Seniority conservation and seniority violation in the $g_{9/2}$ shell

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The $g_{9/2}$ shell of identical particles is the first one for which one can have seniority-mixing effects. We consider three interactions: A delta interaction that conserves seniority, a quadrupole-quadrupole ($Q \cdot Q$) interaction that does not, and a third one consisting of two-body matrix elements taken from experiment (^{98}Cd) that also leads to some seniority mixing. We deal with proton holes relative to a $Z = 50$, $N = 50$ core. One surprising result is that, for a four-particle system with total angular momentum $I = 4$, there is one state with seniority $v = 4$ that is an eigenstate of *any* two-body interaction—seniority conserving or not. The other two states are mixtures of $v = 2$ and $v = 4$ for the seniority-mixing interactions. The same thing holds true for $I = 6$. Another point of interest is that, in the single-j-shell approximation, the splittings $\Delta E = E(I_{\max}) - E(I_{\min})$ are the same for three and five particles with a seniority conserving interaction (a well-known result), but are equal and opposite for a $Q \cdot Q$ interaction. We also fit the spectra with a combination of the delta and $Q \cdot Q$ interactions. The $Z = 40$, $N = 40$ core plus $g_{9/2}$ neutrons (Zr isotopes) is also considered, although it is recognized that the core is deformed.

I. INTRODUCTION

With the advent of nuclei far from stability, there will be more emphasis on identical particles in given shells, in which case the concept of seniority will be even more important than it has been in the past. See for example the work of Lisetskiy *et al*. [1]. Also as part of the revival are works on quasispin and seniority by Rowe and Rosensteel [2,3] and by the present authors [4].

There are several well known formulas for seniority selection rules in the nuclear physics and atomic physics textbooks [5–7]. One of the first things we learn is that for identical particles in a single j shell, seniority is conserved for all shells with $j \leqslant 7/2$, no matter what two-body interaction is used. The first shell, then, where seniority violating effects can be seen is the $g_{9/2}$, and this is the shell we shall consider here.

As noted in Talmi's review article [8], there have been many calculations done in the $g_{9/2}$ region, including calculations with seniority-conserving interactions by Gloeckner and collaborators [9,10], as well as experiment and theory by Oxorn *et al*. [11]. Also to be mentioned are Amusa and Lawson [12,13], and Auerbach and Talmi [14].

Work in a $g_{9/2}$-$p_{1/2}$ model space has been done by Brown and collaborators. In Refs. [15,16], the focus is on the 17/2 state and the possible parity violation, i.e., the admixture of some $17/2^+$ with the $17/2^-$ state. In the present work, we focus on positive-parity states. In Refs. [17,18], Brown *et al*. consider some properties of states in a number of nuclei that are heavier than the ones we consider here.

Our motivation in this work is to see how the effective interaction depends on what "closed shell" is used.

Some of the well-known statements and theorems concerning states of good seniority are

(i) The seniority is roughly the number of identical particles not coupled to zero. Hence, for a single nucleon the seniority v is equal to 1. For two nucleons in a $J = 0$ state we have $v = 0$, but for $J = 2, 4, 6$, etc., $v = 2$. For

three nucleons there is one state with seniority $v = 1$, which must have $J = j$; all other states have seniority $v = 3$.

(ii) The number of seniority-violating interactions is $[(2j - 3)/6]$, where the square brackets mean the largest integer contained therein (see Ref. [19]). For $j = 7/2$ there are no seniority-violating interactions, while for $j = 9/2$ there is one.

(iii) With seniority-conserving interactions, the spectra of states of the same seniority is independent of the particle number.

(iv) At midshell we cannot have any mixing of states with seniorities v and $v + 2$; one can mix v and $v + 4$ states.

There are also well-known results for more general interactions which do not necessarily conserve seniority:

(i) For identical particles in a single j shell, the hole spectrum is the same as the particle spectrum. This will be relevant to ^{93}Tc and ^{97}Ag, and for ^{83}Zr and ^{87}Zr.

(ii) The magnetic moment of a hole is the same as that of a particle. The quadrupole moment of a hole is equal in magnitude but of the opposite sign to that of a particle. This leads to the result that at midshell all static quadrupole moments vanish in the single-j-shell approximation.

There have been many calculations in the past in the "$g_{9/2}$ region," although perhaps due to a lack of data, some but not all of the nuclei we consider here have been addressed. Our intention here is not to do large scale calculations, but rather to see how far we can go with the simplest of shell model approaches. As will be seen, there are some surprises even at this late date, e.g., concerning $g_{9/2}$ coefficients of fractional parentage. Furthermore, while most emphasis has been on seniority-conserving interactions, there are some simplicities even for seniority-nonconserving interactions, such as the quadrupole-quadrupole interaction, which we will utilize to determine the degree of seniority nonconservation. Also newer

data permits us to go further away from the valley of stability than was possible at earlier times. This enables us to start from different cores and to study the core dependance of the effective interaction.

II. OUR CALCULATIONAL METHOD

We used a program given to us by Bayman to calculate cfp's. However, when there is more than one state of a given seniority, the Bayman program does not give the same cfp's as are recorded in the original Bayman-Lande paper [20]. Nevertheless, this is not a cause for concern as will be discussed in the next section.

