Continuum random-phase approximation study of the incoherent $\mu^- - e^-$ conversion rate and its spurious 1^- admixture

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The incoherent transition strength of the exotic $\mu^- - e^-$ conversion in the ^{208}Pb nucleus is investigated by utilizing the continuum random-phase-approximation method, appropriate for the evaluation of the rate that goes to the continuum of the nuclear spectrum. We find that the contribution of resonances lying high in the continuum is not negligible. Special attention is paid to the detailed study of the pronounced 1^- contribution that according to previous calculations, dominates the overall incoherent rate in about all the nuclear targets. The spurious center-of-mass admixture to the partial rate originating from the 1^- excitations is explored, and its elimination is performed by correcting properly the dipole operators. The results found this way show that the greatest portion of the total 1^- contribution to the incoherent rate is spurious.

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I. INTRODUCTION

The exotic neutrinoless conversion of a bound muon to an electron,

$$\mu_b^- + (A, Z) \rightarrow e^- + (A, Z)^*, \tag{1}$$

is an interesting lepton flavor violating process [1–4]. One of its basic characteristics is the possibility of the coherent channel, i.e., the ground-state-to-ground-state transition [5,6]. Experimentally, only the ratio of the coherent rate divided by the total muon capture rate, which is the dominant branching ratio exhausting a great part of the total (μ^-, e^-) rate could be measurable; by a judicious choice of the target nucleus, this channel could be free from the reaction induced background [1–3].

The incoherent (μ^-, e^-) rate is a less significant portion of the total rate and much harder to calculate, but its knowledge is important for determining the fraction of the coherent process to the total (μ^-, e^-) rate, which experimentally is also an interesting quantity. The theoretical calculation of the total rate requires reliable coherent and incoherent nuclear matrix elements [7–10]. Transitions of the reaction (1) have been previously studied by employing various methods, such as (i) closure approximation within the shell model [7] and the quasiparticle random-phase approximation (QRPA) for calculating the average contribution of the transitions to all excited states of the target nucleus, (ii) a Fermi gas method utilizing a relativistic Lindhard function to compute the sum of all partial rates of the incoherent channel [11], (iii) state-by-state calculations by using shell model [8], and various QRPA versions [4,9,10], to construct explicitly the final nuclear states.

Within method (ii), the incoherent rate is calculated by integrating over a continuum of excited states of a local Fermi sea. Therefore, this method is not appropriate for individual calculations of each accessible channel, but it offers the advantage of taking into consideration the part of the rate that goes to the continuum, which is not explicitly included in state-by-state calculations. This is one of the reasons why the incoherent matrix elements obtained with shell model [8] and the various versions of RPA calculations [4,9,10], appear to be smaller compared to those of the method (ii). However, the common RPA and the various refinements of the QRPA offer a relatively simple and detailed state-by-state calculation of all the individual low-lying excitations induced by the $\mu^- - e^-$ conversion operators [4,9].

An important conclusion of the state-by-state calculations is that the contribution of the 1^- states to the incoherent (μ^-, e^-) rate is very large (for most of the isotopes it is the maximum one) compared to that of the other multipolarities [9,10]. The portion of the 1^- contribution was found to be about 50% for all mechanisms leading to the $\mu^- - e^-$ conversion. Therefore, it is essential to properly remove possible spurious contaminations when describing this process and other similar ones.

The methods considered so far for the removal of the spurious center-of-mass (c.m.) admixture from the 1^- contribution [12,13] can be classified in two categories: (i) those that remove from the contaminated Hamiltonian the spurious terms, i.e., those containing the c.m. position (**R**) and the total momentum (**P**) operators and their couplings with the intrinsic Hamiltonian $H_{\rm int}$ [12,13]. The diagonalization of $H_{\rm int}$ obviously gives the real spectrum of the studied nucleus. In this way, the eigenstates of the system separate into the intrinsic nuclear spectrum and the pure c.m. excitation that can be omitted. (ii) Those that construct first a set of purified wave functions to be used for the diagonalization of the contaminated Hamiltonian. The orthonormalization, however, usually necessary in these methods, hinders their use. We should also mention that a recent method by Bes and Civitarese [14], which removes exactly the spurious contaminations of the dipole operator, shows that the spurious portion is much bigger than previously thought and other methods give.

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In QRPA calculations, an effective elimination of spurious components from the 1^- states may be achieved by adjusting the parameters that scale the effective interaction, so that the energy of the first 1^- state becomes equal to zero (equal to the purely spurious c.m. eigenstate). In most of the cases, this requires unphysical values for the parameters that renormalize the individual particle-particle (g_{pp}) and particle-hole (g_{ph}) channels (usually $g_{pp} \approx 1.3$–1.5, $g_{ph} \approx 0.2$–04). Even when the spurious state occurs very close to zero energy, spurious admixtures remaining at higher energies cannot be avoided completely.

