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Two- and three-charged-particle nuclear scattering in momentum space: A two-potential theory and a boundary condition model

Shinsho Oryu

Department of Physics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba 278-8510, Japan

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The two- and three-charged-particle nuclear scattering problems are investigated in momentum space. The three-body equations with a short-range nuclear potential, a three-body force potential, and the long-range Coulomb potential are presented in a mathematically rigorous way within a generalized two-potential theory. To remove the serious singularity in the two- and three-body Coulomb problems in momentum space, we have proposed a novel boundary condition to the phase shift that arises from a potential difference, the so-called auxiliary potential (AP), between the Coulomb potential and the screened Coulomb one: $V^\phi = V^C - V^R$. Furthermore, we point out the importance of the off-shell amplitude for the AP, by which one can uniquely obtain the two-body on-shell and off-shell Coulomb amplitude in momentum space. Therefore, this formulation is also useful for atomic systems, which is another benefit. It is recalled that the traditional phase-shift renormalization theory, in which the screened Coulomb amplitude is sandwiched by renormalized phase factors $e^{i\phi}$, is not consistent with two-potential theory. Some ambiguities or misunderstandings of the traditional methods for handling the Coulomb problem are clarified. Finally, the three-body unitarity relation is proved for the amplitude generated from these three kinds of potential. Moreover, a generalized two-potential theory is presented. It is pointed out that the asymptotic three-body nuclear wave function with the Coulomb potential could not be written by a product of two individual wave functions $\tilde{\psi}_x(\mathbf{x})\tilde{\psi}_y(\mathbf{y})$ defined in terms of Jacobi coordinates $\mathbf{x}$ and $\mathbf{y}$, respectively.

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I. INTRODUCTION

The two-body Lippmann-Schwinger (LS) equation for the Coulomb potential in momentum space suffers a serious numerical difficulty caused by overlapping or coincidence of the Green's function singular pole and the logarithmic singularity of the Coulomb potential [1,2]. Furthermore, the partial wave expansion series of the Rutherford scattering amplitude does not converge, but it does converge as a distribution [3,4]. Although we have an analytic solution for the two-body Coulomb wave function, the three-body analytic Coulomb solution has not been discovered yet [5]. To avoid such difficulties, a naive Coulomb correction was made by sandwiching an amplitude with respect to the short-range potential by the Coulomb phase factors $e^{i\sigma_l}$ [6,7]. It is known that a screened Coulomb potential has been widely used for practical calculations in physics. However, in few-body systems, the calculated results do not always converge by increasing the screening range. One of the most promising methods for the screened Coulomb potential was introduced by Gorshkov in 1960 [8,9], in which the Coulomb phase shift is considered as composed of the screened Coulomb phase shift plus a renormalization phase [10,11]. In 1970, Veselova applied this method to the three-body system at energies below the three-body threshold [12–14]. Alt *et al.* generalized the three-body AGS (Alt-Grassberger-Sandhas) equation [15] for the two-charged-particle system (i.e., ($p + d$ scattering), in which they pointed out an additional important Coulomb interaction between the charged pair and spectator in momentum space [16–21]. Hereafter we denote these works as the Mainz-Bonn model (MBM). A boundary condition method in configuration space was proposed by Merkuriev [22–26]. He described the asymptotic behavior of the three-body breakup wave function by using the eikonal approximation [27]. In [26], numerical results for $p + d$ scattering were compared with previous calculations [16,17]. Furthermore, in [19], a critical discussion was presented in comparison with [26]. Besides these three-body calculations, variational calculations in terms of hyperspherical variables are used by Kievsky and co-workers [28–33]. Many of the approaches for treating three-body Coulomb scattering separate the nonsingular part from the singular Coulomb part. It was proved that the former part safely converges, whereas the latter singular term results in two-body Coulomb scattering, which is composed of two charged fragments; the $p + d$ system is one example. Therefore, the two-charged-particle system in the three-body problem seems to be essentially the two-body Coulomb problem even if the energy is above the three-body breakup. This may be said to belong to the first generation of history in the three-body Coulomb problem. Although many efforts to solve the Coulomb problem were made, most few-body scattering calculations were based on the incomplete Coulomb method of the past half century.

Mukhamedzhanov *et al.* produced a painstaking work involving three charged particles with the same sign as a natural extension of the earlier MBM approach, although numerical calculations have not yet been presented [34,35]. The three-charged-particle problem is very important, not only for atomic systems but also for nuclear reactions. However, the screened Coulomb potential plus the renormalization technique, which is the basis of their method, will encounter trouble when the Coulomb parameter $\eta(k)$ becomes large, which means that the charges or the masses of the charged fragments become large or the energy becomes small i.e., the nuclear three clusters for instance, which were already mentioned in [36]. However, the screened Coulomb potential plus renormalization approach, such as the MBM, is clearer and more straightforward in the three-body Faddeev scheme. Therefore, we will mainly discuss our theory in comparison with the MBM.

We will point out that the phase-shift renormalization is not consistent with two-potential theory. This means that any effort within the MBM method will fail to reach the exact Coulomb amplitude even when one is on the energy shell. The traditional renormalization phase was obtained from the modified wave function, which is given by a screened Coulomb potential, in comparison with the pure Coulomb one [10]. The phase could be obtained by solving the LS equation with respect to the potential difference between the Coulomb potential and the screened Coulomb potential. The two-potential theory, in this case, demands two potentials, consisting of a screened Coulomb potential V^R and the remainder potential $V^\phi = V^C - V^R$. The latter potential creates the renormalization t matrix T^ϕ, which may generate the renormalization phase, whereas the former t matrix consists of a short-range t matrix $T^{R\phi}$ but modified by the latter t matrix T^ϕ. Nevertheless, several authors persist in considering only the $T^{R\phi}$ term but neglecting T^ϕ. This is one of the reasons that the traditional renormalization method fails to reach the genuine Coulomb t matrix. Moreover, the remainder t matrix contains a serious unsolved problem, similar to that of the Coulomb LS equation. The equation also has a notorious overlapping singularity in the kernel. This paper will introduce a novel method that can solve the problem. Using this method, we can obtain not only the Coulomb phase shift but also the off-shell Coulomb t matrix. Although in our former papers we proposed a new definition of the Coulomb amplitude that contained several new aspects for handling a long-range potential, where two-potential theory was often adopted in two- and three-body momentum space, the renormalization amplitude still had a long-range problem caused by the Coulomb interaction [37–39]. Our new scheme can avoid such difficulty by introducing two- and three-body off-shell boundary conditions for a typical auxiliary potential V^ϕ [39].

Section II introduces an "effective auxiliary potential" whose t matrix can be completely obtained by solving the LS equation. By utilizing the auxiliary t matrix catalyst, the fully off-shell Coulomb t matrix is obtained within the framework of two-potential theory, although the Coulomb LS equation cannot be directly solved. Furthermore, the off-shell t matrix for the short-range plus Coulomb potential is also presented. In Sec. III, the full three-body scattering formalism (the nuclear force, the three-body force, and the Coulomb force) is investigated in analogy with the Faddeev-type decomposition and two-body Coulomb method. A summary and discussion are given in Sec. IV. The unitarity for the new three-body amplitude is proven in Appendix A. The two-potential theory for the three-body equation is given in Appendix B. The asymptotic behavior of the three-body Coulomb wave function is discussed in Appendix C.

II. TWO-BODY COULOMB t MATRIX

A. Overlapping singularity

The Coulomb potential with charges Ze and $Z'e$ is given in configuration space by

$$V^C(r) = \frac{ZZ'e^2}{r}. \tag{1}$$

The screened Coulomb potential is given generally as

$$V^R(r) = V^C(r)\xi(r, R), \tag{2}$$

where $\xi(r, R)$ is a damping function. The Heaviside function $\theta(r - R)$ and the exponential functions $e^{-(r/R)^m}(m = 1, 2, \ldots)$ are popular. The Yukawa-type function $m = 1$ and the Gaussian function $m = 2$ are also well known. The momentum representation of Eq. (1) is given by introducing a small damping parameter λ for the integral in the Fourier transformation with $m = 1$. Furthermore, the partial wave expansion can be represented in the following way;

$$\begin{aligned}\langle \vec{p}|V^C|\vec{p}'\rangle &\equiv \lim_{\lambda\to 0}\frac{4\pi ZZ'e^2}{|\vec{p} - \vec{p}'|^2 + \lambda^2}\\ &= \lim_{\lambda\to 0}\sum_{l=0}^{\infty}(2l + 1)V_l^C(p, p'; \lambda)P_l(\cos\theta),\end{aligned} \tag{3}$$

with

$$V_l^C(p, p'; \lambda) \equiv \frac{V_0}{2pp'}Q_l\left(\frac{p^2 + p'^2 + \lambda^2}{2pp'}\right), \tag{4}$$

where $V_0 = 4\pi ZZ'e^2 = 4\pi k\eta(k)/\nu$ with the Coulomb parameter $\eta(k) = ZZ'e^2\nu/k$ and the reduced mass ν.

Then, in the limit of $\lambda \to 0$, the LS equation for this potential is written as

$$\begin{aligned}T_l^C(p, p'; \lambda; z) = V_l^C(p, p'; \lambda) + \int_0^\infty V_l^C(p, p''; \lambda)G_0(p''; z)\\ \times T_l^C(p'', p'; \lambda; z)dp'',\end{aligned} \tag{5}$$

with

$$G_0(p''; z) = \frac{\nu}{\pi^2}\frac{p''^2}{k^2 - p''^2 + i\epsilon}. \tag{6}$$

The kernel has a pole at $p'' = k = \sqrt{2\nu z}$ in the Green's function and a logarithmic singularity at $p = p''$ for $\lambda = 0$. Since both singularities are overlapped or coincide in the limit of $\lambda \to 0$, which we call an "overlapping singularity," the integral equation cannot be solved directly. This is one of the well-known difficulties in solving the LS equation in momentum space for the Coulomb problem. Hereafter, we will suppress λ in the momentum space Coulomb potential except where essential, and we will write the LS equation omitting λs as

$$\begin{aligned}T_l^C(p, p'; z) = V_l^C(p, p')\\ + \int_0^\infty V_l^C(p, p'')G_0(p''; z)T_l^C(p'', p'; z)dp''.\end{aligned} \tag{7}$$

B. Auxiliary potential and analyticity

To avoid the difficulty associated with an overlapping singularity, we define an "auxiliary but effectively long-range potential" (AP), $V^{\phi}(r) \equiv V^C(r) - V^R(r)$, which describes a difference between the Coulomb potential and a "screened Coulomb potential" (SCP) with a critical range $R \equiv R_{cl}$. The singularity at $r = 0$ is completely removed in $V^{\phi}(r)$ but the long-range property remains. One could imagine that $V^{\phi}(r)$ is a shallow long-range potential. If we increase the screening range, then $V_l^{\phi}(p, p') = V_l^C(p, p') - V_l^R(p, p')$ in momentum space becomes smaller and smaller except for the infinite value along the "blade of a knife" on the diagonal line at $p = p'$. Finally, it will vanish, and the t matrix $T_l^{\phi}(p, p'; z)$ for $V_l^{\phi}(p, p')$ could reach a trivial solution that has an effectively "zero phase shift": $\phi_l(R, k) \equiv \phi(R, k, l) = 0$. However, if we could find a very large but appropriate finite range $R = R_{cl}$ that satisfies $\phi(R, k, l) = \pm\pi n (n = 0, 1, 2, \ldots)$, then we have a nontrivial solution $T_l^{\phi}(p, p'; z)$.

Let us start from the Coulomb potential that consists of a short-range SCP and the AP,

$$V^C = V^R + (V^C - V^R) \equiv V^R + V^{\phi}. \tag{8}$$

We obtain for the LS equation with respect to the potential V^{ϕ}

$$T^{\phi} = V^{\phi} + V^{\phi} G_0 T^{\phi} \equiv V^{\phi}\omega^{\phi} \equiv \overline{\omega}^{\phi} V^{\phi}, \tag{9}$$

$$\begin{aligned} \omega^{\phi} &\equiv 1 + G_0 T^{\phi}, \\ \overline{\omega}^{\phi} &\equiv 1 + T^{\phi} G_0. \end{aligned} \tag{10}$$

Therefore, the screened Coulomb t matrix $t^{R\phi}$ given by two-potential theory is

$$t^{R\phi} = V^R + V^R G^{\phi} t^{R\phi} \equiv V^R \omega^R \equiv \overline{\omega}^R V^R, \tag{11}$$

$$\begin{aligned} \omega^R &\equiv 1 + G^{\phi} t^{R\phi}, \\ \overline{\omega}^R &\equiv 1 + t^{R\phi} G^{\phi}, \end{aligned} \tag{12}$$

with

$$G^{\phi} = \frac{1}{z - H_0 - V^{\phi}} = G_0 + G_0 T^{\phi} G_0. \tag{13}$$

Here, the fully off-shell Coulomb t matrix is defined by

$$T^C \Leftarrow T^{R\phi} + T^{\phi} = \overline{\omega}^{\phi} t^{R\phi} \omega^{\phi} + T^{\phi}. \tag{14}$$

It should be noted that the pure Coulomb t matrix is not obtained by solving the Coulomb LS equation but by summing a short-range renormalized t matrix and the long-range AP t matrix. We will see that the on-shell $T_l^{\phi}(k, k; z)$ does not yield the renormalization phase $\phi(R, k) = \eta(k)(\ln 2kR - \gamma/m)$, which was given by the differential equation in the traditional method [10,17] but we obtain the off-shell $T_l^{\phi}(p, p'; z)$, and on-shell $T_l^{\phi}(k, k; z) = 0$ is required. Then we can calculate $t^{R\phi}$ with V^R and G^{ϕ} in Eq. (11) and complete (14). As a consequence, it will be shown that $T^C \neq \overline{\omega}^{\phi} t^R \omega^{\phi}$.

Let us solve the following LS equation;

$$\begin{aligned} T_l^{\phi}(p, p'; z) - V_l^{\phi}(p, p') &= \int_0^{\infty} V_l^{\phi}(p, p'') G_0(p''; z) \\ &\quad \times T_l^{\phi}(p'', p'; z) dp'' \qquad (15) \\ &= \int_0^{\infty} T_l^{\phi}(p, p''; z) G_0(p''; z) \\ &\quad \times V_l^{\phi}(p'', p') dp'', \qquad (16) \end{aligned}$$

where $V_l^{\phi}(p, p')$ is nonsingular except for $p \neq p'$. The $G_0(p''; z)$ has a pole at $p'' = k$.