What is unusual in our approach is that we do not consider a system of identical particles, but rather a system of $(n-1)$ protons and one neutron. When we perform the matrix diagonalization, using only isospin-conserving interactions, we obtain states with isospin $T_{min} = |N - Z|/2$ and also ones with higher isospins. The latter states are analog states of systems of identical particles. If we write the wave function as

$$\Psi^I = \sum_{J_P} D^I(J_P v_P, J_N = j)[(j^{n-1})^{J_P} j]^I, \quad (1)$$

then, for the $T_{min} + 1$ states, the coefficients D^I are coefficients of fractional parentage $(j^{n-1} J_P v_P j|\} j^n I)$. We are also interested in the spectra of T_{min} states, but we will save this for another day.

III. SPECIAL BEHAVIORS FOR $I = 4^+$ AND 6^+ STATES OF THE $g_{9/2}^4$ CONFIGURATION

For a system of four identical nucleons in the $g_{9/2}$ shell, the possible seniorities are $v = 0, 2$, and 4, with $v = 0$ occurring only for a state of total angular momentum $I = 0$. There is also a $v = 4$ state with $I = 0$.

For $I = 4$ and 6, we can have three states, one with seniority $v = 2$ and two with seniority $v = 4$. For the two $v = 4$ states we have at hand, we can construct different sets of $v = 4$ states by taking linear combinations of the original ones. If the original ones are $(4)_1$ and $(4)_2$, we can form

$$(4)_A = a(4)_1 + b(4)_2,$$
$$(4)_B = -b(4)_1 + a(4)_2, \quad (2)$$

with $a^2 + b^2 = 1$. The set $(4)_A$, $(4)_B$ is as valid as the original set.

However, we here note that if we perform a matrix diagonalization with any two-body interaction—seniority conserving or not—one state emerges which does not depend on what the interaction is. The other two states are, in general, mixtures of $v = 2$ and $v = 4$ which do depend on the interaction. The values of the coefficients of fractional parentage (cfp's) of this unique state of seniority 4 are shown in Table I. The states with $J_0 \neq 4.5$ all have seniority $v = 3$. Note that in this special $v = 4$ state there is no admixture of states with $J_0 = j = 9/2$, be they $v = 1$ or $v = 3$. Again, no matter what

TABLE I. A unique $J = 4$, $v = 4$ cfp for $j = 9/2$.

J_0	$(j^3 J_0 j|\} j^4 I = 4, v = 4)$
1.5	0.1222
2.5	0.0548
3.5	0.6170
4.5 ($v = 1$)	0.0000
4.5 ($v = 3$)	0.0000
5.5	−0.4043
6.5	−0.6148
7.5	−0.1597
8.5	0.1853

two-body interaction is used, this $I = 4$ state remains a unique state.

Amusingly, this state *does not appear* in the compilation of seniority-classified cfp's of Bayman and Lande [20] or de Shalit and Talmi [5]. We should emphasize that, although different, the Bayman–Lande cfp's are perfectly correct (as are the ones of de Shalit and Talmi, whose cfp's are also different from those of Bayman and Lande [20]). But then, why do they not obtain the unique state that we have shown above? Bayman and Lande use group theoretical techniques to obtain the cfp's, diagonalizing the following Casimir operator for $Sp(2j + 1)$:

$$G(Sp_{2j+1}) = \frac{1}{2j + 1} \sum_{\text{odd } k=1}^{2j} (-1)^k (2k + 1)^{3/2} [U^k U^k]_0^0, \quad (3)$$

where $U_q^k \equiv \sum_{i=1}^N U_q^k(i)$ and U is the Racah unit tensor operator

$$\langle \Psi_{m'}^{j'} | U_q^k | \Psi_m^j \rangle = \delta_{jj'} (kjqm|j'm'). \quad (4)$$

The two seniority $v = 4$ states are degenerate with such an interaction and, since there is no seniority mixing, we can have arbitrary linear combinations of the 4^+ states. Only by using an interaction which removes the degeneracy and violates seniority, do we learn about the special state in Table I.

IV. THE ENERGY SPLITTING $E(I_{max}) - E(I_{min})$ WITH A $Q \cdot Q$ INTERACTION

A well-known result for identical particles in a single j shell is that, if one uses a seniority-conserving interaction, then the relative spectra of states of the same seniority are independant of the number of particles [5–7]. Thus, for $n = 3$ and $n = 5$, the seniority $v = 3$ states have the same relative spectrum; for $n = 2, 4$, and 6, the seniority $v = 2$ states have the same spectrum. These results hold, in particular, for the delta interaction used here.

Now the $Q \cdot Q$ interaction does not conserve seniority and the above results do not hold. However, we have noticed an interesting result for $n = 3$ and $n = 5$. Consider the splitting $\Delta E = E(I_{max}) - E(I_{min})$, $v = 3$, where for $g_{9/2}$, $I_{max} = 21/2$ and $I_{min} = 3/2$. For a seniority-conserving interaction, $\Delta E(n = 5) = \Delta E(n = 3)$, whereas for a $Q \cdot Q$ interaction, $\Delta E(n = 5) = -\Delta E(n = 3)$. This will be discussed quantitatively later.

TABLE II. Values of the even-J two-body matrix elements (m.e.) for the different interactions mentioned throughout the paper: quadrupole-quadrupole ($Q \cdot Q$), surface delta (SDI), m.e. taken from experimental spectrum of ^{98}Cd [V(^{98}Cd)], and m.e. taken from experimental spectrum of ^{82}Zr [V(^{82}Zr)].

