as needed to generate an orthognal set of DW's, namely, that as $r \to 0$ the exponential piece should be regular and the oscillatory piece should have an energy-independent phase γ. These solutions

[8] This is just the coordinate-space version of the rescaling discussed above.

[9] An alternative way to arrive at this result is to assign the nucleon mass an order $1/Q$ in the power counting [1,46]. This also leads to λ_π being identified as a low-energy scale of order Q.

can be written as

$$\boldsymbol{\psi}_{Li}(p,R) \sim \mathcal{N}_{i+}(\lambda_\pi/p)(pR)^{-1/4}F_+(\lambda_\pi R)\mathbf{e}_+ + \mathcal{N}_{i-}(\lambda_\pi/p)(pR)^{-1/4}F_-(\lambda_\pi R)\mathbf{e}_-, \quad i=1,2, \tag{71}$$

where the short-distance forms of the wave functions are

$$F_+(\lambda_\pi R) = \exp\left[-2\sqrt{\frac{\kappa_+}{x}}\right], \qquad F_-(\lambda_\pi R) = \cos\left[2\sqrt{\frac{\kappa_-}{x}}+\gamma\right], \tag{72}$$

and $\kappa_\pm$ and $\mathbf{e}_\pm$ are the eigenvalues and eigenvectors of the potential matrix $\mathbf{K}_j$, defined in Eqs. (54) and (55).

Matrix elements of the short-range potential between DW's can be written as

$$\langle\psi_{Li}|V_S|\psi_{Lk}\rangle = \sum_{\alpha,\beta=\pm}\mathcal{N}_{i\alpha}(\lambda_\pi/p)^*F_\alpha(\lambda_\pi R)^* \times V_{S\alpha\beta}(p,\lambda_\pi,\Lambda,R)F_\beta(\lambda_\pi R)\mathcal{N}_{k\beta}(\lambda_\pi/p), \tag{73}$$

where the strength has been expressed as a matrix $\mathbf{V}_S(p,\lambda_\pi,\Lambda,R)$ using the basis of the eigenvectors $\mathbf{e}_\pm$.

By analogy with the single-channel case, I define a rescaled potential $\hat{\mathbf{V}}_S$ as a matrix with elements

$$\hat{V}_{S\alpha\beta} = \frac{M_N\Lambda}{2\pi^2}(\Lambda R)^{-1/2}F_\alpha(\lambda_\pi R)^*V_{S\alpha\beta}(p,\lambda_\pi,\Lambda,R)F_\beta(\lambda_\pi R). \tag{74}$$

It is also convenient to define the matrix $\mathbf{N}(\lambda_\pi/p)$ with elements

$$N_{\alpha i} = \mathcal{N}_{i\alpha}(\lambda_\pi/p). \tag{75}$$

Using this in the two-component version of Eq. (58) leads to the RG equation for $\hat{\mathbf{V}}_S$,

$$\Lambda\frac{\partial\hat{\mathbf{V}}_S}{\partial\Lambda} = \hat{p}\frac{\partial\hat{\mathbf{V}}_S}{\partial\hat{p}} + \hat{\lambda}_\pi\frac{\partial\hat{\mathbf{V}}_S}{\partial\hat{\lambda}_\pi} + \frac{1}{2}\hat{\mathbf{V}}_S + \frac{1}{1-\hat{p}^2}\hat{\mathbf{V}}_S\mathbf{N}(\hat{\lambda}_\pi)\mathbf{N}(\hat{\lambda}_\pi)^\dagger\hat{\mathbf{V}}_S. \tag{76}$$

Apart from the additional matrix structure, this equation has the same form as the RG equation (61) that governs the potential in the uncoupled channels. In particular, the coefficient σ, which arises from the power-law behavior of the wave functions at short distances, is the same in all cases (attractive, repulsive, and coupled). This means that the scaling behaviors of perturbations are the same in all cases, although obviously the forms of any nontrivial fixed points are different. In writing down these equations, I have ignored the possibility of bound states in the attractive channels. I discuss below the modifications needed to take account of these, but they do not affect the scaling behavior.