In Ref. [10] an approximate removal of the spurious 1^- components was performed by constructing the properly normalized purely spurious state $|S\rangle$ and evaluating its overlap with all 1^- states involved in the chosen model space. This showed that mostly the lowest lying 1^- state is affected by the translational invariance breaking caused by the use of empirical single particle energies and a truncated model space in RPA. This state was considered as fully spurious and the others were treated as real nuclear excitations. After removing the spurious contributions, a renormalization of the interaction was required for reproducing the energy spectrum of the nucleus with the use of realistic two-body forces. The above method is simple and easy to apply, but it is not as exact as that of Refs. [12,14].

The purpose of the present work is to study in detail the incoherent rate of the $\mu^- - e^-$ conversion, by using the continuum RPA (CRPA) method. We evaluate the contribution of high-lying continuum excitations. Such an explicit calculation has not yet been addressed. As an application, we study extensively the incoherent rate for the heavy nuclear target ^{208}Pb by using Skyrme interactions. To eliminate spurious c.m. contaminations in the dominant 1^- channel, we obtain the corresponding energy distributions by using properly corrected dipole operators, which induce the $\mu^- - e^-$ conversion intrinsic 1^- excitations.

This article is organized as follows. In Sec. II we describe briefly the $\mu^- - e^-$ conversion operators and the formalism of the CRPA method. In Sec. III we calculate the strength distributions as functions of the excitation energy. The elimination of the spurious c.m. contaminations is also discussed. In Sec. IV we summarize our main conclusions.

II. DEFINITIONS AND BRIEF DESCRIPTION OF THE METHOD OF CALCULATION

The inclusive (μ^-, e^-) rate is evaluated by summing the partial contribution of all final states $|f\rangle$. For spherical or nearly spherical nuclei, the vector contribution is given by [4]

$$S_a = \sum_f \left(\frac{q_f}{m_\mu}\right)^2 |\langle f|O_a(\mathbf{q}_f)|0\rangle|^2, \tag{2}$$

where $O_a(\mathbf{q}_f)$ represents the $\mu^- - e^-$ vector-type transition operator resulting in the context of a given mechanism mediated by a photon ($a = \gamma$), a W-boson ($a = W$) or a Z-particle exchange ($a = Z$). Here $\mathbf{q}_f$, with magnitude $q_f = m_\mu - \epsilon_b - E_f$, is the momentum transferred to the nucleus. E_f is the energy of the final state $|f\rangle$ with respect to the ground state $|0\rangle$, ϵ_b is the binding energy of the muon, and m_μ its mass. The transition operators have the form

$$O_a(\mathbf{q}) = \tilde{g}_V f_V \sum_{j=1}^{A} 6c_a(\tau_j) e^{-i\mathbf{q}\cdot\mathbf{r}_j}, \quad c_a(\tau_j) \equiv \tfrac{1}{2} + \tfrac{1}{6}\beta_a \tau_j, \tag{3}$$

where τ_j is the third component of the jth particle's isospin. The parameter $f_V = 1.0$ represents the vector static nucleon form factor and the normalization coefficient $\tilde{g}_V$ takes the value $1/6$ for the photonic case and $1/2$ for the nonphotonic W boson and SUSY Z exchange [5]. The value of β_a depends on the model assumed. (We have taken the relevant values from Ref. [8].) Thus, protons (neutrons) contribute to a given process with a "charge"whose value is determined by $c_a(1/2) = 1/2 + \beta_a/6$ [$c_a(-1/2) = 1/2 - \beta_a/6$]. In the photon and Z case, the isoscalar and isovector components of the transition operator are (almost) equally important, whereas O_W is predominantly isoscalar.

By assuming that the initial and final states are of definite spin and parity, a multipole decomposition of the operators of Eq. (3) into operators T_{aLM} of orbital angular momentum rank L can be carried out. For spherical nuclei we can assume, without loss of generality, $\hat{q} = \hat{z}$. Then, only terms with $M = 0$ survive, for which we obtain

$$T_{aL}(q) \equiv T_{aL0}(q) = \tilde{g}_V f_V \sqrt{4\pi(2L+1)} \times \sum_{j=1}^{A} 6c_a(\tau_j) j_L(qr_j) Y_{L0}(\hat{r}_j). \tag{4}$$

A phase factor $(-\mathrm{i})^L$ has been omitted. The contribution of each multipolarity to the transition rate S_a reads

$$S_{aL} = \sum_f \left(\frac{q_f}{m_\mu}\right)^2 |\langle f|T_{aL}(q_f)|0\rangle|^2. \tag{5}$$

We now rewrite the rate S_{aL} as the integral of a suitable distribution over excitation energy:

$$S_{aL} \equiv \int dE\, R_{aL}(E) \tag{6}$$

with

$$\begin{aligned} R_{aL}(E) &= \sum_f \left(1 - \frac{\epsilon_b + E_f}{m_\mu}\right)^2 \\ &\quad \times |\langle f|T_{aL}(m_\mu - \epsilon_b - E_f)|0\rangle|^2 \delta(E - E_f) \\ &= \left[\left(1 - \frac{\epsilon_b}{m_\mu}\right)^2 \frac{2}{m_\mu^2}(m_\mu - \epsilon_b)E + \frac{1}{m_\mu^2}E^2\right] R'_{aL}(E). \end{aligned} \tag{7}$$