J	$Q \cdot Q$	SDI	V(^{98}Cd)	V(^{82}Zr)
0	0.0000	0.0000	0.0000	0.0000
2	0.3485	2.0063	1.3947	0.4070
4	0.9848	2.3149	2.0823	1.0408
6	1.4848	2.4507	2.2802	1.8879
8	1.1818	2.5415	2.4275	2.9086

V. THE TWO-PARTICLE (TWO-HOLE) SYSTEMS

In order to perform calculations, we must know the two-body matrix elements $E(J) = \langle (g_{9/2}^2)^J | V | (g_{9/2}^2)^J \rangle$. Since in this work we consider only two identical nucleons, we need simply the even-J matrix elements ($J = 0, 2, 4, 6,$ and 8). In Table II and Fig. 1 we give four sets of two-body matrix elements. These are seniority-violating quadrupole-quadrupole interaction, seniority-conserving delta interaction, matrix elements taken from the two-proton-hole system (relative to a $Z = 50$, $N = 50$ core) ^{98}Cd, and matrix elements taken from the two-neutron-particle system (relative to $Z = 40$, $N = 40$) ^{82}Zr.
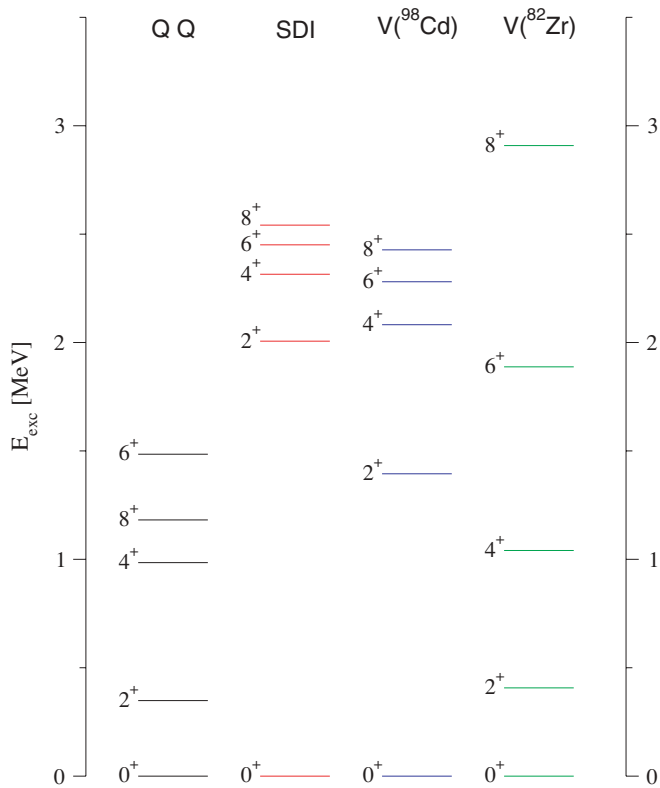

FIG. 1. (Color online) Values of the even-J two-body matrix elements (m.e.) for the following interactions: quadrupole-quadrupole ($Q \cdot Q$), surface delta (SDI), m.e. taken from experimental spectrum of ^{98}Cd [V(^{98}Cd)], and m.e. taken from experimental spectrum of ^{82}Zr [V(^{82}Zr)].

It should be said at the outset, however, that $Z = 40$, $N = 40$ is not a good closed shell. Experiments by Lister *et al.* [21] showed that ^{80}Zr is strongly deformed. Skyrme–Hartree-Fock calculations by Bonche *et al.* [22] and by Zheng and Zamick [23] are in agreement with experiment. In Ref. [23] it was noted that, in the intrinsic deformed ground state of ^{80}Zr, there were 12 nucleons in the $g_{9/2}$ shell. In the spherical limit, there would not be any. For this reason, we will not show single-j-shell fits to the Zr isotopes.

VI. GROUND STATE SPINS

For identical particles in the $g_{9/2}$ shell, the delta interaction yields a ground state spin $I = j = 9/2^+$ for odd-even or even-odd nuclei. In contrast, the $Q \cdot Q$ interaction yields $I = 7/2^+$. Experimentally, it turns out (as will be shown in the next section) that some nuclei have ground-state spins $I = 9/2^+$ and others have $I = 7/2^+$. The latter nuclei are closer to the $Z = 40$, $N = 40$ "closed shell", while the former are closer to the $Z = 50$, $N = 50$ closed shell. This shows that both the delta and $Q \cdot Q$ interactions are important for a proper description of these nuclei.

VII. THE THREE AND FIVE PARTICLE (HOLE) SPECTRA

The nuclei we consider are broken into two groups: One in which $g_{9/2}$ protons are removed from a $Z = 50$, $N = 50$ core and a second in which $g_{9/2}$ neutrons are added to a $Z = 40$, $N = 40$ core. We shall see a significant and systematic difference in the behavior in the two cases. The nuclei we consider and the number of proton holes or neutron particles are shown in Table III.

In Fig. 2 we show the empirical spectra of nuclei for $n = 3, 5,$ and 7 protons removed from the $Z = 50$, $N = 50$ core—these are ^{97}Ag, ^{95}Rh, and ^{93}Tc, respectively. In Fig. 3 we show the corresponding spectra for $n = 3, 5,$ and 7 neutrons relative to a $Z = 40$, $N = 40$ core (which we had pointed out was deformed).