To investigate the right-hand side of (15), we define

$$\begin{aligned} I_l(p, p'; z) &\equiv \int_0^{\infty} V_l^{\phi}(p, p'') G_0(p''; z) T_l^{\phi}(p'', p'; z) dp'' \\ &\equiv I_l^P(p, p'; z) + i I_l^{\delta}(p, p'; z) \\ &= \frac{\nu}{\pi^2} \left[P \int_0^{\infty} \frac{V_l^{\phi}(p, p'') T_l^{\phi}(p'', p'; z)}{k^2 - p''^2} p''^2 dp'' \right. \\ &\qquad \left. - i \frac{\pi k}{2} V_l^{\phi}(p, k) T_l^{\phi}(k, p'; z) \right], \end{aligned} \tag{17}$$

where the integral is the principal value part and the second term comes from the δ function of the Green's function. Calculating the integral equation of (16) gives

$$\begin{aligned} I_l(p, p'; z) &\equiv \int_0^{\infty} T_l^{\phi}(p, p''; z) G_0(p''; z) V_l^{\phi}(p'', p') dp'' \\ &\equiv \overline{I}_l^P(p, p'; z) + i \overline{I}_l^{\delta}(p, p'; z) \\ &= \frac{\nu}{\pi^2} \left[P \int_0^{\infty} \frac{T_l^{\phi}(p, p''; z) V_l^{\phi}(p'', p')}{k^2 - p''^2} p''^2 dp'' \right. \\ &\qquad \left. - i \frac{\pi k}{2} T_l^{\phi}(p, k; z) V_l^{\phi}(k, p') \right]. \end{aligned} \tag{18}$$

Here, the principal parts of Eqs. (17) and (18) are calculated safely by standard procedures,

$$P \int_0^{\infty} \frac{F_l(p'')}{k^2 - p''^2} dp'' = \int_0^{\infty} \frac{F_l(p'') - F_l(k)}{k^2 - p''^2} dp''. \tag{19}$$

In the δ-function part in the integral $I_l(p, p'; z)$, Eqs. (15) and (16) have trouble only at $p = k$ and $p' = k$.

Lemma 1. If the on-shell t matrix $T_l^{\phi}(k, k; z) = 0$ is given, then the half-off-shell t matrices satisfy

$$T_l^{\phi}(p, k; z) = T_l^{\phi}(k, p'; z) = 0.$$

Proof. From Eq. (17), and $T_l^{\phi}(k, k; z) = 0$, we obtain

$$I_l^{\delta}(p, k; z) = -\frac{\nu k}{2\pi} V_l^{\phi}(p, k) T_l^{\phi}(k, k; z) = 0 \quad (\text{for } p \neq k). \tag{20}$$

Using Eqs. (15)–(19) and (20), we have a finite value for

$$I_l(p, k; z) = \overline{I}_l(p, k; z). \tag{21}$$

Since $V_l^{\phi}(k,k)=\infty$, then the half-off shell t matrix should satisfy

$$T_l^{\phi}(p,k;z)=0 \qquad (\text{for } p\neq k). \tag{22}$$

In the same way, $T_l^{\phi}(k,p';z)=0$ is obtained.

Lemma 2. If the relations

$$T_l^{\phi}(k,k;z)=T_l^{\phi}(k,p';z)=T_l^{\phi}(p,k;z)=0 \tag{23}$$

are given, then the off-shell t matrix $T_l^{\phi}(p,p';z)$ is a real function. As a consequence, $T_l^{\phi}(p,p';z)$ is obtained by the K matrix equation.

Proof. Because of the relations

$$\begin{aligned} I_l^{\delta}(p,k;z)&=\bar{I}_l^{\delta}(k,p';z)\\ &=I_l^{\delta}(k,k;z)=I_l^{\delta}(p,p';z)=0. \end{aligned} \tag{24}$$

Eqs. (15) and (16) are written as

$$\begin{aligned} &T_l^{\phi}(p,p';z)-V_l^{\phi}(p,p')\\ &=I_l^{P}(p,p';z)\\ &=P\int_0^{\infty}V_l^{\phi}(p,p'')G_0(p'';z)T_l^{\phi}(p'',p';z)dp''\\ &\equiv\int_0^{\infty}V_l^{\phi}(p,p'')G_0^{P}(p'';z)T_l^{\phi}(p'',p';z)dp''\\ &\equiv K_l^{\phi}(p,p';z)-V_l^{\phi}(p,p'). \end{aligned} \tag{25}$$

Therefore, $T_l^{\phi}(p,p',z)=K_l^{\phi}(p,p',z)$ is a real function that can be given by K matrix theory. The relation between the t matrix and the K matrix is given by, for $p=p'=k$,

$$\begin{aligned} T_l^{\phi}(p,p';z)&=K_l^{\phi}(p,p';z)\\ &\quad+i\rho(k)\frac{K_l^{\phi}(p,k;z)K_l^{\phi}(k,p';z)}{1-i\rho(k)K_l^{\phi}(k,k;z)}\\ &\longrightarrow K_l^{\phi}(p,p';z), \end{aligned} \tag{26}$$

where $\rho(k)=-\nu k/2\pi$.

Lemmas 1 and 2 are true in the Coulomb scattering theory in which the on-shell t matrix vanishes or phase shifts are $\pm\pi n\,(n=0,1,2,\ldots)$ at specific k. The auxiliary amplitude has two parameters R and k, at least, to satisfy the phase shift $\phi_l(R,k)=0$ at points (or on the line) $R=R_{cl}(k)$ for a fixed l. Hence, the off-shell auxiliary LS equation can be safely solved for all energies but by different ranges. Then we can obtain the fully off-shell Coulomb amplitude at any energy not by the Coulomb LS equation but by the two-potential formula.

Consequently, the screening range $R=R_{cl}(k)$ has to be sought to obtain $T_l^{\phi}(k,k,z)=0$ by a proper method. The range will provide "a unique boundary value"; otherwise the off-shell T_l^{ϕ} has no solution as the original Coulomb LS equation has none.

C. The boundary range and a solution method

One of the methods for obtaining $T_l^{\phi}(k,k,z)=0$ could be given by solving the Schrödinger equation with respect to $V_l^{\phi}(R,r)\equiv V_l^{\phi}(r)=V_l^{C}(r)-V_l^{R}(r)$ in which the phase shift satisfies $\phi_l(R,k)=\pm\pi n(n=0,1,2,\ldots)$. We can solve the differential equation for the auxiliary potential $V_l^{\phi}(R_{cl},r)$ with the Coulomb asymptotic wave function

$$(z-H_0)\psi_l^{\phi}(r)=V_l^{\phi}(r)\psi_l^{\phi}(r). \tag{27}$$

From the equation, the auxiliary phase shift $\phi_l(R,k)$ could be easily obtained.

We can also propose another method to solve Eq. (15) in momentum space directly. For a practical calculation, we recall the parameter λ in Eq. (5), taking the limit $\lambda\to 0$ for $0<\lambda\ll 1/R$; then Eq. (15) becomes

$$\begin{aligned} T_l^{\phi}(p,p';\lambda;z)&=V_l^{\phi}(p,p';\lambda)+\int_0^{\infty}V_l^{\phi}(p,p'';\lambda)\\ &\quad\times G_0(p'';z)T_l^{\phi}(p'',p';\lambda;z)dp'', \end{aligned} \tag{28}$$

with

$$V_l^{\phi}(p,p';\lambda)=V_l^{C}(p,p';\lambda)-V_l^{R}(p,p'), \tag{29}$$

where the limit of the auxiliary potential exists,

$$\lim_{\lambda\to 0}V_l^{\phi}(p,p';\lambda)=V_l^{\phi}(p,p'). \tag{30}$$

Therefore, we can calculate Eq. (28) for a sufficiently small λ except for the overlapping singularity at $\lambda=0$ and obtain $R_{cl}(k;\lambda)$ to satisfy $T_l^{\phi}(k,k;\lambda;z)=0$; then we can perform

$$\lim_{\lambda\to 0}T_l^{\phi}(k,k;\lambda;z)\to T_l^{\phi}(k,k;z)=0, \tag{31}$$

with

$$\lim_{\lambda\to 0}R_{cl}(k;\lambda)\to R_{cl}(k), \tag{32}$$

where we will find a specific range $R=R_{cl}(k)$ that satisfies Eq. (31).

Once we find $R_{cl}(k)$, the K-matrix equation is verified with respect to $V_l^{\phi}(p,p')$ by using *Lemmas 1* and *2*; that is, putting $T_l^{\phi}(p,p';z)\equiv K_l^{\phi}(p,p';z)$, we have

$$\begin{aligned} T_l^{\phi}(p,p';z)&=V_l^{\phi}(p,p')+\int_0^{\infty}V_l^{\phi}(p,p'')G_0^{P}(p'';z)\\ &\quad\times T_l^{\phi}(p'',p';z)dp''\\ &=V_l^{\phi}(p,p')+\frac{\nu}{\pi^2}P\int_0^{\infty}V_l^{\phi}(p,p'')\frac{p''^2}{k^2-p''^2}\\ &\quad\times T_l^{\phi}(p'',p';z)dp'', \end{aligned} \tag{33}$$

where no overlapping singularity exists in the kernel of Eq. (33). Using such a unique range R_{cl} we can represent the off-shell $T_l^{\phi}(p,p';z)$ or $K_l^{\phi}(p,p';z)$ without a singular pole of the free Green's function.

Finally, we can conclude that the Coulomb t matrix is not obtained by solving the LS equation (5), but by using (14) and the off-shell solution of (33).

Here, we have to solve (11), that is,

$$t_l^{R\phi}(p,p';z) = V_l^R(p,p') + \int_0^\infty V_l^R(p,p'')G_l^\phi(p'';z) \times t_l^{R\phi}(p'',p';z)dp'' \quad (34)$$
$$= V_l^R(p,p') + \int_0^\infty \mathcal{K}_l^{R\phi}(p,p'';z)G_0(p'';z) \times t_l^{R\phi}(p'',p';z)dp'', \quad (35)$$

with

$$\mathcal{K}_l^{R\phi}(p,p'';z) = \int_0^\infty V_l^R(p,p''')\{\delta(p'''-p'') + G_0^P(p''';z) \times T_l^\phi(p''',p'';z)\}dp''', \quad (36)$$

where the new resolvent is given by $G_l^\phi(z) = G_0(z) + G_0(z)T_l^\phi(z)G_0(z) = \omega_l^\phi(z)G_0(z) = G_0(z)\overline{\omega}_l^\phi(z)$, and $T_l^\phi(z) \equiv K_l^\phi(z)$ is used in the integrand.

The kernel $\mathcal{K}_l^{R\phi}$ is a real function, because in Eq. (36) the δ-function part of $G_0(z)$ ensures that the half-on-shell $K_l^\phi(k,p'';z) = 0$ owing to (23), (26).

Furthermore, the Coulomb phase shift is given by

$$\sigma_l(k) = \tan^{-1}\left(\frac{\mathrm{Im}\left[\overline{\omega}_l^\phi t_l^{R\phi}\omega_l^\phi\right]}{\mathrm{Re}\left[\overline{\omega}_l^\phi t_l^{R\phi}\omega_l^\phi\right]}\right) = \tan^{-1}\left(\frac{\mathrm{Im}\left[t_l^{R\phi}\right]}{\mathrm{Re}\left[t_l^{R\phi}\right]}\right) \equiv \delta_l^{R\phi}(k), \quad (37)$$

where we have used a half-on-shell relations for $\omega_l^\phi(z)$ and $\overline{\omega}_l^\phi(z)$, which follow from (23),

$$\begin{aligned} &\lim_{p\to k}\overline{\omega}_l^\phi(p,p';z) = \delta(k-p'),\\ &\lim_{p'\to k}\overline{\omega}_l^\phi(p,p';z) = \delta(p-k),\\ &\lim_{p\to k}\omega_l^\phi(p,p';z) = \delta(k-p'),\\ &\lim_{p'\to k}\omega_l^\phi(p,p';z) = \delta(p-k). \end{aligned} \quad (38)$$

In this process, the specific range $R(k,l)$ that satisfies $T_l^\phi(k,k;z) = 0$ should be searched for. It is known that the first approximation for such a range is

$$R(k,l) = \frac{1}{2k}\exp\left[\frac{C(k,l)}{\eta(k)}\right] \equiv R_{cl}(k), \quad (39)$$

where $C(k,l) = C_0(k,l) \pm n\pi$ is a factor that depends on the SCP shape and n. This is a kind of boundary condition at $r = R = R_{cl}$ that is not comparable with the usual differential equation. The AP phase shift is taken as $\pm\pi n$.

Finally, one can conclude

$$\phi_l(R,k) \equiv \phi(R,k,l) = \pm\pi n \quad (n = 0,1,2,\ldots). \quad (40)$$

Merkuriev developed practical boundary conditions for differential equations in coordinate space. He investigated the asymptotic behavior of the three-body Coulomb wave function in which the eikonal approximation was widely used [27]. It should be recalled that the asymptotic behavior of the Coulomb three-body breakup wave function is very important because the asymptotic behavior offers a new six-dimensional spherical wave that is completely different from the usual two-body boundary condition framework in the hyperspherical coordinate. The numerical $p + d$ calculation by Merkuriev was critically discussed in [19].

In the Jacobi coordinate, the three-body Coulomb wave function cannot be separable with respect of the coordinates **x** and **y** (see Appendix C). If the separable approximation of the three-body wave function is adopted, then it should be admitted that the method contains a screened Coulomb approximation. Therefore, only for limited cases such as the short-range potential, or below the $p + d$ breakup threshold in the Coulomb field, can one adopt the particular boundary condition for the wave function of **x**, and that for the wave function of **y**, respectively [29]. Generally, the boundary condition for the three-body Coulomb wave function: $\psi(\mathbf{x},\mathbf{y})$ above the $p + d$ breakup threshold as well as the intermediate state could not be separable with respect to **x** or **y**. In other words, the **x**-**y** separated form: $\tilde{\psi}_x(\mathbf{x})\tilde{\psi}_y(\mathbf{y})$ is equivalent to a screend Coulomb potential case [40].

In this paper, we are not concerned with the boundary condition of the differential equation in coordinate space, but we introduced a new boundary condition in momentum space for the integral equation.

In the conventional nuclear reaction calculation within the boundary condition model, it is said that the major wave function is obtained by using a particular long-range SCP that is smoothly continued using the Coulomb asymptotic wave function at a specified range. However, the screened Coulomb wave function should be distorted by the AP with the defined long-range behavior. That is, the SCP wave function could not be connected smoothly with the pure Coulomb wave function at finite range, but it can be done at infinite range. This will be seen in the following discussion.

The Schrödinger equation for the Coulomb potential and the asymptotic behavior of the wave function are given in the AP formalism by $V_l^\phi(r) = V_l^C(r) - V_l^R(r)$, and so

$$\left[z - H_0 - V_l^\phi(r)\right]\psi_l^C(r) = V_l^R(r)\psi_l^C(r), \quad (41)$$

$$\psi_l^C(r) \longrightarrow \exp\left\{i\left[kr - \frac{\pi l}{2} - \eta(k)\ln 2kr + \sigma_l(k)\right]\right\} \quad (\text{for } r\to\infty) \quad (42)$$

$$\equiv \exp\left\{i\left[kr - \frac{\pi l}{2} - \eta(k)\ln 2kr + \delta_l^{R\phi}(k) + \phi(R,k,l)\right]\right\}. \quad (43)$$

The SCP of range R gives

$$(z - H_0)\psi_l^R(r) = V_l^R(r)\psi_l^R(r), \quad (44)$$

$$\psi_l^R(r) \longrightarrow \exp\left\{i\left[kr - \frac{\pi l}{2} + \delta_l^R(k)\right]\right\}. \quad (45)$$

Recall that $\delta_l^{R\phi}(k) \neq \delta_l^R(k)$ is already confirmed by the existence of the T_l^ϕ-modified resolvent $G_l^\phi(z)$ in Eq. (34).