In the above expressions,

$$R'_{aL}(E) = \sum_f |\langle f|T_{aL}(m_\mu - \epsilon_b - E_f)|0\rangle|^2 \delta(E - E_f) \tag{8}$$

stands for the "strength distribution" corresponding to the operator $T_{aL}(q)$, with $q = m_\mu - \epsilon_b - E$. The total rate of

Eq. (6) is subsequently written as

$$S_{aL} = \left(1 - \frac{\epsilon_b}{m_\mu}\right)^2 M_0 - \frac{2}{m_\mu^2}(m_\mu - \epsilon_b)M_1 + \frac{1}{m_\mu^2}M_2,$$

where $M_k \equiv \int R'_{aL}(E)E^k dE$ is the k-moment of $R'_{aL}(E)$. The final states $|f\rangle$, excited by the single-particle operator T_{aL}, are of particle-hole (ph) type. Then, the distribution $R'_{aL}(E)$, and from it $R_{aL}(E)$ and S_{aL}, can be calculated following the standard RPA method.

We consider ph excitations, built on top of the Hartree-Fock (HF) ground state of a closed-shell nucleus and subjected to the ph residual interaction (HF+RPA method). In this work, the quantities introduced above are calculated using a self-consistent Skyrme-Hartree-Fock (SHF) plus CRPA model. The HF equations describing the ground state are derived variationally from the Skyrme energy functional and solved numerically using the code of P.-G. Reinhard [15]. The CRPA is formulated in coordinate space, as described, e.g., in Refs. [16,17] and outlined below.

The main ingredient of the model is the ph Green function $G_L(E)$ in coordinate space. In particular, we are interested in the radial part $G_{Lij}(r\tau, r'\tau'; E)$, which describes the propagation of a fluctuation (or ph state) of multipolarity L and energy E, excited by the operator V_{Li} at the point r, τ and decaying via the operator V_{Lj} at the point $r'\tau'$, where τ or τ' corresponds to the isospin character (proton or neutron) of the fluctuation. $V_{Li(j)}$ stands for one of the rank-L operators Y_L, $[Y_L \otimes (\nabla^2 + \nabla'^2)]_L$, $[Y_{L\pm1} \otimes (\nabla - \nabla')]_L$, $[Y_{L\pm1} \otimes (\nabla + \nabla')]_L$ present in the Skyrme interaction. In practice, the radial coordinates are replaced by points on a discretized mesh. A radial step Δr and a maximum value $r_{\max}$ (larger than the nuclear radius by a factor of about 3 or more, usually) are introduced. Then the Green function $G_L(E)$ can be represented as a supermatrix in coordinate space, isospin character, and operator indices i. The operator $T_{aL}(q)$, Eq. (4), is a multipole operator of the type Y_L—let us label it as V_{L1}. Therefore, within our RPA model, the distribution $R'_{aL}(E)$ of Eq. (8) is given in terms of the RPA Green function $G_L^{\mathrm{RPA}}(E)$ by

$$R'_{aL}(E) = \frac{\mathrm{Im}}{\pi} \mathrm{Tr}\left[T^\dagger_{aL}(q) G^{\mathrm{RPA}}_{L11}(E) T_{aL}(q)\right] \tag{9}$$

in a matrix notation. In practice, this reads

$$\begin{aligned} R'_{aL}(E) = {} & 144 \tilde{g}_V^2 f_V^2 (2L+1) \sum_{\tau,\tau'} c_a(\tau) c_a(\tau') \\ & \times \mathrm{Im} \int j_L(qr) G^{\mathrm{RPA}}_{L11}(r\tau, r'\tau'; E) \\ & \times j_L(qr') dr\, dr', \end{aligned} \tag{10}$$

$q = m_\mu - \varepsilon_b - E$. The integrations are to be understood as numerical ones, carried out by summing over the radial mesh points.

The RPA Green function is given by the equation

$$G_L^{\mathrm{RPA}}(E) = [1 + G_L(E)^0 V_{\mathrm{res}}]^{-1} G_L^0(E), \tag{11}$$

which is solved as a matrix equation in coordinate space, isospin character, and operators V_{Li}. The ph residual interaction V_{res} is zero range, of the Skyrme type, derived self-consistently from the Skyrme-HF energy functional [18,19]. In this work, spin-dependent terms and the Coulomb interaction are omitted from V_{res}. The radial part of the unperturbed ph Green function of multipolarity L is formally given by

$$\begin{aligned} G^0_{Lij}(r\tau, r'\tau'; E) = \delta_{\tau\tau'} \sum_{ph} \Bigg\{ & \frac{\langle p|V_{Li}|h\rangle^*_{r\tau} \langle p|V_{Lj}|h\rangle_{r'\tau}}{\varepsilon_{ph} - E} \\ & \pm \frac{\langle h|V_{Lj}|p\rangle^*_{r'\tau} \langle h|V_{Li}|p\rangle_{r\tau}}{\varepsilon_{ph} + E} \Bigg\}. \end{aligned} \tag{12}$$