In an idealized world where the shell model worked perfectly, we would expect the spectra of the three-particle system to be identical to that of the seven-particle system (i.e., three holes). Hence, ^{97}Ag and ^{93}Tc would have the same spectrum. If furthermore the interaction conserved seniority, then the spectrum of states with $v = 1$ and $v = 3$ would be the same for three particles and five particles.

We could go even further and say that, if the interaction for two proton holes were the same as that for two neutrons, then

TABLE III. Nuclei we consider in this work.

(Z = 50, N = 50) core		(Z = 40, N = 40) core	
No. of holes	Nucleus	No. of particles	Nucleus
2	^{98}Cd	2	^{82}Zr
3	^{97}Ag	3	^{83}Zr
5	^{95}Rh	5	^{85}Zr
7	^{93}Tc	7	^{87}Zr

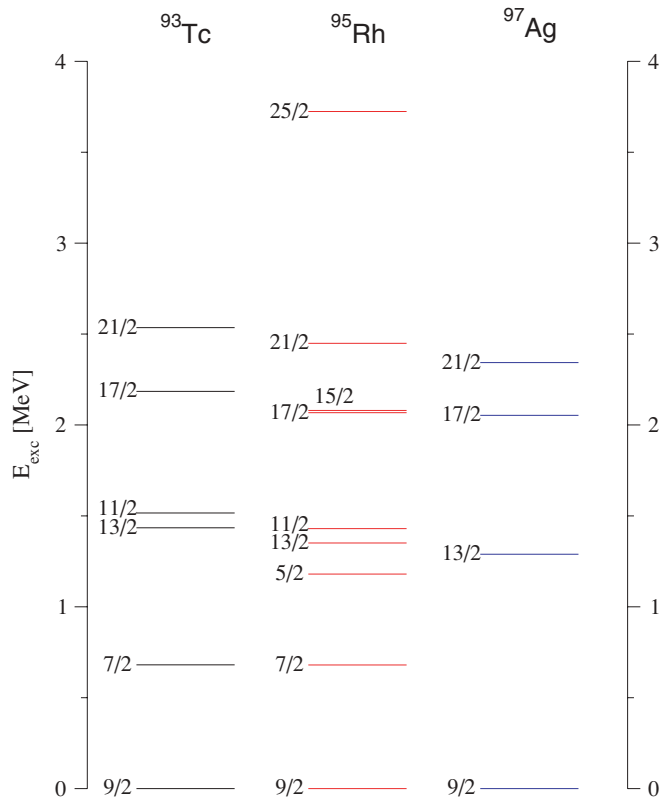FIG. 2. (Color online) Experimental energies of ^{93}Tc, ^{95}Rh, and ^{97}Ag.

FIG. 3. (Color online) Experimental energies of ^{83}Zr, ^{85}Zr, and ^{87}Zr.

the spectra of ^{93}Tc, ^{97}Ag, ^{83}Zr, and ^{87}Zr would all be the same. (If the spectra of ^{93}Tc and ^{83}Zr are different, it does not mean that we have a violation of charge symmetry, of course.) But that is really pushing the envelope.

Looking at the experimental spectra of Fig. 2, we can see that, although the agreement for three holes (^{97}Ag) and seven holes (^{93}Tc) is not perfect, it is still quite good. Also the fact that there is close agreement with ^{95}Rh indicates that we are not far away from the seniority-conserving limit.

Looking at Fig. 3, we see that the spectra of the Zr isotopes are significantly different from those of ^{93}Tc, ^{95}Rh, and ^{97}Ag. This undoubtedly is due to the fact that $Z = 40$, $N = 40$ is deformed. The $J = 7/2^+$ ground state spins of ^{83}Zr and ^{85}Zr agree with the predictions of the $Q \cdot Q$ interaction, but not the delta interaction.

Note the nearly degenerate doublet structure in the experimental spectrum of ^{83}Zr in Fig. 3, taken from the work of Hüttmeier *et al.* [24]. The known doublets have angular momenta $(7/2, 9/2)$, $(11/2, 13/2)$, $(15/2, 17/2)$, ..., up to $(47/2, 49/2)$, although we only show up to $(31/2, 33/2)$. However, for a j^3 configuration of identical particles, there are no states with $J = J_{max} - 1$ or $J > J_{max}$, where $J_{max} = 21/2$ for $j = 9/2$. Hence, those states must have different configurations.