As always, each of these RG equations has a trivial fixed-point solution $\hat{V}_S = 0$. The expansion around this has the form

$$\hat{V}_S = \sum_{m,n} C_{mn}\Lambda^\rho\hat{\lambda}_\pi^m\hat{p}^{2n}, \tag{77}$$

where n and m are nonnegative integers, and the RG eigenvalues of the terms are $\rho = m+2n+\frac{1}{2}$. They are all irrelevant perturbations, in the sense that their eigenvalues are positive, and so the fixed point is stable. The appearance of noninteger anomalous dimensions should not be surprising in the context of the RG. The version developed in Ref. [22] shows that scaling is controlled by the power-law behavior of the wave functions at short distances and in general this is noninteger for potentials with an inverse-power-law form.[10] In the present case, half-integer values appear because $|\psi|^2 \sim (pR)^{-1/2}$.

For comparison, the terms in a pure short-range potential describing weak scattering in an S wave have RG eigenvalues $\rho = 2n+1$ [21]. From this we see that the corresponding terms in the presence of a $1/r^3$ potential have eigenvalues that are smaller by subtraction of $1/2$, and so they vanish more slowly as $\Lambda\to 0$. This means that the effects of these interactions have been enhanced by the $1/r^3$ potential. (In the language of the RG, they are more "relevant.") If we translate these results into the more usual power counting, we find that these terms have orders $Q^{m+2n-1/2}$, and so they have been promoted by half an order compared with the power counting obtained from naive dimensional analysis (Q^{m+2n}).

In the S, P, and D waves, the radii at which the OPE and centrifugal potentials become comparable are of the order of 1 fm. By choosing the factorization radius R to lie significantly inside this, we can ensure that the short-range potential acts in a region where the wave functions have reached their asymptotic WKB form for the $1/r^3$ potential. In the $^3P_1$ and $^3D_2$ channels, for example, this occurs for radii of the order of 0.1 fm or less. At these distances, the wave function in the $^3P_1$ channel is highly suppressed by the repulsive $1/r^3$ potential. Somewhat larger radii, of the order of 0.4 fm, can also be used since the wave functions still have energy-independent forms,[11] at least for energies up to about 250 MeV, but even here the $^3P_1$ wave function is already small, down by a factor of 5–10 compared to its size in the region 0.7–1 fm.

The corresponding short-range potential must therefore be enhanced by a large numerical factor if we choose R less than 0.4 fm. However, one should remember that the potential is not a physical quantity. In any observable, V_S always appears multiplied by two short-distance wave functions. Indeed the radius R is arbitrary, and the form of the R dependence of V_S was chosen to ensure that scattering observables are independent of it. It is the size of the physical effects of a term

[10] Indeed, other examples of noninteger anomalous dimensions have recently been found in the context of three-body systems [50].

[11] By neglecting the x^2 term on the left-hand side and changing variables to $y = \sqrt{\kappa/x}$, Eq. (6) can be put into the form of Bessel's equation. This shows that the energy-independent solutions can be expressed in terms of order-$2l+1$ Bessel functions of y. For large y, these tend to the asymptotic WKB expressions discussed above.

in V_S that matters and, in particular, whether these require that it be iterated to all orders or allow it to be treated perturbatively. Such questions are answered by the power counting that the RG analysis provides.

Note that despite also using a radial cutoff, the approach here is quite different from that of Valderrama and Arriola [16,24,25]: those authors consider the limit as their radial cutoff tends to zero, and so they set all irrelevant perturbations to zero. As a result, their predictions for scattering observables just depend on the long-range pion-exchange potential and a small number of short-distance parameters associated with any relevant or marginal terms.