The sign of the second term depends on the symmetry properties of the operators V_{Li} and V_{Lj} under parity and time-reversal transformations. With $h(p)$ we denote the quantum numbers of the HF hole (particle) state and $\varepsilon_{ph} = \varepsilon_p - \varepsilon_h$ is the energy of the unperturbed ph excitation. A small but finite $\mathrm{Im}E \equiv \Gamma/2$ ensures that bound transitions acquire a finite width. The particle continuum is fully taken into account, as follows Ref. [18]: The summation over the particle states p in Eq. (12) is replaced by the summation over all single-particle states k. The additional hole terms in the first term on the r.h.s will cancel the ones in the second term. Next, ε_k is replaced by the single-particle Hamiltonian. Finally, the completeness of the k states and the properties of the particle Green function are used to replace the sum over k by a closed expression. Therefore, the only truncation introduced is the one of the radial coordinate, $r \leqslant r_{\max}$. The latter is very well justified, because the amplitude of the radial wave functions of the hole states entering expression (12) vanishes at distances much larger than the nuclear radius.

Results derived within this model for the nucleus ^{208}Pb are presented and discussed in the next section.

III. RESULTS AND DISCUSSION

In the following we present results obtained using the SkM* [20] parametrization of the Skyrme force. It describes satisfactorily giant resonances of stable nuclei, and therefore it is suitable for the present study. We have verified that our conclusions do not change when the parametrization SGII [16,21] is used. To test the sensitivity of our results on the interaction used, we have also employed MSk7 [22], which has a large effective mass, thereby shifting most excited states to lower energies compared to the more reliable SkM* and SGII.

For the nucleus ^{208}Pb, the muon binding energy is $\epsilon_b = 10.475$ MeV and the momentum transferred to the nucleus by the outgoing e^- of the $\mu^- - e^-$ conversion ranges from $q = 0.482$ fm^{-1}, when the transition energy E vanishes (namely in the coherent process), to zero, when E reaches the maximum value, i.e., $E_{\max} = m_\mu - \epsilon_b = 95.183$ MeV (namely when all the available energy of the bound μ^- goes to a nuclear excitation). The particle threshold energy E_{thr} is 8.09 MeV in the case of the SkM* force. We have obtained results for $L = 0, 1, \ldots, 6$ and for natural parity, $(-1)^L$. The most important contributions to the incoherent transition rate are expected from $L < 4$ [10]. In all cases we have used $\Gamma = 0.2$ MeV and $r_{\max} = 17$ fm.

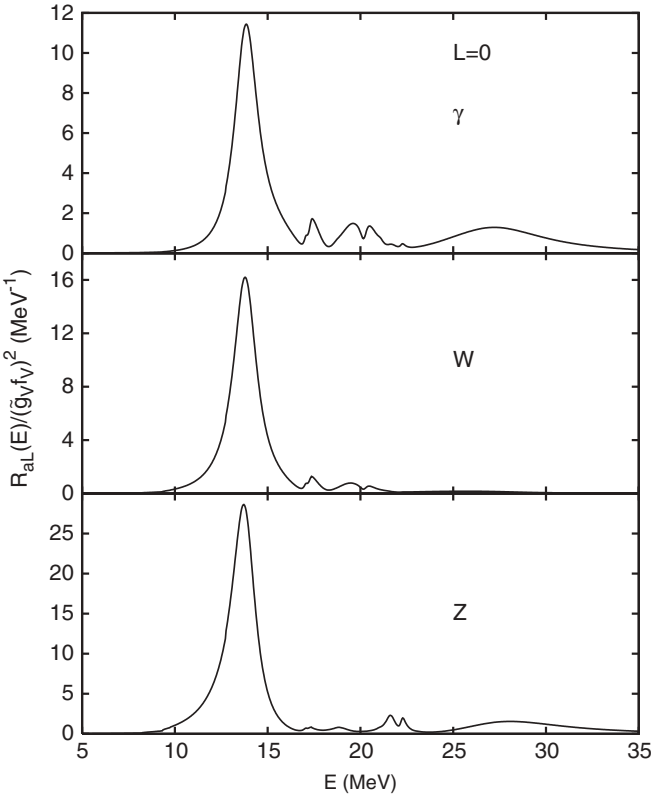

FIG. 1. The distribution $R_{aL}(E)$ in ^{208}Pb for $L = 0$. Skyrme parametrization SkM* has been used.