This idea is not a new one, having been mentioned in the textbook of Jackson [41]; for instance, see Eq. (3.26) on

p. 40. Equation (44) generates $\delta_l^R(k)$ in (45), which refers to the resolvent $G_0(z)$. The smooth matching of the wave function (42) to that of (45) at $r = R$ is given by $\psi_l^C(R) = \psi_l^R(R)$ and $[\psi_l^C(R)]' = [\psi_l^R(R)]'$, which lead to a phase-shift condition $\sigma_l(k) = \delta_l^R(k) + \eta(k)\ln 2kR$ and a momentum relation $k = k - \eta(k)/R$. These relations are only allowed at $r = R \to \infty$. This method is inadequate for our present purpose. Therefore the relation (39) gives a unique screening range $R = R(k,l) \equiv R_{cl}$ at which the asymptotic wave function (42) is smoothly continued with the wave function (43) along with the relation $\sigma_l(k) = \delta_l^{R\phi}(k) + \phi(R,k,l)$. Here the phase shifts $\phi(R,k,l) = \pm\pi n (n = 0,1,2,\ldots)$ are satisfied. Then these give $\sigma_l(k) = \delta_l^{R\phi}(k) \pm \pi n$. Many nuclear reaction calculations have been performed without care being taken to define a unique range.

D. Off-shell Coulomb *t* matrix

Finally, our main purpose is ready for harvest in this subsection. The Coulomb potential is separated into two parts,

$$V^C = V^R + (V^C - V^R) = V^R + V^\phi. \tag{46}$$

Then one can use two-potential theory and obtain by means of Eq. (14)

$$\begin{aligned} T_l^C(p,p';z) &= \int_0^\infty\int_0^\infty \overline{\omega}_l^\phi(p,p'';z) t_l^{R\phi}(p'',p''';z) \\ &\quad \times \omega_l^\phi(p''',p';z)dp''dp''' + T_l^\phi(p,p';z) \\ &= \int_0^\infty\int_0^\infty \big[\delta(p-p'') + T_l^\phi(p,p'';z) \\ &\quad \times G_0^P(p'';z)\big] t_l^{R\phi}(p'',p''';z)\big[\delta(p'''-p') \\ &\quad + G_0^P(p''';z)T_l^\phi(p''',p';z)\big]dp''dp''' \\ &\quad + T_l^\phi(p,p';z), \end{aligned} \tag{47}$$

where $T_l^\phi(p,p';z)$ is already given in the previous section, and $t_l^{R\phi}(p,p';z)$ is calculated numerically by Eqs. (11), (34), and (35), but the details will be shown in the next section.

E. A short-range force and a Coulomb force

Let us consider the system with nuclear and Coulomb forces. The potential is given by

$$\begin{aligned} V^{(C)} &= V^S + V^C = (V^S + V^R) + (V^C - V^R) \\ &= V^{(R)} + V^\phi, \end{aligned} \tag{48}$$

where V^S is a short-range nuclear potential. By analogy with Eq. (9) and Eq. (11), the *t* matrix is given as

$$\begin{aligned} T^{(R)} &= V^{(R)} + V^{(R)}G^\phi T^{(R)} \\ &= \overline{\omega}^R t^{sR}\omega^R + t^{R\phi}, \end{aligned} \tag{49}$$

with

$$t^{sR} = V^S + V^S G^{R\phi} t^{sR}, \tag{50}$$

where $t^{R\phi}$ was given by (11). Then we easily deduce by using Eqs. (9), (11), (14), (48), and (49)

$$\begin{aligned} T^{(C)} &= \overline{\omega}^\phi T^{(R)}\omega^\phi + T^\phi \\ &= \overline{\omega}^\phi(\overline{\omega}^R t^{sR}\omega^R + t^{R\phi})\omega^\phi + T^\phi \\ &= \overline{\omega}^\phi\overline{\omega}^R t^{sR}\omega^R\omega^\phi + \overline{\omega}^\phi t^{R\phi}\omega^\phi + T^\phi \\ &= \overline{\omega}^C t^{sR}\omega^C + T^C, \end{aligned} \tag{51}$$

where the Coulomb Møller operators are defined by $\omega^C = \omega^R\omega^\phi$ and $\overline{\omega}^C = \overline{\omega}^\phi\overline{\omega}^R$. Here we can prove that $t^{sR} = t^{sC}$ by using the resolvent; that is,

$$\begin{aligned} G^{R\phi} &= \frac{1}{z - H_0 - V^R - V^\phi} \\ &= \frac{1}{z - H_0 - V^C} \equiv G^C \end{aligned} \tag{52}$$

$$= G_0 + G_0 T^C G_0. \tag{53}$$

Here it should be stressed that Eq. (52) contains an “important proof” from the screened Coulomb to the pure Coulomb Green’s function (i.e., $G^{R\phi} = G^C$ at a “finite given range R”). If we miss V^ϕ, then we have to take $R \to \infty$ to reach G^C. Therefore, all other calculations should be done at infinite range. If the numerical results converge at finite range, they are inconsistent [40,42,43].

Then Eq. (50) becomes

$$t^{sR} = V^S + V^S G^C t^{sR} \equiv t^{sC} \tag{54}$$

$$= V^S + \mathcal{K}^{SC} G_0 t^{sR}, \tag{55}$$

with the kernel

$$\mathcal{K}^{SC} = V^S(1 + G_0 T^C). \tag{56}$$

Therefore Eq. (51) gives

$$T^{(C)} = \overline{\omega}^C t^{sC}\omega^C + T^C. \tag{57}$$

For the term $\overline{\omega}^C t^{sC}\omega^C$ of Eq. (57), we obtain by using Eqs. (49)

$$\begin{aligned} \overline{\omega}^C t^{sC}\omega^C &= \overline{\omega}^\phi(\overline{\omega}^R t^{sC}\omega^R)\omega^\phi \\ &= \overline{\omega}^\phi(T^{(R)} - t^{R\phi})\omega^\phi. \end{aligned} \tag{58}$$

Here, the on-shell $(T^{(R)} - t^{R\phi})$ converges because of the operators $\overline{\omega}^\phi = \omega^\phi = 1$. The off-shell amplitude will be directly calculated from Eq. (58). In the MBM, $\overline{\omega}^\phi = e^{i\phi}$ and $\omega^\phi = e^{i\phi}$ oscillate very rapidly when $R \to \infty$. However, $e^{i\phi}(T^{(R)} - t^R)e^{i\phi}$ instead of Eq. (58) seems to be converged because the oscillations of both terms could be canceled at a certain range R. Since the later $(T^{(R)} - t^R)$ is not the same as Eq. (58), then the converged values are not proper amplitudes, because they approximate G^ϕ by G_0, which are seen in Eq. (21b) in [43] and Eq. (27) of [40].

We will see that the renormalization, which was inferred from Ref. [19], is not correct but is completed only by using Eq. (14),

$$\begin{aligned} \overline{\omega}^\phi T^{(R)}\omega^\phi &= \overline{\omega}^\phi(\overline{\omega}^R t^{sR}\omega^R + t^{R\phi})\omega^\phi \\ &= \overline{\omega}^\phi\overline{\omega}^R t^{sR}\omega^R\omega^\phi + \overline{\omega}^\phi t^{R\phi}\omega^\phi \\ &= \overline{\omega}^C t^{sC}\omega^C + \overline{\omega}^\phi t^{R\phi}\omega^\phi \\ &= \overline{\omega}^C t^{sC}\omega^C + T^C - T^\phi \\ &\equiv T^{(C)} - T^\phi, \end{aligned} \tag{59}$$

where the existence of T^ϕ is emphasized. Finally, we obtain the following relations from Eq. (59);

$$\overline{\omega}^\phi T^{(R)}\omega^\phi \neq T^{(C)} \equiv \overline{\omega}^C t^{sC}\omega^C + T^C, \quad (60)$$

$$\overline{\omega}^\phi t^{R\phi}\omega^\phi \neq T^C, \quad (61)$$

$$\overline{\omega}^\phi \overline{\omega}^R t^{sR}\omega^R\omega^\phi = \overline{\omega}^C t^{sC}\omega^C. \quad (62)$$

To investigate Eq. (61) practically, let us consider the two-body on-shell amplitude, putting $T_l^C(k,k;z) = -2\pi/\nu f_l^C(k)$. Now, if we adopt the MBM's renormalization technique [17], then $\omega^\phi = \overline{\omega}^\phi \to e^{i\phi}$ is given. Therefore, the left-hand side of Eq. (61) becomes

$$\begin{aligned}\lim_{R\to\infty} \overline{\omega}^\phi t_l^{R\phi}\omega^\phi &= \lim_{R\to\infty} -\frac{2\pi}{\nu} e^{i\phi}\left(\frac{e^{2i\delta_l^{R\phi}}-1}{2ik}\right)e^{i\phi} \\ &= \lim_{R\to\infty} -\frac{2\pi}{\nu}\left(\frac{e^{2i(\delta_l^{R\phi}+\phi)}-e^{2i\phi}}{2ik}\right),\end{aligned}$$

where $\delta_l^{R\phi}(k)$ is the phase shift from Eq. (34). However, the MBM takes $\delta_l^R(k)$ instead of $\delta_l^{R\phi}(k)$ in which G_0 is adopted for G^ϕ in Eq. (34); that is, the MBM employs t_l^R instead of $t^{R\phi}$. Therefore, if and only if $\delta_l^R(k) + \phi(R,k) \to \sigma_l(k)$ is admitted, it gives

$$\lim_{R\to\infty} \omega^\phi t_l^R \omega^\phi = \lim_{R\to\infty} -\frac{2\pi}{\nu}\left(\frac{e^{2i\sigma_l}-e^{2i\phi}}{2ik}\right) \neq T_l^C(k),$$

where $\phi(R,k) = \eta(k)[\log(2kR) - \gamma]$ and $\gamma = 0.5772\ldots$ is the Euler constant. Since $\lim_{R\to\infty}\phi(R,k) \to \infty$, then $e^{2i\phi}$ oscillates very quickly and never converges to the Coulomb t matrix $T_l^C(k)$.

It is also seen that there exists the same confusion in the three-body AGS equation, Eq. (2.17) of [19], which corresponds to our Eq. (59) after making the following replacements;

$$\omega^\phi \to Z_{\alpha,R}^{-1/2}(q_\alpha), \quad (63)$$

$$\overline{\omega} \to Z_{\beta,R}^{-1/2}(q'_\beta), \quad (64)$$

$$T^{(R)} \to \mathcal{T}_{\beta\alpha}^{(R)}(\mathbf{q}'_\beta, \mathbf{q}_\alpha), \quad (65)$$

$$T^C \to \delta_{\beta\alpha}\overline{\delta}_{\alpha 3} t_\alpha^C(\mathbf{q}'_\alpha, \mathbf{q}_\alpha), \quad (66)$$

$$\overline{\omega}^C t^{sC}\omega^C \to \langle \mathbf{q}'_{\beta,C}{}^{(-)} | \mathcal{T}_{\beta\alpha}^{SC}(E+i0) | \mathbf{q}_{\alpha,C}^{(+)}\rangle. \quad (67)$$

Then, one can compare Eq. (59) with Eq. (2.17) of [19] in the elastic scattering limit, using $Z_{\alpha,R}(q_\alpha) = \exp[2i\phi_{\alpha,R}(q_\alpha)] \equiv Z_R(q_e)$. First, let us consider Eq. (60) by using the replacements (63), (64), and (65);

$$\begin{aligned}&\lim_{R\to\infty} Z_R^{-1}(q_e)\mathcal{T}_{dd}^{(R)SL}(q_e,q_e;E+i0) \\ &= \lim_{R\to\infty} Z_R^{-1}(q_e)\big[\mathcal{T}_{dd}^{SR,SL}(q_e,q_e;E+i0) \\ &\quad + t_R^L(q_e,q_e;3q_e^2/4m_N + i0)\big] \\ &= \lim_{R\to\infty} Z_R^{-1}(q_e)\big\{\mathcal{T}_{dd}{}^{SR,SL}(q_e,q_e;E+i0) \\ &\quad + \frac{3}{4\pi i m_N q_e}\exp(-2i\phi_R)[\exp(2i\delta_{R,L})-1]\big\} \\ &= \lim_{R\to\infty} Z_R^{-1}(q_e)\mathcal{T}_{dd}{}^{SR,SL}(q_e,q_e;E+i0) \\ &\quad + \lim_{R\to\infty}\frac{3}{4\pi i m_N q_e}\{\exp[2i(-\phi_R+\delta_{R,L})] \\ &\quad - \exp[-2i\phi_R]\} \quad (68)\\ &\neq \mathcal{T}_{dd}{}^{(C)SL}(q_e,q_e;E+i0). \quad (69)\end{aligned}$$

The second term on the right-hand side of Eq. (68) oscillates very quickly by $e^{-2i\phi_R}$ when R goes to infinity, although the sum $[-\phi_{\alpha,R}(q_\alpha) + \delta_{R,L}]$ may converge. Therefore, the renormalization of this formula never converges. The same relation appears also in other articles; Eq. (2.21) of [17], Eq. (3.29) in [18], Eq. (4) in [20], Eq. (17) in [21], Eq. (21a) in [43], etc. This is a misunderstanding of the traditional screening method, which is clear from Eq. (59) and the related discussion.

Second, Eq. (62) for the on-shell limit corresponds to (in [19])

$$\begin{aligned}&\lim_{R\to\infty} Z_R^{-1}(q_e)\mathcal{T}_{dd}{}^{SR,SL}(q_e,q_e;E+i0) \\ &= \lim_{R\to\infty}\frac{-3}{4\pi i m_N q_e}\exp[2i(-\phi_R+\delta_{R,L})] \\ &\quad \times [\exp(2i^{2S+1}\delta_{SR,L})-1] \\ &= \frac{-3}{4\pi i m_N q_e}e^{2i\sigma_L}[\exp(2i^{2S+1}\delta_{SC,L})-1] \\ &= \mathcal{T}_{dd}{}^{SC,SL}(q_e,q_e;E+i0). \quad (70)\end{aligned}$$

It converges. This contradiction is only resolved by the definition of Eq. (14) with the "auxiliary t matrix," which is seen in Eqs. (60)–(62).