In the Hüttmeier reference [24], a theoretical analysis using a Woods-Saxon cranking model was performed. A triaxial shape was predicted, which was the main cause of a signature
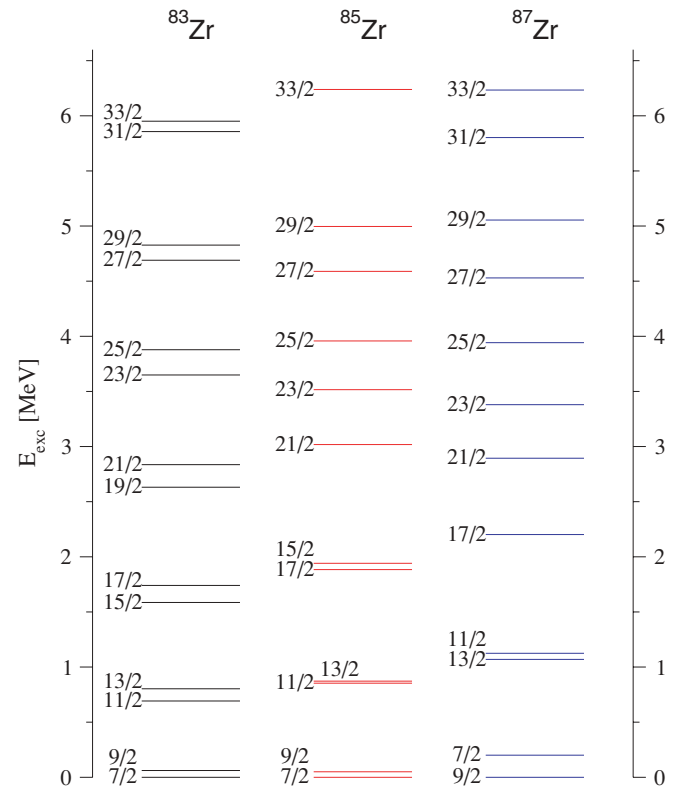
splitting that leads to the deviation from a simple rotational spectrum. Discussions of signature splitting for triaxial nuclei can be found in several places, e.g., Mottelson [25], Chen *et al.* [26], and Hamamoto and Mottelson [27].

VIII. CALCULATIONS WITH MATRIX ELEMENTS FROM EXPERIMENT

The success and limitations of the "Talmi" procedure of taking matrix elements from experiment can be discussed in terms of Goldstone diagrams. For two nucleons, the experimental spectrum includes not only the bare interaction, but effectively all sorts of higher order corrections, including hard-core correlations on the one hand and the exchange of a Bertsch-Kuo-Brown bubble on the other hand. If we have three-nucleon lines, we see that something is missing. This is a diagram in which two particles interact and one gets excited to a higher shell. Then the excited nucleon is deexcited by the third nucleon. This looks like an effective three-body interaction. Evidently, for the states we are considering here with an $N = 50$ core, these effective three-body effects are not sufficiently strong to destroy the simple shell-model picture, but in general there is no *a priori* reason why they cannot be large. Indeed, in ^{43}Ca there is a significant deviation of the excitation energy of the $J = 3/2^-$ state from the single-*j*-shell
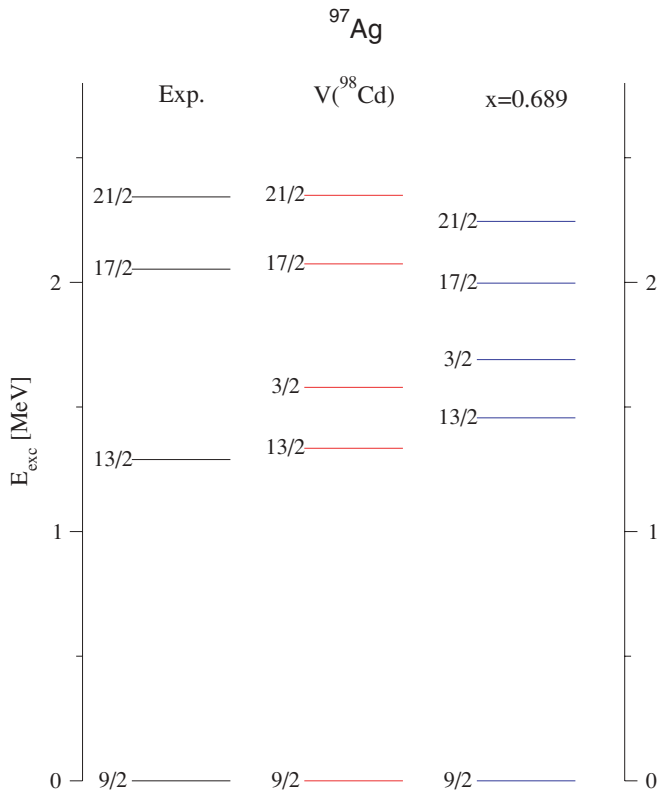
FIG. 4. (Color online) Experimental and calculated spectra of ^{97}Ag.
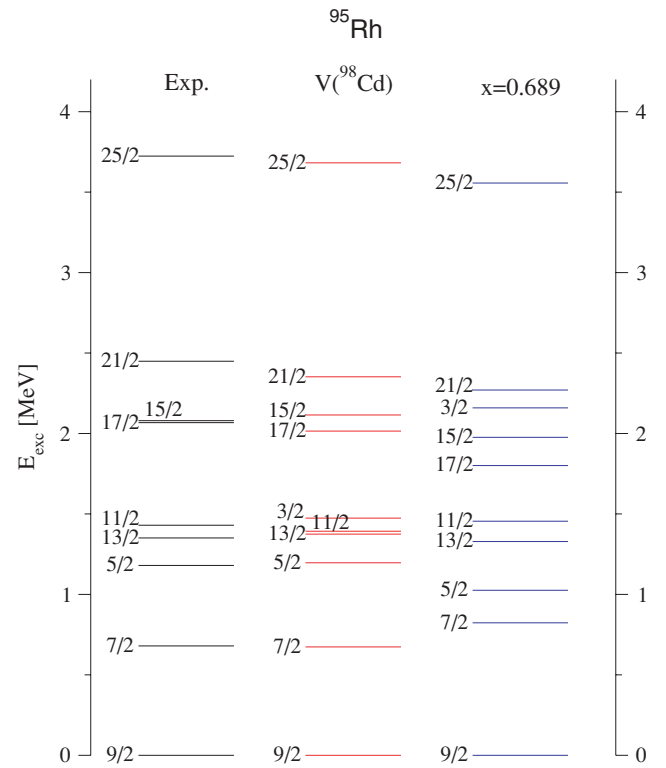
FIG. 5. (Color online) Experimental and calculated spectra of ^{95}Rh.

prediction in which two-body matrix elements are taken from the spectrum of ^{42}Ca.