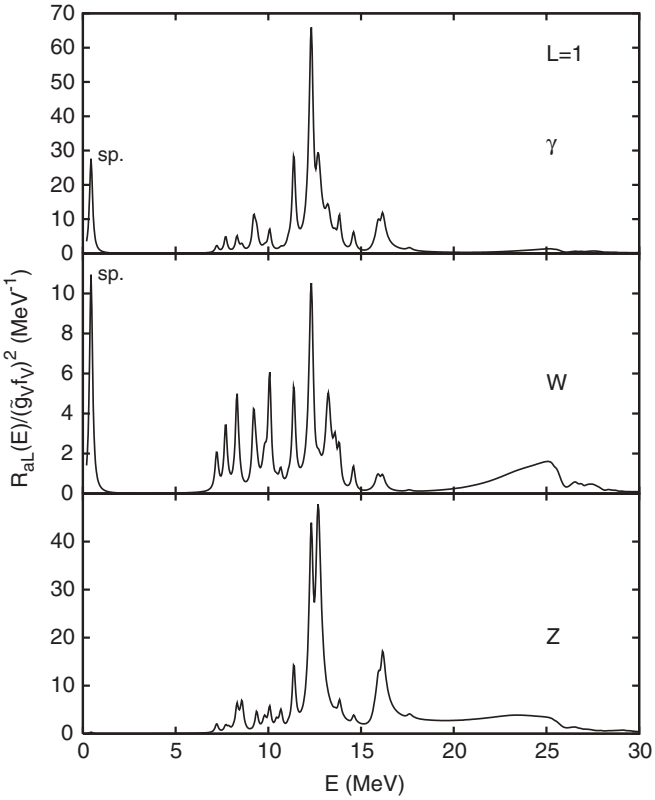

FIG. 2. Same as Fig. 1, for $L = 1$.

A. Incoherent transition-strength distributions versus excitation energy

In Figs. 1–4 the distribution $R_{aL}(E)$ is plotted as a function of E, for $L = 0, 1, 2, 4$, and for natural-parity transitions. In the monopole case, $L = 0$, the isoscalar (IS) giant monopole resonance (GMR) is the main peak. For γ and Z, there is considerable contribution coming from higher energies (20–35 MeV), i.e., the isovector (IV) GMR region. For $L = 1$, the IV giant dipole resonance (GDR) corresponds to the strength clustered around $E \approx 12$ MeV. For W and Z, important contribution seems to come from higher energies (above 20 MeV), in particular, the IS GDR. For W exchange, the region below 10 MeV contributes significantly. In this region we find the oscillation of the neutron skin against the nuclear core (pygmy dipole resonance) [23]. The transition density is isoscalar in the interior of the nucleus, whereas on the surface the proton contribution vanishes. In the quadrupole case, $L = 2$, the IS giant quadrupole resonance (GQR) is the second peak, close to 11 MeV. The collective low-lying peak is strong as well. There is some contribution from energies higher than 15 MeV, i.e., from the IV GQR region, especially in the cases γ and Z. For $L = 3$ (not shown) the strength is mostly concentrated in the collective octupole state at low energy. For $L > 3$, as shown in Fig. 4 for $L = 4$, the calculated strength is quite fragmented and most of it lies below 20 MeV.

A finite momentum transfer q_f can result in the excitation of overtones of giant resonances [24,25]. With the exception of the IS GDR, such states lie typically above 30 MeV, for ^{208}Pb, where the corresponding momentum transfer is less than 0.33 fm^{-1}. As a result, the possible contribution of such states was too small to be identified and visible on graphs.

The dipole results presented in Fig. 2 were obtained with corrected dipole operators—see Sec. III B for more details regarding this special case. A small amount of spurious strength remains close to zero energy (labeled sp. on the figure), but it is well separated from the rest of the distributions.

In Fig. 5 we plot the fraction of the total strength S_{aL} coming from states below the particle threshold ($S_{aL,\mathrm{thr}}$) and in Fig. 6 the fraction coming from states below 20 MeV ($S_{aL,20\,\mathrm{MeV}}$), vs. the multipolarity L. In the dipole case, the remaining strength of the spurious state is not taken into account when evaluating these fractions.

We see that for low multipoles $L = 0, 1, 2$ only a small portion of the strength originates from energies below particle threshold. The trend followed is similar for all mechanisms and, as we have verified, independent of the interaction used. For even multipoles $L = 2, 4$ a big portion of the contribution is pushed to higher energies as compared to the neighboring odd ones. Another interesting feature is the fact that, for some multipoles ($L = 0$ for photonic mechanism, $L = 1$ for W-boson exchange), a significant portion of the strength comes from above $E = 20$ MeV.