The results above apply to all partial waves where tensor OPE is treated nonperturbatively. Hence in waves with nonzero orbital angular momentum, where the leading terms are naively of order Q^{2l}, the orders of these terms are much lower than dimensional analysis would suggest. This agrees with the conclusion of Nogga, Timmermans, and van Kolck [15] that, based on their numerical analysis, short-range terms must be promoted in channels where the tensor potential is attractive. The RG analysis here makes quantitative the degree of promotion involved by determining the power counting for all terms in the double expansion in powers of energy (p^2) and the coupling scale (λ_π). It also shows that the effect is present in repulsive as well as attractive channels.

The scattering in the ${}^3S_1$–${}^3D_1$ channel is strong at low energies, and so the trivial fixed point is not an appropriate starting point. Instead, we need to find a nontrivial fixed point and expand around it. This is most easily done by rewriting the RG equation as in Eq. (62). The linear nature of this equation makes it straightforward to find the perturbations around the fixed point that scale with definite powers of Λ. The resulting expansion is

$$\frac{1}{\hat{V}_S} = \frac{1}{\hat{V}_{S0}} - \sum_{m,n} C_{mn} \Lambda^{m+2n-1/2} \hat{\lambda}_\pi^m \hat{p}^{2n}, \qquad (78)$$

where n and m are again nonnegative integers. As in the case of a pure short-range potential [21], the nontrivial fixed point is unstable, with one negative eigenvalue. Terms in this expansion can be related to the terms in a DW effective-range expansion, analogously to the examples studied in Refs. [22,23]. The RG eigenvalue for a general term in Eq. (78) is $\rho = m + 2n - 1/2$, which should be compared with $\rho = 2n - 1$ for the pure short-range case [21]. This shows that the terms in the expansion here have been demoted by half an order (that is, they are less important) compared with the corresponding terms without the long-range potential.

Although strictly the RG equation (61) only applies to channels where the tensor force is repulsive, the scaling behavior is in fact the same for the attractive and coupled channels. The only difference is that attractive $1/r^3$ potentials give rise to deeply bound states that lie outside the domain of our EFT. We should therefore cut them off at $E = -\Lambda^2/M_N$, as in the case of the attractive inverse square potential [23]. This adds δ-function terms to the RG equation at the values of Λ where bound states fall outside the cutoff. These lead to step discontinuities in $\hat{V}_S$ at these points, which can be thought of as jumps to different branches of the fixed-point potential $\hat{V}_{S0}$.

The existence of multiple branches of the potential is a consequence of the oscillatory nature of the short-distance wave functions. In order to make these well defined, we had to choose a particular self-adjoint extension of the long-distance Hamiltonian by fixing the phase of these oscillations. As in the $1/r^2$ case, the energy-independent short-range interaction has the effect of changing the self-adjoint extension [23]. However, in that example, the scale-free nature of the potential and the associated Efimov effect [51] (an infinite tower of geometrically spaced bound states) mean that the different choices lie on a limit cycle of the RG. As a result, the leading short-range term forms a marginal perturbation which changes the starting point on that cycle. In contrast, only a discrete set of extensions of the $1/r^3$ Hamiltonian lead to scale-free systems with bound states at zero energy. The different branches of $\hat{V}_{S0}$ correspond to this set of extensions.

The use of the DW basis in this RG analysis makes it straightforward to expand the potential in terms of perturbations that scale with definite powers of Λ. In Ref. [15] the cutoff was applied to a plane-wave basis. Such a cutoff has two effects: it regulates the short-distance interaction, and it removes the singularity of the long-range potential at the origin. The second aspect means that changing the cutoff has the effect of changing the self-adjoint extension that determines the long-range behavior of the DW's. After fitting to low-energy scattering observables, the resulting short-distance potential displays dramatic oscillations due to the changing number of bound states of the long-range potential. Similar oscillations are also seen with the radial cutoff of Ref. [25]. However, with care, it should still be possible to determine the power counting using such cutoffs, as illustrated by the analysis of attractive three-body systems in Ref. [52].