In [19], the renormalization phase $-\phi_R = \sigma_L - \delta_{L,R}$ is obtained by the Yukawa-type screened Coulomb potential $[m = 1 : \xi(r,R) = e^{-r/R}$ in Eq. (2)], which appears in the textbook of Goldberger-Watson [10] (p. 265),

$$\begin{aligned}\phi_{\alpha,R}(q_\alpha) = &-\frac{e^2 M_\alpha}{q_\alpha}[\log(2q_\alpha R) - \gamma] \\ &+ O\left(\frac{\log(q_\alpha R)}{q_\alpha R}\right). \quad (71)\end{aligned}$$

Unfortunately, Eq. (71) is not a good representation for a large Coulomb parameter $\eta(q_\alpha) = ZZ'e^2M_\alpha/q_\alpha$ with heavy masses and large charges and at very low energies [36]. Another attempt with respect to $m > 1$ in Eq. (2) exists, for which $\phi_{\alpha,R}(q_\alpha) = -\eta(q_\alpha)[\ln 2q_\alpha R - \gamma/m]$ is obtained, but it is still an approximation.

Another way finding $\phi_{\alpha,R}$ may be possible with the aid of relations $-\phi_{\alpha,R}(q_\alpha) = \sigma_L(q_\alpha) - \delta_{L,R}(q_\alpha)$ in which $\delta_{L,R}$ is given numerically for proper R, and σ_L is obtained analytically. However, this attempt will also fail, because $\delta_{L,R}$ should be distorted by $\phi_{\alpha,R}$ or G^ϕ in Eq. (34).

In our theory, we use the notation $(\overline{R}, q_0)$ instead of (R, q_α) in [19], and we take $\delta_L^{\overline{R}\Phi}(q_0)$ for $\delta_{L,R}(q_\alpha)$ and $\Phi(\overline{R}, q_0, L)$ for $-\phi_{\alpha,R}(q_\alpha)$, respectively. Here, three phases are linked by the relation between (42) and (43); that is, $\sigma_L(q_0) = \delta_L^{\overline{R}\Phi}(q_0) + \Phi(\overline{R}, q_0, L)$ [41]. To cut the linkage, we have to adopt the on-shell three-body auxiliary matrix $X_{\alpha\alpha,L}^\Phi(q_0,q_0;E)(=0)$ in (134) with the phase shift $\Phi(\overline{R},q_0,L)(=\pm\pi m : m = 0,1,2\ldots)$ in Eq. (153). However, the off-shell auxiliary matrix

$X^{\Phi}_{\alpha\alpha,L}(q,q';E)$ $(\neq 0)$ takes an important part to make the three-body off-shell t matrix as well as the two-body auxiliary matrix $T^{\phi}_{l}(p,p';z)(\neq 0)$ did.

Therefore, the off-shell extension of the three-body scattering amplitude by the phase-shift renormalization of the MBM is not correct. Then any calculations for the breakup amplitude by the traditional phase-shift renormalization will fail. Details will be given later.

III. THREE-BODY SCATTERING EQUATION

A. The three-charged-particle t matrix in nuclear systems

Our "regulation" for the two-body Coulomb t matrix in Eq. (14) and (51) is obviously written for the three-body transition t matrix $T^{(C)}$ for a nuclear system in which the full potential is given by

$$\begin{aligned} V^{(C)} &= V^S + W^0 + V^C \\ &= (V^S + W^0 + V^R) + (V^C - V^R) \\ &= V^{(R)} + V^{\phi}, \end{aligned} \tag{72}$$

where V^S, W^0, and V^C are a nuclear force, a short-range three-body force, and the Coulomb force, respectively. Here we introduce the three-body Jacobi-coordinate channels α, β, and γ or 1, 2, and 3. The two-body potentials V are given by V_α, V_β, and V_γ, whereas the three-body force W^0 could be presented by $W^0_{\alpha\beta}$. Therefore, Eq. (72) indicates

$$\begin{aligned} V^{(C)}_{\alpha\beta} &= V^S_\alpha\delta_{\alpha\beta} + W^0_{\alpha\beta} + V^C_\alpha\delta_{\alpha\beta} \\ &= \left(V^S_\alpha\delta_{\alpha\beta} + W^0_{\alpha\beta} + V^R_\alpha\delta_{\alpha\beta}\right) + \left(V^C_\alpha - V^R_\alpha\right)\delta_{\alpha\beta} \\ &= V^{(R)}_{\alpha\beta} + V^{\phi}_\alpha\delta_{\alpha\beta}. \end{aligned} \tag{73}$$

Hereafter we suppress the indices for simplicity except when necessary. Then the formal equation for such a three-body t matrix could be represented by

$$\begin{aligned} T^{(C)} &= V^{(C)} + V^{(C)}G_0T^{(C)} \\ &= (V^{(R)} + V^{\phi}) + (V^{(R)} + V^{\phi})G_0T^{(C)}. \end{aligned} \tag{74}$$

However, the three-body t matrix can also be decomposed using two-potential theory as

$$\begin{aligned} T^{(C)} &= \overline{\omega}^{\phi}T^{(R)}\omega^{\phi} + T^{\phi} \\ &= \overline{\omega}^{\phi}(\overline{\omega}^{R}T^{sR}\omega^{R} + T^{R})\omega^{\phi} + T^{\phi} \end{aligned} \tag{75}$$

$$\begin{aligned} &= \overline{\omega}^{\phi}[\overline{\omega}^{R}(\overline{\Omega}^{0}T\Omega^{0} + T^{0})\omega^{R} + T^{R}]\omega^{\phi} + T^{\phi} \\ &= \overline{\omega}^{C}\overline{\Omega}^{0}T\Omega^{0}\omega^{C} + \overline{\omega}^{C}T^{0}\omega^{C} + T^{C}, \end{aligned} \tag{76}$$

where these t matrices are given by the three-body Jacobi channels α, β, and γ;

$$T^{\phi} \Rightarrow T^{\phi}_{\alpha\beta} = V^{\phi}_\alpha\delta_{\alpha\beta} + \sum_\gamma V^{\phi}_\alpha\delta_{\alpha\gamma}G_0T^{\phi}_{\gamma\beta} \tag{77}$$

$$= T^{\phi}_\alpha\delta_{\alpha\beta} + \sum_\gamma T^{\phi}_\alpha\overline{\delta}_{\alpha\gamma}G_0T^{\phi}_{\gamma\beta} \tag{78}$$

$$\equiv T^{\phi}_\alpha\omega^{\phi}_{\alpha\beta} \equiv \overline{\omega}^{\phi}_{\alpha\beta}T^{\phi}_{\beta}, \tag{79}$$

$$T^{R} \Rightarrow T^{R}_{\alpha\beta} = V^{R}_\alpha\delta_{\alpha\beta} + \sum_\gamma V^{R}_\alpha G^{\phi}_{\alpha\gamma}T^{R}_{\gamma\beta}, \tag{80}$$

$$\begin{aligned} T^{sR} \Rightarrow T^{sR}_{\alpha\beta} &= \left(V^S_\alpha\delta_{\alpha\beta} + W^0_{\alpha\beta}\right) \\ &\quad + \sum_{\gamma,\delta}\left(V^S_\alpha\delta_{\alpha\gamma} + W^0_{\alpha\gamma}\right)G^C_{\gamma\delta}T^{sR}_{\delta\beta} \\ &\equiv \sum_{\gamma,\delta}\overline{\Omega}^{0}_{\alpha\gamma}T_{\gamma\delta}\Omega^0_{\delta\beta} + T^0_{\alpha\beta}, \end{aligned} \tag{81}$$

$$\begin{aligned} T^0 \Rightarrow T^0_{\alpha\beta} &= W^0_{\alpha\beta} + \sum_{\gamma,\delta}W^0_{\alpha\gamma}G^C_{\gamma\delta}T^0_{\delta\beta} \\ &\equiv \sum_\gamma W^0_{\alpha\gamma}\Omega^0_{\gamma\beta} \equiv \sum_\gamma\overline{\Omega}^{0}_{\alpha\gamma}W^0_{\gamma\beta}, \end{aligned} \tag{82}$$

$$T \Rightarrow T_{\alpha\beta} = V^S_\alpha\delta_{\alpha\beta} + \sum_\gamma V^S_\alpha G^H_{\alpha\gamma}T_{\gamma\beta}, \tag{83}$$

where $\overline{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$ is defined. Equation (78) is the three-body Faddeev equation for the AP in which $\omega^{\phi}_{\alpha\beta}$ and $\overline{\omega}^{\phi}_{\alpha\beta}$ are defined.

The resolvents of the operator forms are given by

$$G_0 = \frac{1}{E - H_0}, \tag{84}$$

$$G^{\phi} = \frac{1}{E - H_0 - V^{\phi}} = G_0 + G_0T^{\phi}G_0, \tag{85}$$

$$\begin{aligned} G^C &= \frac{1}{E - H_0 - V^R - V^{\phi}} = \frac{1}{E - H_0 - V^C} \\ &= G_0 + G_0T^CG_0, \end{aligned} \tag{86}$$

$$\begin{aligned} G^H &= \frac{1}{E - H_0 - V^C - W^0} \\ &= G^C + G^CT^0G^C. \end{aligned} \tag{87}$$

These are also represented by the matrix elements of the Jacobi-channel notation,

$$G^{\phi} \Rightarrow G^{\phi}_{\alpha\beta} = G_0\delta_{\alpha\beta} + G_0T^{\phi}_{\alpha\beta}G_0, \tag{88}$$

$$G^{C} \Rightarrow G^{C}_{\alpha\beta} = G_0\delta_{\alpha\beta} + G_0T^{C}_{\alpha\beta}G_0, \tag{89}$$

$$G^{H} \Rightarrow G^{H}_{\alpha\beta} = G^C_{\alpha\beta} + \sum_{\gamma,\delta}G^C_{\alpha\gamma}T^0_{\gamma\delta}G^C_{\delta\beta}. \tag{90}$$

Here, $T^R_{\alpha\beta}$ can be obtained from Eqs. (80) and (88) and by adopting $T^R_\alpha \equiv t^{R\phi}$ of Eq. (11);

$$\begin{aligned} T^R_{\alpha\beta} &= V^R_\alpha\delta_{\alpha\beta} + \sum_\gamma V^R_\alpha G^{\phi}_{\alpha\gamma}T^R_{\gamma\beta} \\ &= T^R_\alpha\delta_{\alpha\beta} + \sum_\gamma T^R_\alpha\left(G^{\phi}_{\alpha\gamma} - G^{\phi}_{\alpha}\delta_{\alpha\gamma}\right)T^R_{\gamma\beta} \\ &= T^R_\alpha\delta_{\alpha\beta} + \sum_\gamma T^R_\alpha G_0U^{\phi}_{\alpha\gamma}G_0T^R_{\gamma\beta} \end{aligned} \tag{91}$$

$$\equiv T^R_\alpha\omega^R_{\alpha\beta} \equiv \overline{\omega}^R_{\alpha\beta}T^R_\beta, \tag{92}$$

$$\begin{aligned} T^R_\alpha &= V^R_\alpha + V^R_\alpha G^{\phi}_\alpha T^R_\alpha = V^R_\alpha + T^R_\alpha G^{\phi}_\alpha V^R_\alpha \\ &\equiv V^R_\alpha\omega^R_\alpha \equiv \overline{\omega}^R_\alpha V^R_\alpha, \end{aligned} \tag{93}$$

where $U^{\phi}_{\alpha\beta}$ is a completely connected amplitude that is related to the AGS operator $u^{\phi}_{\alpha\beta}$ for the AP,

$$U^{\phi}_{\alpha\beta} \equiv T^{\phi}_{\alpha\beta} - T^{\phi}_\alpha\delta_{\alpha\beta} = T^{\phi}_\alpha G_0u^{\phi}_{\alpha\beta}G_0T^{\phi}_\beta. \tag{94}$$

The pure three-body Coulomb t matrix or the Rutherford transition matrix, defined by analogy with two-body case, is

$$T^C_{\alpha\beta} = \sum_{\gamma,\delta=1}^{3} \overline{\omega}^{\phi}_{\alpha\gamma} T^R_{\gamma\delta} \omega^{\phi}_{\delta\beta} + T^{\phi}_{\alpha\beta}, \tag{95}$$

where the three-body AP t matrix $T^{\phi}_{\alpha\beta}$ could be solved using the Faddeev equation (78) with the two-body boundary condition, and also with the three-body boundary condition, which will be shown later.

Now the Coulomb Møller operators with α, β, and γ channels are defined by

$$\omega^C = \omega^R \omega^{\phi} \Rightarrow \omega^C_{\zeta\beta} = \sum_{\gamma=1}^{3} \omega^R_{\zeta\gamma} \omega^{\phi}_{\gamma\beta}, \tag{96}$$

where $\omega^{\phi}_{\gamma\beta}$ and $\omega^R_{\zeta\gamma}$ are matrix elements of the three-body renormalization Møller wave operator and the screened Coulomb one, respectively. Therefore, we can solve Eq. (91) by using (78). Then, $T^C_{\alpha\beta}$ of (95) can be obtained as well as $\omega^C_{\alpha\beta}$ of (96).

Finally, the three-body channel representation of Eqs. (75) and (76) with α, β, and γ becomes

$$T^{(C)} = \sum_{\alpha,\beta=1}^{3} \sum_{\eta,\zeta=1}^{3} \sum_{\gamma,\delta=1}^{3} \left[\overline{\omega}^C_{\alpha\eta} \left(\overline{\Omega}^0_{\eta\gamma} T_{\gamma\delta} \Omega^0_{\delta\zeta} + T^0_{\eta\zeta} \right) \omega^C_{\zeta\beta} + T^C_{\alpha\beta} \right]. \tag{97}$$

By solving (82) and (83), and substituting into (97), we obtain the three-charged particle amplitude with the nuclear force and the three-body force.

More details about Eq. (97) will be discussed in the next section.

B. A new Faddeev-type equation

In this section we derive a new Faddeev-type three-body equation with a short-range potential

$$V^S = V_1 + V_2 + V_3, \tag{98}$$

where V_1 stands for a short-range two-body potential V_{23}, V_2 for V_{31}, and so on.

Let us start from Eq. (83),

$$T = (V_1 + V_2 + V_3) + (V_1 + V_2 + V_3) G^H T. \tag{99}$$

Consequently, the Faddeev-like reduction can be carried out by separating the three-body T matrix into three channels:

$$T = T^1 + T^2 + T^3 = \sum_{\alpha=1}^{3} T^{\alpha} = \sum_{\alpha,\beta} T_{\alpha\beta}\,. \tag{100}$$

Then we have

$$T^{\alpha} = V_{\alpha} + V_{\alpha} G^H \sum_{\beta=1}^{3} T^{\beta}, \tag{101}$$

$$T_{\alpha\beta} = V_{\alpha}\delta_{\alpha\beta} + \sum_{\gamma} V_{\alpha} G^H_{\alpha\gamma} T_{\gamma\beta}, \tag{102}$$

where α and β stand for the particle channels, which run from one to three, respectively.