We have also calculated the fraction of the total strength S_{aL}, coming from states below 50 MeV, for L up to 6. In all cases, the fraction is practically equal to unity. This means that the discretized versions of RPA and QRPA for the examined

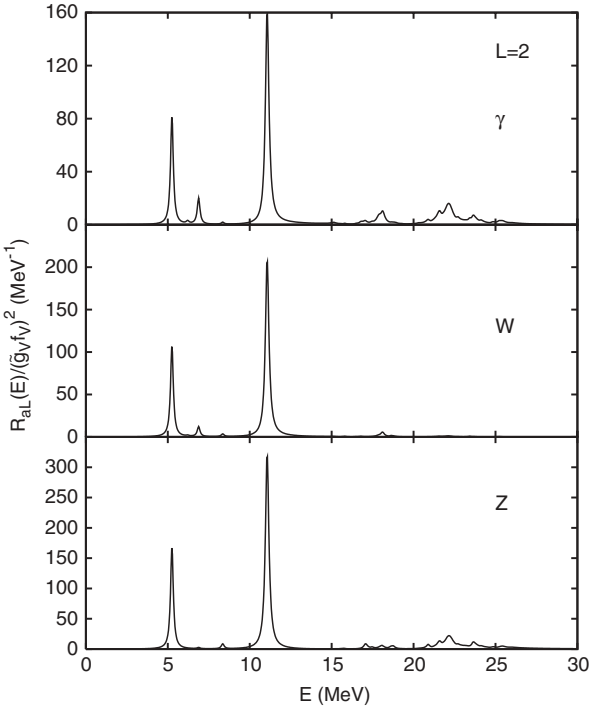

FIG. 3. Same as Fig 1, for $L = 2$.

nucleus ^{208}Pb are safe to use if the energy cutoff is large enough to sufficiently account for transitions below this value.

B. Dipole strength and the spurious c.m. motion

It is well known [12,13] that the 1^- excitations contain admixtures of the spurious excitation of the c.m. of the nucleus

$$\mathbf{R} = \frac{1}{A}\sum_{j=1}^{A} \mathrm{r}_j \tag{13}$$

corresponding to a situation in which the unexcited nucleus moves as a whole around the localized fictitious potential well. Normally, these spurious components are separated out by the RPA methods. However, the use of a truncated model space and non-self-consistent single-particle energies in ordinary RPA and the other versions of QRPA destroys the translational invariance and inserts spurious excitations into the spectrum. Thus, the spurious CM state is not completely separated from the real (intrinsic) nuclear excitations, and in addition its energy eigenvalue is not zero.

In CRPA models with Skyrme interactions it has been possible to achieve a high degree of self-consistency, i.e., the same interaction is used for the HF calculation of ground-state properties and for the residual interaction. In addition, no truncation is involved. However, because of the formulation of the model in coordinate space, it is common practice to exclude the Coulomb and spin-orbit contribution (at least) to the residual interaction. Therefore, self-consistency is violated and, even in cases where the spurious state appears very close to zero energy, some spurious strength may remain at higher energies. Restoration methods like those considered in Refs. [12–14] cannot be applied because the CRPA is not formulated in configuration space.

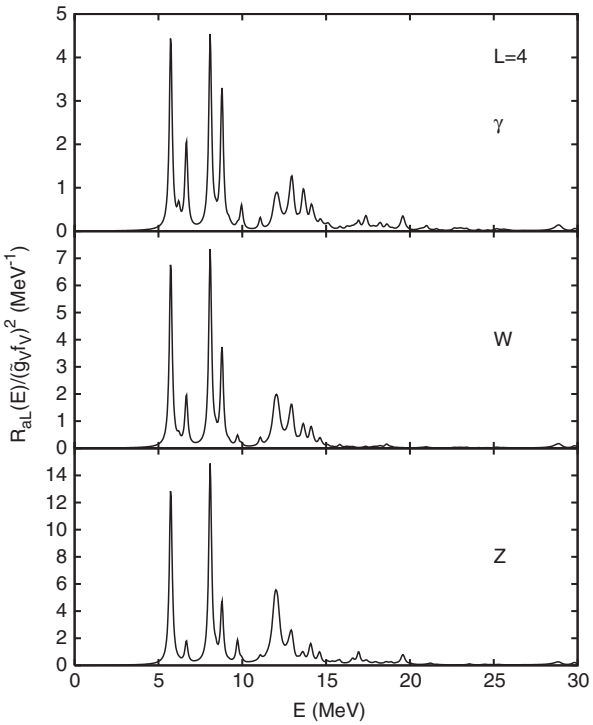

FIG. 4. Same as Fig 1, for $L = 4$.