In spin-triplet channels without low-energy bound or virtual states (in other words, all except ${}^3S_1$–${}^3D_1$), we can define the DW basis by picking an initial extension that does not produce a low-energy bound or virtual state, and then we can expand the short-range potential around the trivial fixed point. The terms in this expansion are all irrelevant. This implies that, provided we pick an initial extension that gives weak low-energy scattering, any dependence of the low-energy phase shifts on this choice must be small. Otherwise, the leading irrelevant perturbation could not be equivalent to a change in the extension.

This weak dependence of scattering observables on the choice of extension, at least well away from the ones that generate low-energy bound states, can be seen in the numerical results of Ref. [15], particularly in Figs. 10 and 12. These contain long "plateau regions" where the short-distance interactions are small and only weakly dependent on Λ. These values of the cutoff correspond to extensions that generate only deeply bound states, outside the scope of the EFT. In contrast, there are also narrow ranges of Λ where the regulated tensor potential produces a low-energy bound or virtual state. To describe the observed weak scattering, the potential in these regions must be supplemented by the nontrivial fixed point and its relevant (unstable) perturbation, hence the very large counterterms needed.

The results so far can be applied to the S, P, and D waves where, as we saw in Sec. II, the tensor OPE needs to be treated nonperturbatively. As discussed above, the wave functions in these channels attain their energy-independent short-distance forms for radii of 0.4 fm or less. These forms are controlled by the $1/r^3$ tensor potential, which is stronger than the centrifugal barrier for radii smaller than about 1 fm. In F waves and above, in contrast, the OPE and centrifugal potentials become comparable at radii of the order of 0.1–0.2 fm or less. The asymptotic forms of the waves controlled by the tensor potential are thus reached only for radii much less than 0.1 fm, far beyond the domain of validity of our EFT. Moreover, the amplitudes of the wave functions for such small radii will be strongly suppressed by the centrifugal barrier, at least for momenta of the order of m_π.[12] In effect, the strong centrifugal barrier in high partial waves "protects" low-energy waves from probing the nonperturbative region.

We can take advantage of this by choosing our factorization scale R for the high partial waves to be of the order of 0.4 fm, so that it lies in the region where the centrifugal potential dominates over both the tensor OPE and the on-shell energy. This allows us to treat OPE as a perturbation and to expand the DW's in Eq. (58) in powers of $V_{\pi T}$. In this region, the waves still have the r^l form produced by the centrifugal barrier and so scaling behavior of the short-range potential is the same as that in the presence of the $1/r^2$ centrifugal potential alone [22]. In particular, the resulting power counting is just the usual one given by naive dimensional analysis. Of course, at high enough energies, the waves will penetrate the barrier and this perturbative treatment will break down. However, in the high partial waves, the results in Sec. II C show that this only happens at scales that lie outside the domain of our low-energy effective theory.

IV. CONCLUSIONS

The chiral limit of the tensor OPE potential has a $1/r^3$ form. In this work, I have studied it using techniques developed in atomic physics for solving the Schrödinger equation with inverse-power-law potentials [17–19]. These lead to analytic solutions constructed as expansions in Bessel functions whose orders satisfy an eigenvalue equation. In each channel, there is a critical value of the product of the momentum and coupling strength above which these eigenvalues become complex. This sets a limit on the range of energies for which the tensor potential can be treated perturbatively. These values are the same for both attractive and repulsive $1/r^3$ potentials.

I have determined the critical dimensionless couplings for the tensor interaction in low-lying partial waves. In the ${}^3S_1$–${}^3D_1$ and ${}^3P_0$ channels, the corresponding breakdown scales are $\sim m_\pi$ or less. In the other P and D waves, the scales are of the order of 400 MeV. These results imply that for the energies relevant to nuclear physics, OPE should be treated nonperturbatively in these channels. They explain why Fleming, Mehen, and Stewart [12] found that the perturbative KSW treatment breaks down in these cases. In contrast, the scales for the higher partial waves all lie well above 1 GeV, and so perturbation theory should be valid for them.