Two-body amplitudes in the three-body Hilbert space in the Coulomb field satisfy (54), by putting $t^{sC} \to T_{\alpha}$, $V^S \to V_{\alpha}$, etc.;

$$\begin{aligned} T_{\alpha} &= V_{\alpha} + V_{\alpha} G^C_{\alpha} T_{\alpha} \\ &= V_{\alpha}\left(1 + G^C_{\alpha} T_{\alpha}\right) = \left(1 + T_{\alpha} G^C_{\alpha}\right) V_{\alpha}, \end{aligned} \tag{103}$$

with a resolvent for the α channel of

$$G^C_{\alpha} = \left(E - H_0 - V^C_{\alpha}\right)^{-1} = G_0 + G_0 T^C_{\alpha} G_0. \tag{104}$$

Multiplying the factor $(1 + T_{\alpha} G^C_{\alpha})$ to Eq. (101) from the left, we obtain

$$T^{\alpha} = T_{\alpha} + T_{\alpha} \overline{G}^H_{\alpha} T^{\alpha} + T_{\alpha} G^H \sum_{\beta} \overline{\delta}_{\alpha\beta} T^{\beta}, \tag{105}$$

$$T_{\alpha\beta} = T_{\alpha}\delta_{\alpha\beta} + T_{\alpha}(G^H_{\alpha\alpha} - G^C_{\alpha}) T_{\alpha\beta} + \sum_{\gamma} T_{\alpha} G^H_{\alpha\gamma} \overline{\delta}_{\alpha\gamma} T_{\gamma\beta}, \tag{106}$$

where we use the notation $G^H_{\alpha\beta}$ in (90) and

$$\overline{G}^H_{\alpha} = G^H_{\alpha\alpha} - G^C_{\alpha}, \tag{107}$$

$$\overline{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}. \tag{108}$$

It should be noted that $T_{\alpha\beta}$ is the same t matrix seen in Eq. (97). Now we define a new matrix $U_{\alpha\beta}$ by

$$U_{\alpha\beta} = T_{\alpha\beta} - T_{\alpha}\delta_{\alpha\beta}. \tag{109}$$

Then the Faddeev-like equation (106) is reduced to the following form:

$$\begin{aligned} U_{\alpha\beta} &= T_{\alpha}\left(\overline{G}^H_{\alpha}\delta_{\alpha\beta} + G^H_{\alpha\beta}\overline{\delta}_{\alpha\beta}\right) T_{\beta} \\ &\quad + \sum_{\gamma} T_{\alpha}\left(\overline{G}^H_{\alpha}\delta_{\alpha\gamma} + G^H_{\alpha\gamma}\overline{\delta}_{\alpha\gamma}\right) U_{\gamma\beta}, \end{aligned} \tag{110}$$

where the Born term and the kernel are reduced by using the relations

$$G^H_{\alpha\beta} = G^C_{\alpha\beta} + G^C_{\alpha\gamma} T^0_{\gamma\delta} G^C_{\delta\beta}, \tag{111}$$

$$\begin{aligned} G^C_{\alpha\beta} &= G_0\delta_{\alpha\beta} + G_0 T^C_{\alpha\beta} G_0 \\ &= \omega^C_{\alpha\beta} G_0 = G_0 \overline{\omega}^C_{\alpha\beta}, \end{aligned} \tag{112}$$

$$\begin{aligned} \overline{G}^H_{\alpha}\delta_{\alpha\beta} + G^H_{\alpha\beta}\overline{\delta}_{\alpha\beta} &= G^C_{\alpha\gamma} T^0_{\gamma\delta} G^C_{\delta\beta} \\ &\quad + \left(G^C_{\alpha\beta} - G^C_{\alpha}\right)\delta_{\alpha\beta} + G^C_{\alpha\beta}\overline{\delta}_{\alpha\beta} \\ &= G_0\big[G_0^{-1}\overline{\delta}_{\alpha\beta} + \left(T^C_{\alpha\beta} - T^C_{\alpha}\delta_{\alpha\beta}\right) \\ &\quad + \overline{\omega}^C_{\alpha\gamma} T^0_{\gamma\delta} \omega^C_{\delta\beta}\big] G_0, \end{aligned} \tag{113}$$

where, in Eqs. (111) and (113), we used dummy indices without summation signs "$\sum_{\gamma}$, $\sum_{\gamma,\delta}$" for the inner channels between two operators, and hereafter we will use the same notation for simplicity without further indication. We define the new AGS operator $u_{\alpha\beta}$ by

$$U_{\alpha\beta} \equiv T_{\alpha} G_0 u_{\alpha\beta} G_0 T_{\beta}. \tag{114}$$

Then we obtain $T_{\alpha\beta}$ from (109)

$$T_{\alpha\beta} = T_\alpha G_0 u_{\alpha\beta} G_0 T_\beta + T_\alpha \delta_{\alpha\beta}. \tag{115}$$

Therefore, by using Eqs. (110) and (113), a new AGS equation is given as follows;

$$\begin{aligned} u_{\alpha\beta} &= \left[G_0^{-1}\overline{\delta}_{\alpha\beta} + \left(T^C_{\alpha\beta} - T^C_\alpha \delta_{\alpha\beta}\right) + \overline{\omega}^C_{\alpha\gamma} T^0_{\gamma\delta}\omega^C_{\delta\beta}\right] \\ &\quad + \left[G_0^{-1}\overline{\delta}_{\alpha\gamma} + \left(T^C_{\alpha\gamma} - T^C_\alpha \delta_{\alpha\gamma}\right) + \overline{\omega}^C_{\alpha\delta} T^0_{\delta\zeta}\omega^C_{\zeta\gamma}\right] G_0 \\ &\quad \times T_\gamma G_0 u_{\gamma\beta}. \end{aligned} \tag{116}$$

For the two-body t matrix T_γ, substituting $V_\gamma = |\vec{\gamma}\rangle\lambda^s_\gamma\langle\vec{\gamma}|$ into Eq. (103), then we obtain

$$T_\gamma = V_\gamma + V_\gamma G^C_\gamma T_\gamma = |\vec{\gamma}\rangle\tau^{sC}_\gamma\langle\vec{\gamma}|, \tag{117}$$

where λ^s_γ denotes a matrix of specified rank and $|\vec{\gamma}\rangle$ is the corresponding vector.

Now, τ^{sC}_γ is a two-body propagator given by

$$\begin{aligned} \tau^{sC}_\gamma(z_\gamma) &\equiv \tau^{sC}_\gamma\left(E - q^2_\gamma/2\mu_\gamma\right) \\ &= \left[1 - \lambda^s_\gamma\langle\vec{\gamma}|G^C_\gamma(z_\gamma)|\vec{\gamma}\rangle\right]^{-1}\lambda^s_\gamma && (118) \\ &= \frac{S^{sC}_\gamma(z_\gamma)}{z_\gamma(k_\gamma) - z_\gamma\left(k^0_\gamma\right)} && (119) \\ &= \frac{S^{sC}_\gamma(z_\gamma)}{\left[E - z_\gamma\left(k^0_\gamma\right)\right] - q^2_\gamma/2\mu_\gamma} && (120) \\ &\equiv \frac{S^{sC}_\gamma(z_\gamma)}{E\left(q^0_\gamma\right) - H_0(q_\gamma)}, && (121) \end{aligned}$$

with

$$S^{sC}_\gamma(z_\gamma) = \lim_{k_\gamma\to k^0_\gamma}\left[z_\gamma(k_\gamma) - z_\gamma\left(k^0_\gamma\right)\right]\tau^{sC}_\gamma(k_\gamma), \tag{122}$$

$$\begin{aligned} E\left(q^0_\gamma\right) &= E - z_\gamma\left(k^0_\gamma\right), \\ H_0(q_\gamma) &= q^2_\gamma/2\mu_\gamma, \end{aligned} \tag{123}$$

where it has a two-body pole at energy $z_\gamma = z_\gamma(k^0_\gamma)$. Furthermore, the two-body wave function in the three-body Hilbert space is given by $|\psi^s_\gamma\rangle = G_0|\vec{\gamma}\rangle$. Therefore, by substituting (117) into Eq. (115), the three-body t matrix is given by

$$T_{\alpha\beta} = |\vec{\alpha}\rangle\tau^{sC}_\alpha\langle\psi^s_\alpha|u_{\alpha\beta}|\psi^s_\beta\rangle\tau^{sC}_\beta\langle\vec{\beta}| + |\vec{\alpha}\rangle\tau^{sC}_\alpha\langle\vec{\alpha}|\delta_{\alpha\beta}. \tag{124}$$

Finally, we obtain the integral equation for the transition matrix by sandwiching the new AGS operator between the wave functions:

$$\begin{aligned} &\langle\psi^s_\alpha|u_{\alpha\beta}|\psi^s_\beta\rangle \\ &\quad = \langle\psi^s_\alpha|\left[G_0^{-1}\overline{\delta}_{\alpha\beta} + \left(T^C_{\alpha\beta} - T^C_\alpha\delta_{\alpha\beta}\right) + \overline{\omega}^C_{\alpha\gamma}T^0_{\gamma\delta}\omega^C_{\delta\beta}\right]|\psi^s_\beta\rangle \\ &\quad\quad + \langle\psi^s_\alpha|\left[G_0^{-1}\overline{\delta}_{\alpha\gamma} + \left(T^C_{\alpha\gamma} - T^C_\alpha\delta_{\alpha\gamma}\right) + \overline{\omega}^C_{\alpha\delta}T^0_{\delta\zeta}\omega^C_{\zeta\gamma}\right]|\psi^s_\gamma\rangle \\ &\quad\quad \times \tau^{sC}_\gamma\langle\psi^s_\gamma|u_{\gamma\beta}|\psi^s_\beta\rangle. \end{aligned} \tag{125}$$

In Eq. (125), the Born term as well as the kernel contains a weakly connected Coulomb-like potential $\mathcal{V}^C_\alpha$ between a pair and the third particle. We can separate the Born term into two parts by using Eqs. (107) and (113):

$$\begin{aligned} &\langle\psi^s_\alpha|\left[G_0^{-1}\overline{\delta}_{\alpha\beta} + \left(T^C_{\alpha\beta} - T^C_\alpha\delta_{\alpha\beta}\right) + \overline{\omega}^C_{\alpha\gamma}T^0_{\gamma\delta}\omega^C_{\delta\beta}\right]|\psi^s_\beta\rangle \\ &\quad = \langle\alpha|G^H_{\alpha\beta}\overline{\delta}_{\alpha\beta}|\beta\rangle + \langle\alpha|\left(G^H_{\alpha\beta} - G^C_\alpha\right)\delta_{\alpha\beta}|\beta\rangle \\ &\quad \equiv \mathcal{B}^C_{\alpha\beta} + \mathcal{V}^C_\alpha\delta_{\alpha\beta}, \end{aligned} \tag{126}$$

where the first term of Eq. (126) is a nondiagonal term, and the second term is a diagonal one and manifests a Coulomb-like potential, which was pointed out in the MBM [16,17]. That is, we have

$$\begin{aligned} \mathcal{B}^C_{\alpha\beta} &= \langle\alpha|G^H_{\alpha\beta}|\beta\rangle\overline{\delta}_{\alpha\beta} \\ &= \langle\psi^s_\alpha|\left(G_0^{-1} + T^C_{\alpha\beta} + \overline{\omega}^C_{\alpha\gamma}T^0_{\gamma\delta}\omega^C_{\delta\beta}\right)|\psi^s_\beta\rangle\overline{\delta}_{\alpha\beta}, \end{aligned} \tag{127}$$

where we can easily confirm that the term is completely connected. While, the diagonal term is given by

$$\begin{aligned} \mathcal{V}^C_\alpha\delta_{\alpha\beta} &= \langle\alpha|\left(G^H_{\alpha\beta} - G^C_\alpha\right)|\beta\rangle\delta_{\alpha\beta} \\ &= \langle\psi^s_\alpha|\left(T^C_{\alpha\beta} - T^C_\alpha + \overline{\omega}^C_{\alpha\gamma}T^0_{\gamma\delta}\omega^C_{\delta\beta}\right)|\psi^s_\beta\rangle\delta_{\alpha\beta} \\ &= \left[\mathcal{V}^{\overline{R}}_\alpha + \left(\mathcal{V}^C_\alpha - \mathcal{V}^{\overline{R}}_\alpha\right)\right]\delta_{\alpha\beta} \equiv \left(\mathcal{V}^{\overline{R}}_\alpha + \mathcal{V}^\Phi_\alpha\right)\delta_{\alpha\beta}, \end{aligned} \tag{128}$$

where $\mathcal{V}^\Phi_\alpha$ is the three-body AP with a screening range $\rho = \overline{R}$ in which ρ is the coordinate between the c.m. of a pair and the spectator particle. $\mathcal{V}^{\overline{R}}_\alpha$ stands for a harmless screened Coulomb term but $\mathcal{V}^\Phi_\alpha$ has a Coulomb-like structure. The three-body long-range behavior can be treated the same way as those in the two-body case.