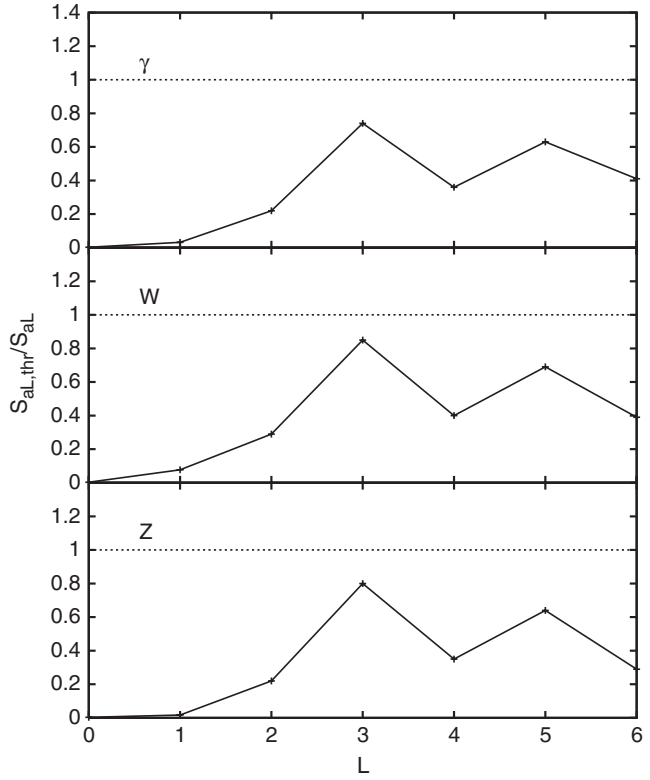

FIG. 5. Fraction of the total strength S_{aL}, coming from states below the particle threshold vs. the multipolarity L. Skyrme parametrization SkM* has been used. (Lines are drawn to guide the eye.)

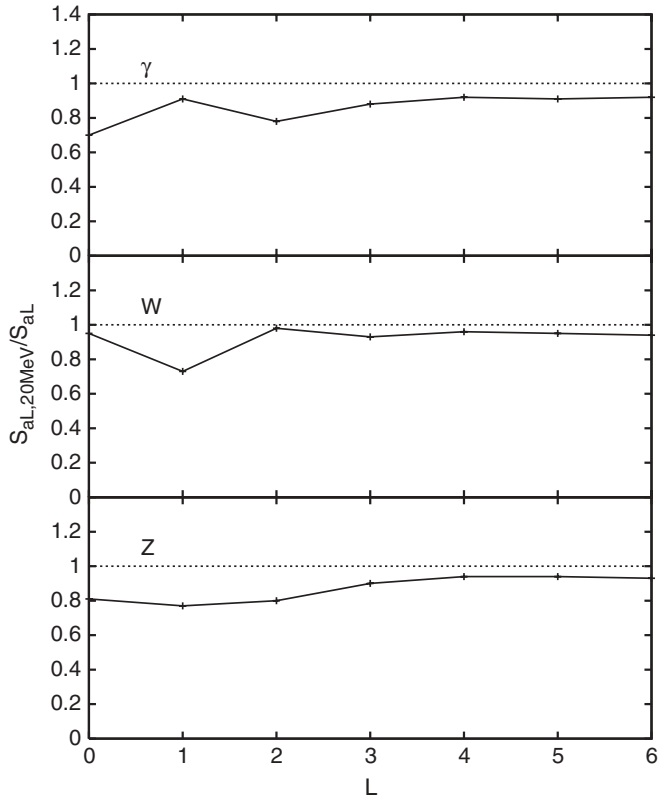

FIG. 6. As in Fig. 5, fraction of strength S_{aL}, coming from states below 20 MeV.

For electric dipole excitations, the problem is usually treated by using effective charges [26]. Similarly, in the case of IS dipole excitations, effective transition operators are used [16,27], which minimize the spurious admixture in the strength distribution. The effect on the IS dipole excitations of ^{208}Pb was examined in detail in Ref. [28]. Here we present a similar prescription for the operators involved in $\mu^- - e^-$ conversion.

We begin with a dipole excitation operator of the generic form

$$\Omega_1 = \sum_{j=1}^{A} c(\tau_j) f(r_j) Y_{10}(\hat{r}_j) = \sum_{j=1}^{A} c(\tau_j) \frac{f(r_j)}{r_j} \sqrt{\frac{3}{4\pi}} z_j. \quad (14)$$

As far as intrinsic excitations are concerned, this operator is equivalent to the corresponding "corrected" operator

$$\Omega_1^{\rm corr} \equiv \Omega_1 - \tilde{\eta} R_z = \sum_{j=1}^{A} [c(\tau_j) f(r_j) - \eta r_j] Y_{10}(\hat{r}_j), \quad (15)$$

where R_z stands for the z component of the c.m. space vector $\mathbf{R}$ of Eq. (13) and

$$\eta = \frac{1}{A} \sqrt{\frac{4\pi}{3}} \tilde{\eta}.$$

Our task is to determine the parameter η so as to eliminate the spurious c.m. excitation. This can be achieved within the collective model, by imposing the translational-invariance condition on the transition density characterizing the collective state induced by $\Omega_1^{\rm corr}$ [16,26]. A condition on η is thus obtained analytically. The result is

$$\eta = \frac{1}{3} \left[\frac{c(1/2)Z}{A} \left\langle \frac{1}{r^2} \frac{d}{dr} [f(r) r^2] \right\rangle_p + \frac{c(-1/2)N}{A} \left\langle \frac{1}{r^2} \frac{d}{dr} [f(r) r^2] \right\rangle_n \right], \quad (16)$$

where the mean values are defined as

$$\langle g(r) \rangle_{p,n} = \frac{\int_0^\infty g(r) \rho_{p,n}(r) r^2 dr}{\int_0^\infty \rho_{p,n}(r) r^2 dr}$$

and $\rho_p(r)$, $\rho_n(r)$, are the proton and neutron densities in the nuclear ground state. They are normalized so that $4\pi \int_0^\infty \rho_{p,n}(r) r^2 {\rm d}r = Z, N$.