In the context of the RG, the nonperturbative treatment of OPE can be justified if we identify the scale λ_π controlling its strength as a low-energy scale. The resulting RG analysis of the ${}^1S_0$ channel, where only the central Yukawa piece contributes, leads to power countings that are similar to those found for pure short-range interactions. There is a nontrivial fixed point which describes systems with strong scattering at low energies. The terms in the expansion around this have RG eigenvalues $\rho = -1, 0, +1, \ldots$ and so are of order Q^d, where $d = \rho - 1 = -2, -1, 0, \ldots$ [22]. This is similar to the power counting for the expansion around the nontrivial fixed point for pure short-range forces [7,21]. In both cases, the terms in the expansion correspond directly to terms in an effective-range expansion.

Here I have used this RG method to study the scaling behavior in the spin-triplet channels. This is controlled by the power-law dependence of the DW's near the origin which, in turn, follows from the singularity of the long-range potential at the origin. It is the same for attractive and repulsive $1/r^3$ potentials. In the S-, P-, and D-wave channels, where the tensor OPE force needs to be treated nonperturbatively, the expansion around the trivial fixed point leads to RG eigenvalues $\rho = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, corresponding to orders Q^d with $d = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$. These are promoted by half an order compared with naive dimensional analysis in an S wave and by many more orders in higher partial waves. This provides a quantitative measure of the effect observed in Ref. [15].

In the ${}^3S_1$–${}^3D_1$ channel, we need to expand around the nontrivial fixed point. The corresponding RG eigenvalues are $\rho = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$, and so there is one relevant perturbation. This is similar to the pure short-range case, except that the terms in the expansion have been demoted by half an order.

Finally, in the higher spin-triplet waves, one can treat the tensor OPE potential perturbatively at low energies. The corresponding scaling behavior is determined by the region where the centrifugal barrier dominates, and so the power counting is just that given by naive dimensional analysis.

These results show that it is possible to set up a consistent EFT embodying the WvK scheme where OPE is treated nonperturbatively in low partial waves. However, as already remarked, a central element of this is the identification of λ_π as a low-energy scale. Since this scale is built out of quantities that are treated as high-energy scales in ChPT, this analysis leaves open the question of how to make this theory consistent with chiral expansions of other effective operators, such as those for EM or weak couplings. Further work is needed to address this problem.

[12]The only exception would be if we were to choose an extension of the attractive $1/r^3$ potential that leads to a low-energy bound or virtual state which is trapped inside the barrier. However, there is no reason to do so in the context of nucleon-nucleon scattering in high partial waves.