Let us separate the Born term of Eq. (126) into two parts;

$$\begin{aligned} \mathcal{B}^C_{\alpha\beta} + \mathcal{V}^C_\alpha\delta_{\alpha\beta} &= \left(\mathcal{B}^C_{\alpha\beta} + \mathcal{V}^{\overline{R}}_\alpha\delta_{\alpha\beta}\right) + \left(\mathcal{V}^C_\alpha - \mathcal{V}^{\overline{R}}_\alpha\right)\delta_{\alpha\beta} && (129) \\ &\equiv \mathcal{B}^{(\overline{R})}_{\alpha\beta} + \mathcal{V}^\Phi_\alpha\delta_{\alpha\beta}, && (130) \end{aligned}$$

$$\mathcal{B}^{(\overline{R})}_{\alpha\beta} \equiv \mathcal{B}^C_{\alpha\beta} + \mathcal{V}^{\overline{R}}_\alpha\delta_{\alpha\beta}, \tag{131}$$

$$\mathcal{V}^\Phi_\alpha \equiv \mathcal{V}^C_\alpha - \mathcal{V}^{\overline{R}}_\alpha. \tag{132}$$

The three-body rearrangement amplitude $\langle\psi^s_\alpha|u_{\alpha\beta}|\psi^s_\beta\rangle$ is also separated into two parts in the same way as Eq. (14), by using two-potential theory (see Appendix B). We define $\langle\psi^s_\alpha|u_{\alpha\beta}|\psi^s_\beta\rangle \equiv X^{(C)}_{\alpha\beta}$; then we have in Eq. (125)

$$X^{(C)}_{\alpha\beta} = \left(\mathcal{B}^{(\overline{R})}_{\alpha\beta} + \mathcal{V}^\Phi_\alpha\delta_{\alpha\beta}\right) + \left(\mathcal{B}^{(\overline{R})}_{\alpha\gamma} + \mathcal{V}^\Phi_\alpha\delta_{\alpha\gamma}\right)\tau^{sC}_\gamma X^{(C)}_{\gamma\beta}, \tag{133}$$

where the three-body AP t matrix satisfies

$$X^\Phi_{\alpha\alpha} = \mathcal{V}^\Phi_\alpha + \mathcal{V}^\Phi_\alpha\tau^{sC}_\alpha X^\Phi_{\alpha\alpha} \equiv \mathcal{V}^\Phi_\alpha\Omega^\Phi_\alpha \equiv \overline{\Omega}^\Phi_\alpha\mathcal{V}^\Phi_\alpha, \tag{134}$$

with

$$X^{(C)}_{\alpha\beta} = \overline{\Omega}^\Phi_\alpha X^{(\overline{R})}_{\alpha\beta}\Omega^\Phi_\beta + X^\Phi_{\alpha\alpha}\delta_{\alpha\beta}, \tag{135}$$

$$\begin{aligned} X^{(\overline{R})}_{\alpha\beta} &= \mathcal{B}^{(\overline{R})}_{\alpha\beta} + \mathcal{B}^{(\overline{R})}_{\alpha\gamma}\tau^{sC\Phi}_\gamma X^{(\overline{R})}_{\gamma\beta} \\ &= \left(\mathcal{B}^C_{\alpha\beta} + \mathcal{V}^{\overline{R}}_\alpha\delta_{\alpha\beta}\right) + \left(\mathcal{B}^C_{\alpha\gamma} + \mathcal{V}^{\overline{R}}_\alpha\delta_{\alpha\gamma}\right)\tau^{sC\Phi}_\gamma X^{(\overline{R})}_{\gamma\beta} && (136) \\ &= \overline{\Omega}^{\overline{R}}_\alpha\chi^{sC}_{\alpha\beta}\Omega^{\overline{R}}_\beta + X^{\overline{R}}_{\alpha\alpha}\delta_{\alpha\beta}, && (137) \end{aligned}$$

where the submatrices, sub-Møller operators, and the extended propagators are defined by

$$X^{\overline{R}}_{\alpha\alpha} = \mathcal{V}^{\overline{R}}_\alpha + \mathcal{V}^{\overline{R}}_\alpha\tau^{sC\Phi}_\alpha X^{\overline{R}}_{\alpha\alpha} \equiv \mathcal{V}^{\overline{R}}_\alpha\Omega^{\overline{R}}_\alpha \equiv \overline{\Omega}^{\overline{R}}_\alpha\mathcal{V}^{\overline{R}}_\alpha, \tag{138}$$

with

$$\tau_\alpha^{sC\Phi} = \frac{S_\alpha^{sC}(z_\alpha)}{E(q_\alpha^0) - H_0(q_\alpha) - \mathcal{V}_\alpha^\Phi} = \tau_\alpha^{sC} + \tau_\alpha^{sC} X_{\alpha\alpha}^\Phi \tau_\alpha^{sC}. \tag{139}$$

The core t matrix $\chi_{\alpha\beta}^{sC}$ satisfies

$$\chi_{\alpha\beta}^{sC} = \mathcal{B}_{\alpha\beta}^C + \mathcal{B}_{\alpha\gamma}^C \mathcal{T}_\gamma^C \chi_{\gamma\beta}^{sC}, \tag{140}$$

where a new resolvent, given by using Eq. (139), is

$$\mathcal{T}_\gamma^C = \frac{S_\gamma^{sC}(z_\gamma)}{E(q_\gamma^0) - H_0(q_\gamma) - \mathcal{V}_\alpha^\Phi - \mathcal{V}_\alpha^{\overline{R}}} = \frac{S_\gamma^{sC}(z_\gamma)}{E(q_\gamma^0) - H_0(q_\gamma) - \mathcal{V}_\alpha^C} = \tau_\gamma^{sC} + \tau_\gamma^{sC} X_{\gamma\gamma}^C \tau_\gamma^{sC}. \tag{141}$$

Here the resolvent $\mathcal{T}_\gamma^C$ is nonsingular at the singular point of $\mathcal{V}_\alpha^C$; therefore, Eq. (140) is solvable.

The three-body rearrangement t matrix, given by a sandwiched form by substituting (137) into (135), is

$$\begin{aligned} X_{\alpha\beta}^{(C)} &= \overline{\Omega}_\alpha^\Phi \big(\overline{\Omega}_\alpha^{\overline{R}} \chi_{\alpha\beta}^{sC} \Omega_\beta^{\overline{R}} + X_{\alpha\alpha}^{\overline{R}} \delta_{\alpha\beta}\big) \Omega_\beta^\Phi + X_{\alpha\alpha}^\Phi \delta_{\alpha\beta} \\ &= \overline{\Omega}_\alpha^C \chi_{\alpha\beta}^{sC} \Omega_\beta^C + \big(\overline{\Omega}_\alpha^\Phi X_{\alpha\alpha}^{\overline{R}} \Omega_\beta^\Phi + X_{\alpha\alpha}^\Phi\big)\delta_{\alpha\beta} \\ &= \overline{\Omega}_\alpha^C \chi_{\alpha\beta}^{sC} \Omega_\beta^C + X_{\alpha\alpha}^C \delta_{\alpha\beta}, \end{aligned} \tag{142}$$

where the Coulomb Møller wave operators are defined by

$$\begin{aligned} \Omega_\alpha^C &\equiv \Omega_\alpha^{\overline{R}} \Omega_\alpha^\Phi, \\ \overline{\Omega}_\alpha^C &\equiv \overline{\Omega}_\alpha^\Phi \overline{\Omega}_\alpha^{\overline{R}}. \end{aligned} \tag{143}$$

Then the pure three-body Coulomb t matrix is given by

$$X_{\alpha\alpha}^C \equiv \overline{\Omega}_\alpha^\Phi X_{\alpha\alpha}^{\overline{R}} \Omega_\alpha^\Phi + X_{\alpha\alpha}^\Phi. \tag{144}$$

Finally, we obtain the three-body rearrangement t matrix from

$$\langle \psi_\alpha^s | u_{\alpha\beta} | \psi_\beta^s \rangle = \overline{\Omega}_\alpha^C \chi_{\alpha\beta}^{sC} \Omega_\beta^C + X_{\alpha\alpha}^C \delta_{\alpha\beta} \equiv X_{\alpha\beta}^{(C)}. \tag{145}$$

Therefore, the t matrix in Eq. (124) can be rewritten as

$$T_{\alpha\beta} = |\vec{\alpha}\rangle \tau_\alpha^{sC} \big(\overline{\Omega}_\alpha^C \chi_{\alpha\beta}^{sC} \Omega_\beta^C + X_{\alpha\alpha}^C \delta_{\alpha\beta}\big) \tau_\beta^{sC} \langle \vec{\beta}| + |\vec{\alpha}\rangle \tau_\alpha^{sC} \langle \vec{\alpha}| \delta_{\alpha\beta}. \tag{146}$$

Consequently, the [3] → [3] t matrix is obtained by substituting (146) into (97);

$$\begin{aligned} T^{(C)} &= \sum_{\alpha,\beta=1}^{3} \sum_{\eta,\zeta=1}^{3} \sum_{\gamma,\delta=1}^{3} \Big(\overline{\omega}_{\alpha\eta}^C \big\{\overline{\Omega}_{\eta\gamma}^0 \big[|\vec{\gamma}\rangle \tau_\gamma^{sC} \big(\overline{\Omega}_\gamma^C \chi_{\gamma\delta}^{sC} \Omega_\delta^C \\ &\quad + X_{\gamma\gamma}^C \delta_{\gamma\delta}\big) \tau_\delta^{sC} \langle \vec{\delta}| + |\vec{\gamma}\rangle \tau_\gamma^{sC} \langle \vec{\gamma}| \delta_{\gamma\delta}\big] \Omega_{\delta\zeta}^0 + T_{\eta\zeta}^0 \big\} \\ &\quad \times \omega_{\zeta\beta}^C + T_{\alpha\beta}^C\Big) \\ &= \sum_{\alpha,\beta=1}^{3} \sum_{\gamma,\delta=1}^{3} \sum_{\eta,\zeta=1}^{3} \overline{\omega}_{\alpha\eta}^C \overline{\Omega}_{\eta\gamma}^0 |\gamma\rangle \tau_\gamma^{sC} \mathcal{A}_{\gamma\delta} \tau_\delta^{sC} \langle \delta | \Omega_{\delta\zeta}^0 \omega_{\zeta\beta}^C \\ &\quad + \sum_{\alpha,\beta=1}^{3} \left(\sum_{\gamma,\delta=1}^{3} \overline{\omega}_{\alpha\gamma}^C T_{\gamma\delta}^0 \omega_{\delta\beta}^C + T_{\alpha\beta}^C \right), \end{aligned} \tag{147}$$

where the first term originates from the new AGS term, the second one is the Rutherford term, and the last one is the direct propagation. One finds that an onion-like structure appears in a multipotential t matrix. The label $\mathcal{A}_{\gamma\delta}$ is defined by

$$\mathcal{A}_{\gamma\delta} = \overline{\Omega}_\gamma^C \chi_{\gamma\delta}^{sC} \Omega_\delta^C + X_{\gamma\gamma}^C \delta_{\gamma\delta} + \big(\tau_\gamma^{sC}\big)^{-1} \delta_{\gamma\delta}. \tag{148}$$

This term stands for the [2] to [2] rearrangement amplitude in terms of the full interaction.

More clearly explicating this formula term by term, one can show that by using implied summations over indices, and also by (97), (115), (95), (124), (96), (146), and (143),

$$\begin{aligned} T^{(C)} &= \overline{\omega}_{\alpha\xi}^\phi \overline{\omega}_{\xi\eta}^R \overline{\Omega}_{\eta\gamma}^0 T_\gamma G_0 u_{\gamma\delta} G_0 T_\delta \Omega_{\delta\zeta}^0 \omega_{\zeta\rho}^R \omega_{\rho\beta}^\phi \\ &\quad + \overline{\omega}_{\alpha\xi}^\phi \overline{\omega}_{\xi\eta}^R \overline{\Omega}_{\eta\gamma}^0 T_\gamma \Omega_{\gamma\zeta}^0 \omega_{\zeta\rho}^R \omega_{\rho\beta}^\phi + \overline{\omega}_{\alpha\xi}^\phi \overline{\omega}_{\xi\eta}^R T_{\eta\zeta}^0 \omega_{\zeta\rho}^R \omega_{\alpha\beta}^\phi \\ &\quad + \overline{\omega}_{\alpha\xi}^\phi T_{\xi\rho}^R \omega_{\rho\beta}^\phi + T_{\alpha\beta}^\phi \\ &= \overline{\omega}_{\alpha\xi}^C \overline{\Omega}_{\xi\gamma}^0 \big(|\gamma\rangle \tau_\gamma^{sC} \langle \gamma | G_0 u_{\gamma\delta} G_0 | \delta \rangle \tau_\delta^{sC} \langle \delta | + T_\gamma\big) \Omega_{\gamma\zeta}^0 \omega_{\rho\beta}^C \\ &\quad + \overline{\omega}_{\alpha\xi}^C T_{\eta\zeta}^0 \omega_{\alpha\beta}^C + T_{\alpha\beta}^C \end{aligned} \tag{149}$$

$$\begin{aligned} &= \overline{\omega}_{\alpha\xi}^C \overline{\Omega}_{\xi\gamma}^0 |\gamma\rangle \big(\tau_\gamma^{sC} \overline{\Omega}_\gamma^C \chi_{\gamma\delta}^{sC} \Omega_\delta^C \tau_\delta^{sC} + \tau_\gamma^{sC} X_{\gamma\gamma}^C \delta_{\gamma\delta} \tau_\delta^{sC} \\ &\quad + \tau_\gamma^{sC} \delta_{\gamma\delta}\big) \langle \delta | \Omega_{\delta\zeta}^0 \omega_{\rho\beta}^C + \overline{\omega}_{\alpha\gamma}^C T_{\gamma\delta}^0 \omega_{\delta\beta}^C + T_{\alpha\beta}^C \end{aligned} \tag{150}$$

$$= \overline{\omega}_{\alpha\xi}^C \overline{\Omega}_{\xi\gamma}^0 |\gamma\rangle \tau_\gamma^{sC} \mathcal{A}_{\gamma\delta} \tau_\delta^{sC} \langle \delta | \Omega_{\delta\zeta}^0 \omega_{\rho\beta}^C + \overline{\omega}_{\alpha\gamma}^C T_{\gamma\delta}^0 \omega_{\delta\beta}^C + T_{\alpha\beta}^C. \tag{151}$$

The amplitude in the core of the onion-like structure satisfies a Faddeev-type integral equation with a screened Coulomb potential that is completely prescribed by the three-body boundary conditions, which will be seen in the next section. Therefore, all of the calculations are performed without any ambiguity using short-range potential scattering methods.

C. Three-body boundary condition

In the three-body system, we must define two boundary conditions, not only for the two-body relative coordinate r but also for the Jacobi coordinate ρ between the c.m. of two particles and the spectator to make a specific range $\rho = \overline{R}$. Here, the Coulomb phase shift of the α channel, $\overline{\sigma}_L(q_0)$, is not given by the screening phase shift plus the renormalization phase, $\overline{\sigma}_L(q_0) = \delta_L^{\overline{R}\Phi}(q_0) + \Phi(\overline{R}, q_0, L)$, but by the phase shift $\delta_L^{\overline{R}\Phi}(q_0)$ with respect to the on-shell t matrix of Eq. (138) under the condition that the on-shell auxiliary t matrix of (134) vanishes, where q_0 and L are the corresponding momentum and the orbital angular momentum, respectively.

Furthermore, in Eq. (134), we demand that the on-shell and half-on-shell auxiliary $X^\Phi(E)$ matrix vanish; that is,

$$\begin{aligned} X_{\alpha\alpha}^\Phi(q_0, q_0; E) &= 0, \\ X_{\alpha\alpha}^\Phi(q_0, q'; E) &= 0, \\ X_{\alpha\alpha}^\Phi(q, q_0; E) &= 0, \end{aligned} \tag{152}$$

where q_0 stands for the on-shell momentum of the α channel. This leads to the auxiliary phase that satisfies

$$\Phi(\overline{R}, q_0, L) = \pm\pi m \quad (m = 0, 1, 2, \ldots.), \tag{153}$$

and also the half-on and half-off-shell operators satisfy $\overline{\Omega}_\alpha^\Phi = \Omega_\alpha^\Phi = 1$. Therefore, we obtain the phase shift from Eqs. (138) and (144);

$$\overline{\sigma}_L(q_0) = \tan^{-1}\left[\frac{\mathrm{Im}\left(\overline{\Omega}_\alpha^\Phi X_{\alpha\alpha}^{\overline{R}} \Omega_\alpha^\Phi\right)}{\mathrm{Re}\left(\overline{\Omega}_\alpha^\Phi X_{\alpha\alpha}^{\overline{R}} \Omega_\alpha^\Phi\right)}\right] = \tan^{-1}\left[\frac{\mathrm{Im}\left(X_{\alpha\alpha}^{\overline{R}}\right)}{\mathrm{Re}\left(X_{\alpha\alpha}^{\overline{R}}\right)}\right] \tag{154}$$

$$\equiv \delta_L^{\overline{R}\Phi}(q_0). \tag{155}$$

Therefore, Eq. (134) is rewritten as

$$X_{\alpha\alpha}^\Phi(q, q'; E) - \mathcal{V}_\alpha^\Phi(q, q'; E) = P\int_0^\infty \mathcal{V}_\alpha^\Phi(q, q''; E)\tau_\alpha^{sC}(q''; E) X_{\alpha\alpha}^\Phi(q'', q'; E)dq'' \equiv J_{\alpha\beta}^P(q, q'; E), \tag{156}$$

where P stands for the principal part of the singular integral that comes from the pole of $\tau_\alpha^{sC}(q''; E)$. Therefore, $J_{\alpha\beta}^P(q, q'; E)$ is free from the singularity coincidence, and it is converged as well as that in the two-body case which, is given by Lemmas 1 and 2.