We can get two-body matrix elements for a $Z = 50$, $N = 50$ core from the two-proton-hole spectrum of ^{98}Cd—we will call it V(^{98}Cd). If the excitation energy of the lowest state of angular momentum J ($J = 0, 2, 4, 6$, and 8) is $E(J)$, then we make the association $\langle(\bar{g}_{9/2}^2)^J|V|(\bar{g}_{9/2}^2)^J\rangle = E(J)$. And now we can proceed to do calculations for n holes, with $n > 2$. Note that, except for an overall constant, the hole-hole spectrum is the same as the particle-particle spectrum.

In Figs. 4, 5, and 6, we show a comparison of the calculated spectra for V(^{98}Cd) with experiment for $n = 3, 5$, and 7 proton holes corresponding to ^{97}Ag, ^{95}Rh, and ^{93}Tc, respectively. The results, although not perfect, are quite reasonable considering the simplicity of the model.

IX. LINEAR COMBINATION OF A DELTA AND $Q \cdot Q$ INTERACTION

The formula for the two-body matrix elements of the $Q \cdot Q$ interaction is

$$\langle[jj]^J|V_{Q \cdot Q}|[jj]^J\rangle = (-1)^J V_0 \frac{5}{4\pi} \langle r^2\rangle_1 \langle r^2\rangle_2 (2j+1)$$

$$\times \left(j2\frac{1}{2}0\bigg|j\frac{1}{2}\right)^2 \begin{Bmatrix} j & j & J \\ j & j & 2 \end{Bmatrix}. \quad (5)$$
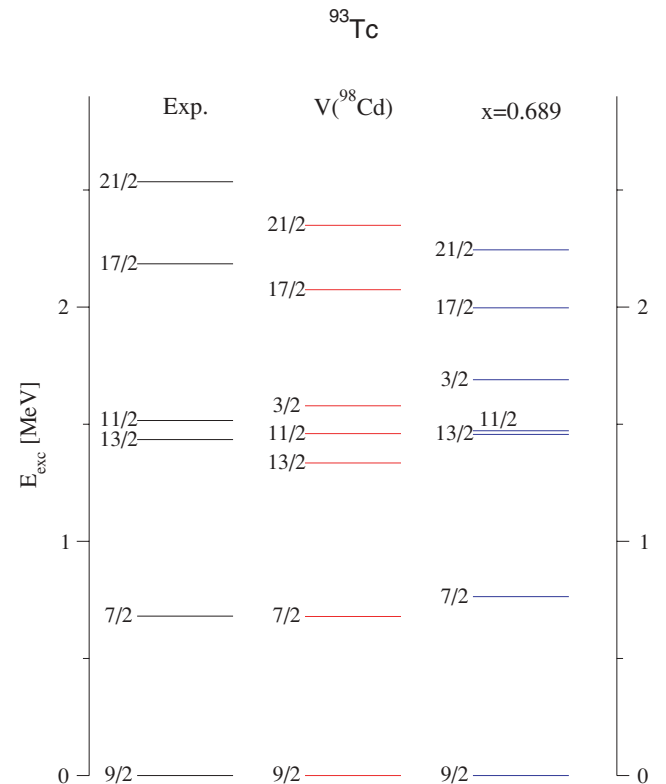
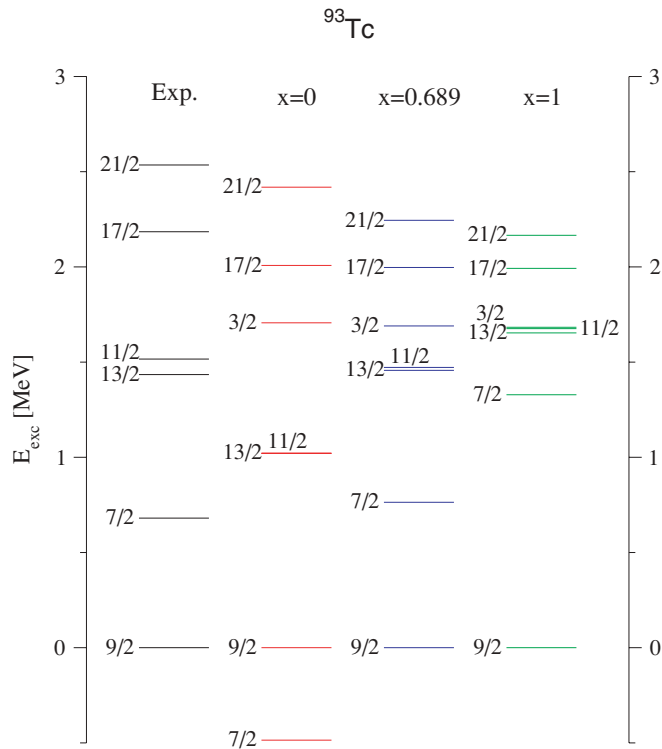FIG. 6. (Color online) Experimental and calculated spectra of ^{93}Tc.