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[1] S. Weinberg, Phys. Lett. **B251**, 288 (1990); Nucl. Phys. **B363**, 3 (1991).
[2] S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, in *At the Frontier of Particle Physics: Handbook of QCD*, edited by M. Shifman (World Scientific, Singapore, 2001), Vol. 1, p. 133.
[3] P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002).
[4] C. Ordoñez, L. Ray, and U. van Kolck, Phys. Rev. C **53**, 2086 (1996).
[5] S. R. Beane, M. Malheiro, D. R. Phillips, and U. van Kolck, Nucl. Phys. **A656**, 367 (1999).
[6] S. Scherer, Adv. Nucl. Phys. **27**, 277 (2003).
[7] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. **B424**, 390 (1998); Nucl. Phys. **B534**, 329 (1998).
[8] P. F. Bedaque and U. van Kolck, Phys. Lett. **B428**, 221 (1998).
[9] U. van Kolck, Nucl. Phys. **A645**, 273 (1999).
[10] H. A. Bethe, Phys. Rev. **76**, 38 (1949).
[11] R. G. Newton, *Scattering Theory of Waves and Particles* (Springer, New York, 1982).
[12] S. Fleming, T. Mehen, and I. W. Stewart, Nucl. Phys. **A677**, 313 (2000).
[13] S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. **A700**, 377 (2002).
[14] V. G. J. Stoks, R. A. M. Klomp, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. C **48**, 792 (1993).
[15] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005).
[16] M. Pavón Valderrama and E. Ruíz Arriola, Phys. Rev. C **72**, 054002 (2005).
[17] M. J. Cavagnero, Phys. Rev. A **50**, 2841 (1994).
[18] B. Gao, Phys. Rev. A **58**, 1728 (1998).
[19] B. Gao, Phys. Rev. A **59**, 2778 (1999).
[20] K. G. Wilson and J. G. Kogut, Phys. Rep. **12**, 75 (1974).
[21] M. C. Birse, J. A. McGovern, and K. G. Richardson, Phys. Lett. **B464**, 169 (1999).
[22] T. Barford and M. C. Birse, Phys. Rev. C **67**, 064006 (2003).
[23] T. Barford and M. C. Birse, J. Phys. A. **38**, 697 (2005).
[24] M. Pavón Valderrama and E. Ruíz Arriola, Phys. Lett. **B580**, 149 (2004); Phys. Rev. C **70**, 044006 (2004).
[25] M. Pavón Valderrama and E. Ruíz Arriola, nucl-th/0506047.
[26] H. Sadeghpour *et al*., J. Phys. B **33**, R93 (2000).
[27] W. M. Frank, D. J. Land, and R. M. Spector, Rev. Mod. Phys. **43**, 36 (1971).
[28] W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941).
[29] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part I* (McGraw-Hill, New York, 1963), pp. 555–562.
[30] S. Fubini and R. Stroffolini, Nuovo Cimento **37**, 1812 (1965); R. Stroffolini, Nuovo Cimento A **2**, 793 (1971).
[31] G. Esposito, J. Phys. A **31**, 9493 (1998).
[32] N. A. W. Holzwarth, J. Math. Phys. **14**, 191 (1973).
[33] D. B. Khrebtukov, J. Phys. A **26**, 6357 (1993).
[34] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, 1999).
[35] S. Wolfram, *Mathematica 5* (Wolfram Media, Champaign, 2003).
[36] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics Non-relativistic Theory* (Butterworth-Heinemann, Oxford, 1958), Sec. 132 and Problem 132.4.
[37] J. Shakeshaft, J. Phys. B **5**, L115 (1972).
[38] S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire, and U. van Kolck, Phys. Rev. A **64**, 042103 (2001).
[39] M. Bawin and S. A. Coon, Phys. Rev. A **67**, 042712 (2003).
[40] E. Braaten and D. Phillips, hep-th/0403168.
[41] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999); Nucl. Phys. **A646**, 444 (1999); **A676**, 357 (2000).
[42] K. G. Wilson, talk presented at the Institute for Nuclear Theory, Seattle, 2000 (unpublished); R. F. Mohr, Ph.D. thesis, Ohio State University, 2003, nucl-th/0306086.
[43] E. Braaten and H.-W. Hammer, Phys. Rev. Lett. **91**, 102002 (2003).
[44] T. Barford, Ph.D. thesis, University of Manchester, 2004, nucl-th/0404072.
[45] S. Scherer and H. W. Fearing, Phys. Rev. D **52**, 6445 (1995); H. W. Fearing, Phys. Rev. Lett. **81**, 758 (1998).
[46] E. Epelbaoum, W. Glöckle, and Ulf.-G. Meissner, Nucl. Phys. **A637**, 107 (1998).
[47] H. van Haeringen and L. P. Kok, Phys. Rev. A **26**, 1218 (1982).
[48] X. Kong and F. Ravndal, Nucl. Phys. **A665**, 137 (2000).
[49] J. V. Steele and R. J. Furnstahl, Nucl. Phys. **A645**, 439 (1999).
[50] H. Griesshammer, Nucl. Phys. **A760**, 110 (2005).
[51] V. N. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971); **29**, 546 (1979).
[52] P. F. Bedaque, H. W. Griesshammer, H.-W. Hammer, and G. Rupak, Nucl. Phys. **A714**, 589 (2003).