From (153) and (154), one could say that the three-body renormalization formula of the MBM is an approximation because, for instance, in Eq. (70), $Z_R^{-1}(q_e)$ is not fixed at unity but oscillates for $R \to \infty$.

Here, one can obtain an energy-dependent range for the α channel of

$$\overline{R}(q_0, L) = \frac{1}{2q_0}\exp\left[\frac{\overline{C}(q_0, L)}{\overline{\eta}(q_0)}\right] \equiv \overline{R}_{cL}(q_0). \tag{157}$$

At this boundary, the effective Coulomb t matrix between a charged cluster and the charged spectator is obtained. We can obtain the inverse functions for Eqs. (39) and (157), as $k = R_{cl}^{-1}(R)$ and $q_0 = \overline{R}_{cL}^{-1}(\overline{R})$, respectively. A closed-circuit relation between the two- and three-body boundary conditions exists;

$$\frac{1}{2\nu_\alpha}\left[R_{cl}^{-1}(R)\right]^2 + \frac{1}{2\mu_\alpha}\left[\overline{R}_{cL}^{-1}(\overline{R})\right]^2 = E, \tag{158}$$

where E is the three-body energy and ν_α and μ_α are the reduced masses for the corresponding channel, respectively.

If we have a simple energy-dependent form such as $R = A_l/k$ and $\overline{R} = B_L/q_0$ with constants A_l and B_L, then energy conservation for the three-body system defines a boundary region

$$\frac{1}{2\nu_\alpha}\left(\frac{A_l}{R}\right)^2 + \frac{1}{2\mu_\alpha}\left(\frac{B_L}{\overline{R}}\right)^2 = E. \tag{159}$$

On the closed circuit (158) or (159), the two- and the three-body renormalization phases $\phi(R, k, l)$ and $\Phi(\overline{R}, q_0, L)$ become $\pm\pi m (m = 0, 1, 2, \ldots)$.

Finally, we can conclude that the two- and three-body boundary conditions are mutually linked in the long-range Coulomb field.

IV. SUMMARY AND DISCUSSION

To obtain a rigorous Coulomb t matrix in momentum space, we have divided the Coulomb potential into a major screened Coulomb potential and a minor but effectively long-range auxiliary potential. We proposed a new boundary condition model on-shell and half-on-shell AP t matrix which is zero at the range $r = R_{cl}$, but the off-shell t matrix is not. Then the AP phase shift becomes zero or $\pm\pi n$. Here the off-shell K-matrix equation for the AP is solved easily at this boundary because no singularities coincide in the integral kernel. It should be noted that the LS equation for the SCP has a T^ϕ-modified resolvent that is not that of the free Green's function. Therefore, the SCP phase shift is not the phase shift resulting from the pure SCP. As a result, our boundary condition allows the SCP amplitude to yield the rigorous Coulomb phase shift. Hence, the long-range difficulty in obtaining the Coulomb amplitude is completely removed within any required accuracy. Furthermore, the fully off-shell Coulomb t matrix is given by the off-shell K matrixes of the AP and the SCP using two-potential theory. One could say that the Coulomb LS equation cannot be solved but that the solution can be obtained by using the AP K matrix and the specified boundary condition, although it is not the typical boundary condition of the differential equation in r space.

As a consequence, we can reliably calculate the amplitude without any difficulty after we have required zero on-shell and half-on-shell amplitudes.

For the practical calculation in two-body problems, it is useful to recall our algorithm:

(i) Find the screening range $R_{cl}(k)$ of Eq. (39) and obtain the fully off-shell t- or K matrix T^ϕ with R_{cl} and calculate the kernel (36) as well as G^ϕ.
(ii) Solve the LS equation (34) for the short-range amplitude $t^{R\phi}$ with potential V^R.
(iii) Find the on-shell amplitude to obtain $\delta^{R\phi}$, which becomes the Coulomb phase shift.
(iv) Calculate Eq. (47), which is the fully off-shell Coulomb t matrix.
(v) Substitute (47) into (56) to generate this kernel.
(vi) By using the kernel, solve Eqs. (54) and (55), and substitute into (51). Then we obtain the two-body nuclear amplitude.

In the case of a separable short range potential, the nuclear two-body propagator is also represented by a Coulomb modified form, but it has no trouble with the long range difficulty. This fact is mathematically proved by using the AP-formulation.

In the three-charged particle system, the three-body Faddeev equation is only verified for the AP, but the major SCP leads to another Faddeev-like equation. This equation is also calculated with the results of the AP-Faddeev amplitude. As a consequence, the three-charged particle Coulomb t matrix is obtained by using the generalized two-potential theory which is mentioned in Appendix B.

The previously proposed phase shift renormalization method [16–19] can not improve the precision, in principle, because the phase of (71) has no unique value for a large Coulomb parameter. Moreover, such a method can not supply a rigorous off-shell amplitude. Furthermore, the method is not adequate for the three-body break-up reaction because of the missing Coulomb three-body break-up Møller operator $\omega^C_{\alpha\beta}$ even if the three-body force break-up operator $\Omega^0_{\alpha\beta}$ is omitted. Therefore, our formulation presents the entire three-body amplitude for the first time, which has ever been seen before.

It may be useful to summarize the algorism advocated to obtain the pure Coulomb three-body t matrix in practice:

(i) Prepare T^{ϕ}_{α} from Eq. (33). Substitute it into (78) to obtain $T^{\phi}_{\alpha\beta}$.
(ii) Solve (11) or (34) to obtain $t^{R\phi} \equiv T^R_{\alpha}$. Substitute it into (91) to obtain $T^R_{\alpha\beta}$.
(iii) Calculate (95) to obtain the pure Coulomb three-body t matrix $T^C_{\alpha\beta}$.

This process is for the pure three-body Coulomb problem.

Next for the three-charged "nuclear problems", we have to take into account the above results and the nuclear short range forces by the following routine;

(iv) Obtain $|\overrightarrow{\gamma} >$ and $\tau^{sC}_{\gamma}(z)$ from (117), and (118).
(v) Obtain $X^C_{\alpha\alpha}$ by Eqs. (134), (138), and (144).
(vi) Substitute $X^C_{\alpha\alpha}$ and $\tau^{sC}_{\gamma}(z)$ into (141) to obtain $\mathcal{T}^C_{\gamma}$.
(vii) Calculate $T^0_{\alpha\beta}$ from (82). Substitute $T^0_{\alpha\beta}$ and $T^C_{\alpha\beta}$ and $\omega^C_{\alpha\beta}$ of (96) into (127) to obtain $\mathcal{B}^C_{\alpha\beta}$.
(viii) Solve (140) using $\mathcal{B}^C_{\alpha\beta}$ and $\mathcal{T}^C_{\gamma}$ to obtain $\chi^{sC}_{\alpha\beta}$.
(ix) Obtain $\langle u_{\alpha\beta}\rangle$ from (145) by using Ω^C_{α} of (143).
(x) From (150) and (151), one can calculate any physical amplitude.

Finally, our three-body t matrix formalism is presented in a universal style following the mathematically rigorous manner to use it in practical applications involving the Coulomb force, a three-body force, and a short-range force. It includes the three-particle to three-particle amplitude [3] → [3], two to three [2] → [3] (breakup), or vice versa [3] → [2] (absorption), two to two [2] → [2] (rearrangement), and cascade [1] → [2] and [1] → [3] (photo-disintegration), respectively. These amplitudes include the initial- and final-state interactions exactly, interactions in which the two- and three-body forces and the Coulomb force mutually interfere.

One can include interference effects in Coulomb breakup problems, for instance. Furthermore, it could be applicable even for photoabsorption, photoemission, and photodisintegration problems, that are frequently occurring three-body problems. This paper includes some points from previously published articles [37] and [38] by the author that are very useful in terms of highlighting new aspects of physics, but the author must admit that some parts in the articles [37] and [38] involve an approximation with respect to the Møller operator and its phase-shift representation $e^{i\phi}$. Consequently, these approximations are corrected in the present article. Practical calculations will be presented elsewhere.

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APPEENDIX A: THE UNITARITY RELATION

Let us prove the unitary relation for the new three-body AGS-like Eq. (116). Consider the matrix form of Eq. (116),

$$[\mathbf{u}(E)]_{\alpha\beta} \equiv u_{\alpha\beta}(E), \tag{A1}$$

$$[\mathbf{u}_0(E)]_{\alpha\beta} \equiv \left(G_0^{-1}\overline{\delta}_{\alpha\beta}\right) + \left(T^C_{\alpha\beta} - T^C_{\alpha}\delta_{\alpha\beta}\right) + \left(\overline{\omega}^C_{\gamma\delta}T^0_{\gamma\delta}\omega^C_{\delta\beta}\right) \tag{A2}$$

$$(\mathbf{\Gamma})_{\alpha\beta} \equiv G_0 T_{\alpha} G_0 \delta_{\alpha\beta}. \tag{A3}$$

Then Eq. (116) becomes

$$\mathbf{u}(E) = \mathbf{u}_0(E) + \mathbf{u}_0(E)\mathbf{\Gamma}(E)\mathbf{u}(E). \tag{A4}$$

Let us multiply $\mathbf{u}_0^{-1}(E)$ from the left in Eq. (A4) and $\mathbf{u}^{-1}(E)$ from the right to obtain

$$\mathbf{u}^{-1}(E) = \mathbf{u}_0^{-1}(E) - \mathbf{\Gamma}(E). \tag{A5}$$

Here the amplitude operator $\mathbf{u}(E)$ stands for $E = E^+ = E + i\varepsilon$, and $\mathbf{u}(E)^\dagger = \mathbf{u}(E - i\varepsilon)$. Therefore, we have

$$\left[\mathbf{u}^{-1}(E)\right]^\dagger = \left[\mathbf{u}_0^{-1}(E)\right]^\dagger - \left[\mathbf{\Gamma}(E)\right]^\dagger. \tag{A6}$$

Now we define

$$\Delta\mathbf{u}^{-1} \equiv \left[\mathbf{u}^{-1}(E)\right]^\dagger - \left[\mathbf{u}^{-1}(E)\right], \tag{A7}$$

$$\Delta\mathbf{u}_0^{-1} \equiv \left[\mathbf{u}_0^{-1}(E)\right]^\dagger - \left[\mathbf{u}_0^{-1}(E)\right], \tag{A8}$$

$$\Delta\mathbf{\Gamma} \equiv \mathbf{\Gamma}^\dagger - \mathbf{\Gamma}. \tag{A9}$$

Consequently, we obtain from (A5) and (A6)

$$\Delta\mathbf{u}^{-1} = \Delta\mathbf{u}_0^{-1} - \Delta\mathbf{\Gamma}. \tag{A10}$$

Sandwiching the equation between $\mathbf{u}^\dagger$ and $\mathbf{u}$, we get

$$\begin{aligned}\mathbf{u}^\dagger\left(\mathbf{u}^{\dagger -1} - \mathbf{u}^{-1}\right)\mathbf{u} &= \mathbf{u} - \mathbf{u}^\dagger \\ &= \mathbf{u}^\dagger\left(\Delta\mathbf{u}_0^{-1}\right)\mathbf{u} - \mathbf{u}^\dagger\Delta\mathbf{\Gamma}\mathbf{u} \\ &\equiv -\Delta\mathbf{u}.\end{aligned} \tag{A11}$$

In the same way, we can calculate $\Delta\mathbf{u}_0^{-1}$:

$$\mathbf{u}_0^\dagger\left(\mathbf{u}_0^{\dagger -1} - \mathbf{u}_0^{-1}\right)\mathbf{u}_0 = \mathbf{u}_0 - \mathbf{u}_0^\dagger \equiv -\Delta\mathbf{u}_0. \tag{A12}$$

Therefore (A11) becomes

$$\Delta\mathbf{u} = \mathbf{u}^\dagger\left(\mathbf{u}_0^{-1}\right)^\dagger\Delta\mathbf{u}_0\left(\mathbf{u}_0^{-1}\right)\mathbf{u} + \mathbf{u}^\dagger\Delta\mathbf{\Gamma}\mathbf{u}. \tag{A13}$$

Before we calculate $\Delta\mathbf{u}_0$ we must redefine $\mathbf{u}_0$ (A2);

$$\begin{aligned}(\mathbf{u}_0)_{\alpha\beta} &= G_0^{-1}\overline{\delta}_{\alpha\beta} - T^C_{\alpha}\delta_{\alpha\beta} + T^C_{\alpha\beta} + \overline{\omega}^C_{\alpha\gamma}T^0_{\gamma\delta}\omega^C_{\delta\beta} \\ &= G_0^{-1}\overline{\delta}_{\alpha\beta} - T^C_{\alpha}\delta_{\alpha\beta} + \mathcal{T}^C_{\alpha\beta},\end{aligned} \tag{A14}$$

with

$$\mathcal{T}^C \equiv T^C + \overline{\omega}^C T^0 \omega^C = (V^C + W_0) + (V^C + W_0) G_0 \mathcal{T}^C. \quad (A15)$$

Then we replace the discontinuity of $\mathbf{u}_0$ by

$$\begin{aligned}(\Delta \mathbf{u}_0)_{\alpha\beta} &= (\mathbf{u}_0^\dagger - \mathbf{u}_0)_{\alpha\beta} \\ &= -\left(T_\alpha^{C\dagger} - T_\alpha^C\right)\delta_{\alpha\beta} + (\mathcal{T}^{C\dagger} - \mathcal{T}^C)_{\alpha\beta} \\ &= -2\pi i T_\alpha^{C\dagger}\delta(E - H_0) T_\alpha^C \delta_{\alpha\beta} \\ &\quad + 2\pi i \left[\mathcal{T}^{C\dagger}\delta(E - H_0)\mathcal{T}^C\right]_{\alpha\beta}. \end{aligned} \quad (A16)$$