FIG. 7. (Color online) Experimental and calculated spectra of ^{93}Tc. $x = 0$ means pure $Q \cdot Q$ interaction; $x = 1$, pure delta interaction; and $x = 0.689$ is our best linear combination of both interactions for this nucleus.

FIG. 8. (Color online) Same as Fig. 7 for ^{95}Rh.

The formula for the surface delta interaction of Moszkowski and collaborators [28–30] is

$$\langle [jj]^{JT} | V_{\text{SD}} | [jj]^{JT} \rangle = -W_0 \frac{(2j+1)^2}{4(2J+1)}$$

$$\times \left[\{1 + (-1)^T\} \left(jj \frac{1}{2} \frac{1}{2} \middle| J1 \right)^2 \right.$$

$$\left. + \{1 - (-1)^{J+T}\} \left(jj \frac{1}{2} -\frac{1}{2} \middle| J0 \right)^2 \right].$$

(6)

In a single j shell, there is no distinction between a delta interaction and a surface delta interaction.

In Figs. 7–9, we give results for the $Q \cdot Q$ and delta interactions, choosing optimum V_0 and W_0, respectively. Then, we form the linear combination $[x V_{Q \cdot Q} + (1 - x)V_{\text{SD}}]$ and show the optimum x to fit experiment. Thus, $x = 0$ in the figures corresponds to pure $Q \cdot Q$, while $x = 1$ means pure delta. The values of V_0, W_0, and x are shown in Table IV.

Because ^{80}Zr is deformed, we do not show figures for single j shell fits to ^{83,85,87}Zr. However, from Figs. 7–9 (proton holes relative to $Z = 50$, $N = 50$), we can get some feeling for what is happening. In Fig. 7 (^{93}Tc) we focus on the pure $Q \cdot Q$ ($x = 0$) and surface delta ($x = 1$) limits.

Whereas in ^{93}Tc the $J = 9/2^+$ is the lowest state, in ^{83}Zr the $J = 7/2^+$ is the lowest. The $Q \cdot Q$ interaction displays
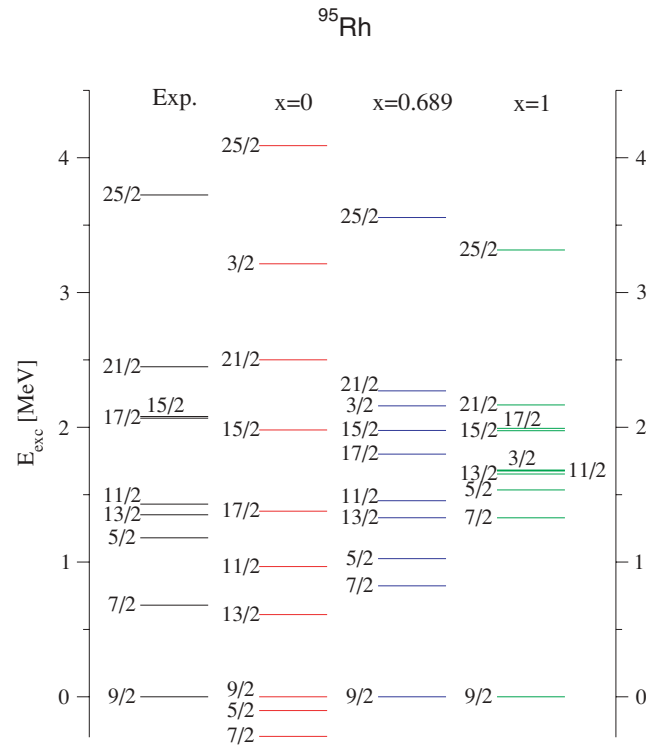
this feature—$E(7/2) < E(9/2)$—, but the surface delta does not. So, if we were naively to try to fit the ^{83}Zr spectrum with a $g^3_{9/2}$ configuration, we would need much more $Q \cdot Q$ than
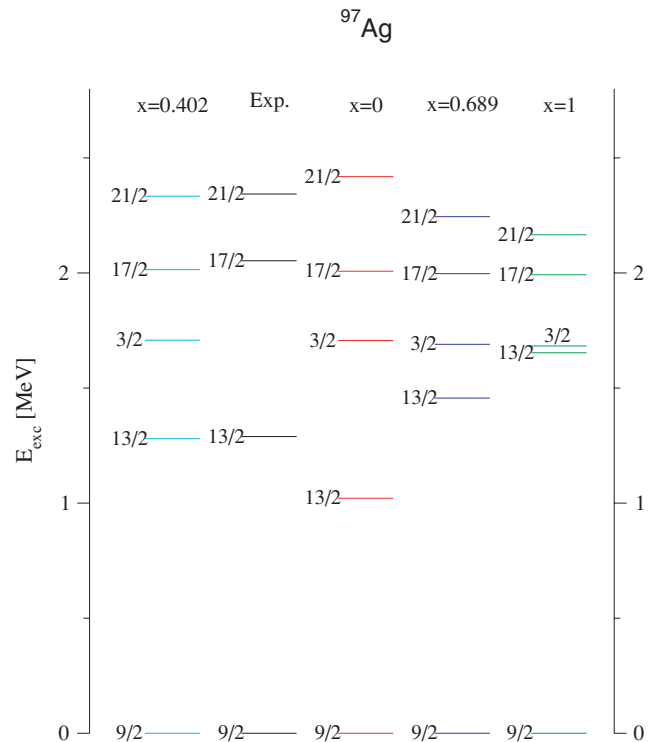

FIG. 9. (Color online) Same as Fig. 7 for ^{97}Ag.