Finally, we obtain $\Delta\mathbf{\Gamma}$. Because of Eq. (117) and Eq. (A3), $\mathbf{\Gamma}$ satisfies the following equation, with $(\mathbf{C})_{\alpha\beta} \equiv G_0 V_\alpha^s G_0 \delta_{\alpha\beta}$ and $(\mathbf{K})_{\alpha\beta} \equiv G_0 V_\alpha^s \omega_\alpha^C \delta_{\alpha\beta}$;

$$\mathbf{\Gamma} = \mathbf{C} + \mathbf{K}\mathbf{\Gamma}. \quad (A17)$$

Multiplying $\mathbf{C}^{-1}$ from the left in (A17) and $\mathbf{\Gamma}^{-1}$ from the right, we get

$$\mathbf{\Gamma}^{-1} = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{K}. \quad (A18)$$

Here, $\mathbf{C}^{-1}\mathbf{K}$ becomes

$$\begin{aligned}(\mathbf{C}^{-1}\mathbf{K})_{\alpha\beta} &= \left(G_0 V_\alpha^s G_0\right)^{-1} G_0 V_\alpha^s \omega_\alpha^C \delta_{\alpha\beta} \\ &= G_0^{-1}\delta_{\alpha\beta} + T_\alpha^C \delta_{\alpha\beta}. \end{aligned} \quad (A19)$$

Then we obtain

$$(\mathbf{\Gamma}^{-1})_{\alpha\beta} = \left(\mathbf{C}^{-1}\right)_{\alpha\beta} - G_0^{-1}\delta_{\alpha\beta} - T_\alpha^C \delta_{\alpha\beta}. \quad (A20)$$

Therefore, we have

$$(\Delta\mathbf{\Gamma}^{-1})_{\alpha\beta} = (\Delta\mathbf{C}^{-1})_{\alpha\beta} - \Delta T_\alpha^C \delta_{\alpha\beta}. \quad (A21)$$

Sandwiching $\Delta\mathbf{\Gamma}^{-1}$ between $\mathbf{\Gamma}^\dagger$ and $\mathbf{\Gamma}$, we obtain

$$\mathbf{\Gamma}^\dagger(\mathbf{\Gamma}^{\dagger^{-1}} - \mathbf{\Gamma}^{-1})\mathbf{\Gamma} = -\Delta\mathbf{\Gamma}, \quad (A22)$$

$$\begin{aligned}\Delta\mathbf{\Gamma} &= \mathbf{\Gamma}^\dagger(\mathbf{C}^{-1})^\dagger \Delta\mathbf{C}(\mathbf{C}^{-1})\mathbf{\Gamma} \\ &\quad + 2\pi i \mathbf{\Gamma}^\dagger\left[T_\alpha^{C\dagger}\delta(E - H_0) T_\alpha^C \delta_{\alpha\beta}\right]\mathbf{\Gamma}. \end{aligned} \quad (A23)$$

Since $\Delta\mathbf{C}$ is zero for the real potential V_α^s, the new AGS operator satisfies the unitarity condition

$$\begin{aligned}\Delta\mathbf{u} &= 2\pi i \mathbf{u}^\dagger \mathbf{u}_0^{-1^\dagger} \mathcal{T}^{C\dagger}\delta(E - H_0)\mathcal{T}^C \mathbf{u}_0^{-1}\mathbf{u} \\ &\quad - 2\pi i \mathbf{u}^\dagger(\mathbf{u}_0^{-1})^\dagger\left[T_\alpha^{C\dagger}\delta(E - H_0) T_\alpha^C \delta_{\alpha\beta}\right]\mathbf{u}_0^{-1}\mathbf{u} \\ &\quad + 2\pi i \mathbf{u}^\dagger \mathbf{\Gamma}^\dagger\left[T_\alpha^{C\dagger}\delta(E - H_0) T_\alpha^C \delta_{\alpha\beta}\right]\mathbf{\Gamma}\mathbf{u}. \end{aligned} \quad (A24)$$

APPENDIX B: GENERALIZED TWO-POTENTIAL THEORY

To derive the generalized two-potential formulation, let us start from the following LS-type equation:

$$\begin{aligned}X_{\alpha\beta}(q_\alpha, q'_\beta; E) &= \left[V_{\alpha\beta}(q_\alpha, q'_\beta; E) + V^0_{\alpha\beta}(q_\alpha, q'_\beta; E)\right] \\ &\quad + \int_0^\infty dq''_\gamma \int_0^\infty dq'''_\delta \left[V_{\alpha\gamma}(q_\alpha, q''_\gamma; E)\right. \\ &\quad \left. + V^0_{\alpha\gamma}(q_\alpha, q''_\gamma; E)\right] G^0_{\gamma\delta}(q''_\gamma, q'''_\delta; E) \\ &\quad \times X_{\delta\beta}(q'''_\delta, q'_\beta; E). \end{aligned} \quad (B1)$$

We can put this in matrix form by omitting the momentum variables as well as the signs of the integral and the summation;

$$\mathbf{X} = (\mathbf{V} + \mathbf{V}^0) + (\mathbf{V} + \mathbf{V}^0)\mathbf{G}^0\mathbf{X}, \quad (B2)$$

with

$$\mathbf{X} \equiv X_{\alpha\beta}(q_\alpha, q'_\beta; E), \quad (B3)$$

$$\mathbf{V} \equiv V_{\alpha\beta}(q_\alpha, q'_\beta; E), \quad (B4)$$

$$\mathbf{V}^0 \equiv V^0_{\alpha\beta}(q_\alpha, q'_\beta; E), \quad (B5)$$

$$\mathbf{G}^0 \equiv G^0_{\alpha\beta}(q_\alpha, q'_\beta; E). \quad (B6)$$

Here we take a LS-type equation for $\mathbf{V}^0$,

$$\mathbf{X}^0 = \mathbf{V}^0 + \mathbf{V}^0\mathbf{G}^0\mathbf{X}^0 \quad (B7)$$

$$\equiv \mathbf{V}^0\mathbf{\Omega}^0 \equiv \overline{\mathbf{\Omega}}^0\mathbf{V}^0, \quad (B8)$$

with

$$\begin{aligned}\mathbf{\Omega}^0 &\equiv \mathbf{1} + \mathbf{G}^0\mathbf{X}^0, \\ \overline{\mathbf{\Omega}}^0 &\equiv \mathbf{1} + \mathbf{X}^0\mathbf{G}^0. \end{aligned} \quad (B9)$$

Or, we can rewrite these in the concrete form

$$\begin{aligned}\Omega^0_{\alpha\beta} &= \delta_{\alpha\beta} + G^0_{\alpha\gamma}X^0_{\gamma\beta}, \\ \overline{\Omega}^0_{\alpha\beta} &= \delta_{\alpha\beta} + X^0_{\alpha\gamma}G^0_{\gamma\beta}. \end{aligned} \quad (B10)$$

Multiplying Eq. (B2) by $\overline{\mathbf{\Omega}}^0$ from the left, and defining $\mathbf{X} \equiv \mathbf{X}^0 + \mathbf{Y}$, we obtain

$$\begin{aligned}\mathbf{Y} &= \overline{\mathbf{\Omega}}^0\mathbf{V} + \overline{\mathbf{\Omega}}^0\mathbf{V}\mathbf{G}^0(\mathbf{X}^0 + \mathbf{Y}) \\ &= \overline{\mathbf{\Omega}}^0\mathbf{V}\mathbf{\Omega}^0 + \overline{\mathbf{\Omega}}^0\mathbf{V}\mathbf{G}^0\mathbf{Y}. \end{aligned} \quad (B11)$$

If we adopt a sandwiched form for $\mathbf{Y}$(i.e., $\mathbf{Y} \equiv \overline{\mathbf{\Omega}}^0\mathcal{Y}\mathbf{\Omega}^0$), we have a symmetric equation,

$$\overline{\mathbf{\Omega}}^0\mathcal{Y}\mathbf{\Omega}^0 = \overline{\mathbf{\Omega}}^0\mathbf{V}\mathbf{\Omega}^0 + \overline{\mathbf{\Omega}}^0\mathbf{V}\mathbf{G}^0\overline{\mathbf{\Omega}}^0\mathcal{Y}\mathbf{\Omega}^0. \quad (B12)$$

Then $\mathcal{Y}$ satisfies a kind of LS-type equation,

$$\mathcal{Y} = \mathbf{V} + \mathbf{V}\mathbf{G}\mathcal{Y}, \quad (B13)$$

with

$$\mathbf{G} \equiv \mathbf{G}^0\overline{\mathbf{\Omega}}^0 = \mathbf{G}^0 + \mathbf{G}^0\mathbf{X}^0\mathbf{G}^0. \quad (B14)$$

Finally, we obtain the solution $\mathbf{X}$ by

$$\mathbf{X} = \mathbf{Y} + \mathbf{X}^0 = \overline{\mathbf{\Omega}}^0 \mathcal{Y} \mathbf{\Omega}^0 + \mathbf{X}^0. \tag{B15}$$

APPENDIX C: THE ASYMPTOTIC BEHAVIOR OF THE THREE-BODY WAVE FUNCTION

To prove that the asymptotic three-body wave function is not always separable, let us recall the three-body kinematics in the three-body c.m. system:

$$\begin{aligned}\mathbf{x} &\equiv \mathbf{x}_3 = \mathbf{r}_2 - \mathbf{r}_1,\\ \mathbf{y} &\equiv \mathbf{y}_3 = -\frac{m_1(m_1+m_2+m_3)}{m_3(m_1+m_2)}\mathbf{r}_1,\\ &\quad -\frac{m_2(m_1+m_2+m_3)}{m_3(m_1+m_2)}\mathbf{r}_2,\end{aligned} \tag{C1}$$

where $\mathbf{r}_\alpha(\alpha = 1, 2, 3)$ and m_α are the coordinates of the individual particles and masses, respectively, and $\mathbf{x}$ and $\mathbf{y}$ are the Jacobi coordinates. Then the individual coordinates and two-body relative one are given with $\mu_3^{-1} \equiv m_3^{-1} + (m_1 + m_2)^{-1}$,

$$\begin{aligned}\mathbf{r}_1 &= \frac{-\mu_3\mathbf{y} - m_2\mathbf{x}}{m_1+m_2}\\ &= \frac{-m_3}{m_1+m_2+m_3}\mathbf{y} - \frac{m_2}{m_1+m_2}\mathbf{x},\\ \mathbf{r}_2 &= \frac{-\mu_3\mathbf{y} + m_1\mathbf{x}}{m_1+m_2}\\ &= \frac{-m_3}{m_1+m_2+m_3}\mathbf{y} + \frac{m_1}{m_1+m_2}\mathbf{x}\end{aligned} \tag{C2}$$

and

$$\mathbf{x}_1 \equiv \mathbf{r}_3 - \mathbf{r}_2 = \mathbf{y} - \frac{m_1}{m_1+m_2}\mathbf{x} \equiv \alpha\mathbf{y} + \beta\mathbf{x}, \tag{C3}$$

$$\mathbf{x}_2 \equiv \mathbf{r}_1 - \mathbf{r}_3 = -\mathbf{y} - \frac{m_2}{m_1+m_2}\mathbf{x} \equiv \gamma\mathbf{y} + \delta\mathbf{x}. \tag{C4}$$

By using Eqs. (C3) and (C4), the three-body potential is represented as follows:

$$\begin{aligned}&V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2) + V_3(\mathbf{x}_3)\\ &\quad = V_1(\mathbf{x},\mathbf{y}) + V_2(\mathbf{x},\mathbf{y}) + V_3(\mathbf{x})\\ &\quad = V_1(\alpha\mathbf{y}+\beta\mathbf{x}) + V_2(\gamma\mathbf{y}+\delta\mathbf{x}) + V_3(\mathbf{x}).\end{aligned} \tag{C5}$$

Then, the three-body Schrödinger equation is given by

$$\begin{aligned}&\left[-\frac{\Delta_x}{2\nu_3} - \frac{\Delta_y}{2\mu_3} + V_1(\mathbf{x},\mathbf{y}) + V_2(\mathbf{x},\mathbf{y}) + V_3(\mathbf{x})\right]\psi(\mathbf{x},\mathbf{y})\\ &\quad = E\psi(\mathbf{x},\mathbf{y}),\end{aligned} \tag{C6}$$

where ν_3 and μ_3 are the reduced masses with respect to the Jacobi coordinates $\mathbf{x}$ and $\mathbf{y}$, respectively.

Now, the asymptotic forms of the potentials are given by three cases.

Case 1: $\{\mathbf{x} \to \infty, \mathbf{y} < \infty\}$. The potential becomes

$$V_1(\mathbf{x},\mathbf{y}) + V_2(\mathbf{x},\mathbf{y}) + V_3(\mathbf{x}) \to \tilde{V}_1(\delta\mathbf{x}) + \tilde{V}_2(\beta\mathbf{x}) + V_3(\mathbf{x}); \tag{C7}$$

then, the asymptotic wave function will be given by multiplying the two-body wave function with respect to the coordinate $\mathbf{x}$ and the plane wave for the relative momentum $\mathbf{q}$ regarding, the Jacobi coordinate $\mathbf{y}$,

$$\psi(\mathbf{x},\mathbf{y}) \to \tilde{\phi}_x(\mathbf{x})\exp(i\mathbf{q}\mathbf{y}). \tag{C8}$$

Case 2: $\{\mathbf{y} \to \infty, \mathbf{x} < \infty\}$. This leads to the asymptotic potential

$$V_1(\mathbf{x},\mathbf{y}) + V_2(\mathbf{x},\mathbf{y}) + V_3(\mathbf{x}) \to \tilde{V}_1(\alpha\mathbf{y}) + \tilde{V}_2(\gamma\mathbf{y}) + V_3(\mathbf{x}); \tag{C9}$$

then, the asymptotic wave function will be

$$\psi(\mathbf{x},\mathbf{y}) \to \tilde{\psi}_x(\mathbf{x})\tilde{\psi}_y(\mathbf{y}). \tag{C10}$$

This case lead a separable wave function in the asymptotic region.

Case 3: $\{\mathbf{x} \to \infty, \mathbf{y} \to \infty\}$. Here the potential becomes

$$\begin{aligned}&V_1(\mathbf{x},\mathbf{y}) + V_2(\mathbf{x},\mathbf{y}) + V_3(\mathbf{x})\\ &\quad \to \tilde{V}_1(\alpha\mathbf{y}+\beta\mathbf{x}) + \tilde{V}_2(\gamma\mathbf{y}+\delta\mathbf{x}) + V_3(\mathbf{x}).\end{aligned} \tag{C11}$$

One finds that the potential does not change on one side of the coordinates. This fact leads to a nonseparable but mixed asymptotic wave function

$$\psi(\mathbf{x},\mathbf{y}) \neq \tilde{\psi}_x(\mathbf{x})\tilde{\psi}_y(\mathbf{y}). \tag{C12}$$

Finally, one can conclude that the relative coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot be separable with respect to $\mathbf{x}$ and $\mathbf{y}$; only the cases $\{\mathbf{x} \to \infty, \mathbf{y} < \infty\}$ and $\{\mathbf{y} \to \infty, \mathbf{x} < \infty\}$ leads to a separable wave function. This situation will occur not only for the three-charged-particle systems but also for a two-charged-particle system in the three-body problem.

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