TABLE IV. Values of the optimum V_0, W_0, and x (see text) for each isotope considered in the paper; $F(x)$ gives an estimate of how well our calculated energies fit experiment (the closer to zero, the better); g.s. stands for the experimental ground state.

Z	N	x	$F(x)$	W_0	V_0	g.s.
43	50	0.6286	0.0830	0.4969	0.0296	$9/2^+$
45	50	0.6887	0.1865	0.5066	0.0270	$9/2^+$
47	50	0.4021	0.0016	0.5023	0.0272	$9/2^+$
40	43	0.0471	0.0307	0.3037	0.0190	$(7/2^+)$
40	45	0.0495	0.4521	0.9060	0.0271	$7/2^+$
40	47	0.2808	0.2774	0.4624	0.0316	$9/2^+$

we needed for the fit to ^{93}Tc. On the other hand, some of the near doublet structure seen in the experimental spectrum of ^{83}Zr (11/2, 13/2) and (15/2, 17/2) is also a property of the delta interaction. However, introducing a lot of $Q \cdot Q$ will destroy the near degeneracy of these doublets. Thus, it is essentially impossible to fit both features of the ^{83}Zr spectrum—a $J = 7/2^+$ ground state and nearly degenerate doublets—with a combination of $Q \cdot Q$ and delta in a $g_{9/2}^3$ configuration.

The best way to explain why $N = 50$ is a better closed shell than $N = 40$ is to invoke the Nilsson model. Relative to a spherical $N = 40$, the first unoccupied Nilsson level is $g_{9/2, K=1/2}$. As a function of prolate deformation, this level goes rapidly down in energy. There is a crossover with the level emanating from $p_{1/2, K=1/2}$ and so it pays to transfer many particles to the positive-parity orbit. However, for $N = 50$ we have filled the $g_{9/2}$ shell. As we try to turn on the deformation, some newly occupied levels go up in energy. Furthermore, excitations from lower shells are blocked.

X. THE $E(I_{\max}) - E(I_{\min})$ SPLITTING FOR $n = 3$ AND $n = 5$: ^{97}AG VERSUS ^{95}RH AND ^{83}ZR VERSUS ^{85}ZR

As mentioned in a previous section, the splitting $\Delta E = E(I_{\max} = 21/2^+) - E(I_{\min} = 3/2^+)$ is the same for three particles as it is for five particles (or three holes and five holes) if one has a seniority-conserving interaction. However, for a pure $Q \cdot Q$ interaction, we have $\Delta E(n = 5) = -\Delta E(n = 3)$.

Using the V(^{98}Cd) interaction for ^{97}Ag and ^{95}Rh, we find

$$\Delta E(n = 3) = 0.77058 \text{ MeV},$$
$$\Delta E(n = 5) = 0.87818 \text{ MeV}.$$

They are both positive, an indication that the seniority-conserving delta interaction is much more important than the seniority-violating $Q \cdot Q$ interaction.

Talmi had previously concluded, from an analysis of $h_{11/2}$ nuclei with a closed shell of neutrons ($N = 82$), that seniority conservation held to a high degree [7,31].

Unfortunately, for the $g_{9/2}$ nuclei that we are here considering (^{93}Tc, ^{95}Rh, ^{97}Ag, as well as the zirconium isotopes ^{83}Zr, ^{85}Zr, and ^{87}Zr), although the high spin states including $I = 21/2^+$ have been identified, the $I = 3/2^+$ states have not been found yet. So our analysis provides very strong motivation for an experimental search for the $I = 3/2^+$ states in ^{97}Ag, ^{95}Rh, and ^{93}Tc, as well as for the Zr isotopes.

For ^{83}Zr and ^{85}Zr, with a fitted interaction (despite misgivings of using a single j model space), we find for $\Delta E = E(I_{\max}) - E(I_{\min})$

$$\Delta E(^{83}\text{Zr}) = 0.48742 \text{ MeV},$$
$$\Delta E(^{85}\text{Zr}) = -0.59355 \text{ MeV}.$$

They have opposite signs, which shows that for these fitted interactions the $Q \cdot Q$ interaction is much more important for this case—neutrons beyond a $Z = 40$, $N = 40$ core—than it is for the case of proton holes relative to a $Z = 50$, $N = 50$ core.

But it should be emphasized that the $I = 3/2^+$ state is not part of the fit because it has not been identified experimentally. If more levels were known in the Zr isotopes, and in particular the low spin level $I = 3/2^+$ (but also $5/2^+$ and $1/2^+$), then the picture might change. We strongly urge that experimental work be done on all the nuclei considered here in order to locate the missing states, especially $I = 3/2_1^+$ and also $5/2_1^+$.

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