By recalling from Eq. (4) the operator T_{a1} that induces the dipole 1^- excitations in $\mu^- - e^-$, we find that it can be cast in the form (14) with

$$f(r) = 6 \tilde{\rm g}_V f_V \sqrt{12\pi}\, j_1(qr)$$

and $c(\tau) = c_a(\tau)$. Then, using recursion relations of the Bessel functions, we find that

$$\frac{1}{r^2} \frac{d}{dr} [f(r) r^2] = 6 \tilde{g}_V f_V \sqrt{12\pi} q j_0(qr). \quad (17)$$

Notice that the proton, neutron form factor $F_{p,n}(q) = \langle j_0(qr) \rangle_{p,n}$. Under the above circumstances Eq. (16) reads

$$\eta_a = \tilde{g}_V f_V 4\sqrt{3\pi} q \left[\frac{c_a(1/2)Z}{A} F_p(q) + \frac{c_a(-1/2)N}{A} F_n(q) \right]. \quad (18)$$

The form factors are calculated numerically using the HF ground-state densities. The subscript in η_a denotes that the correction is different for each one of the three operators T_{a1} considered.

The above prescription is a generalization of the method used in Ref. [16], where a purely isoscalar field $[c(1/2) = c(-1/2)]$ was assumed and a specific form of the function $f(r)$ was utilized, i.e., $f(r) = r^3$. In Eq. (16) the values of $c(\pm 1/2)$ and the form of $f(r)$ are arbitrary. However, for the above mentioned isoscalar field, the present prescription leads to that given in Refs. [16,27].

In Fig. 7, we plot the dipole distributions $R_{a1}(E)$ of Eq. (8), for photon, W- and Z-boson exchange diagrams, calculated by using the corrected and uncorrected operator (they have been obtained with the SkM*force and for $\Gamma = 0.2$ MeV). One can see that most of the spurious strength below $\approx$6 MeV has been removed. The strength distributions above 20 MeV are practically unaffected. The strength between 6 and 20 MeV appears somewhat redistributed. The effect of the correction appears strongest in the case of the W-boson exchange mechanism. We should note that the radial mesh used in the CRPA calculation, with $\Delta r = 0.34$ fm, may not be fine enough to yield completely converged results in this energy region [28]. A result that persists when other interactions are employed, is that the pygmy dipole state below 10 MeV is strongly affected by the correction in the photonic and W cases.

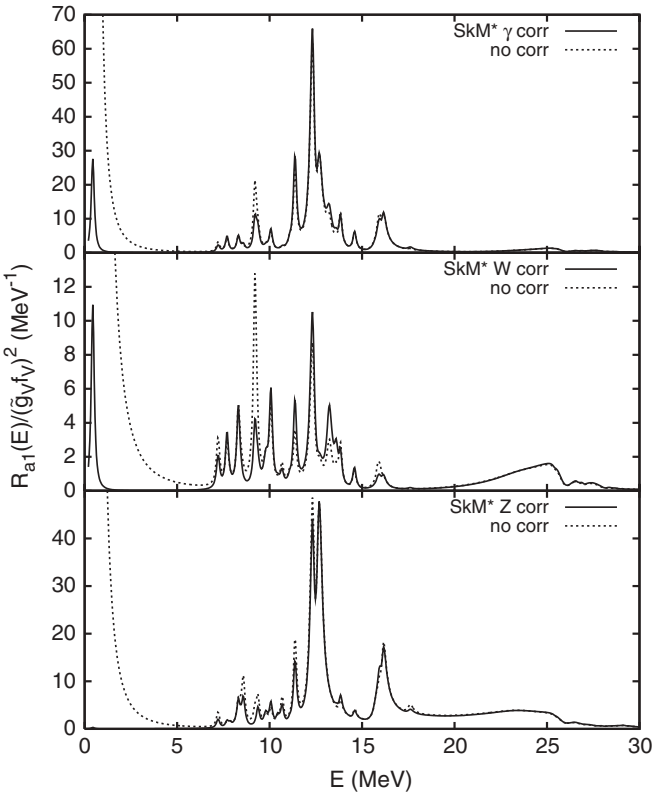

FIG. 7. The dipole distributions $R_{a1}(E)$ for γ-photon and W-boson exchange diagrams of the $\mu^- \to e^-$ conversion in ^{208}Pb. The results have been calculated for dipole operator T_{a1} (dotted line). To estimate the spurious c.m. contribution of this operator, we also show the distribution strength (full line) of the corresponding corrected operator given by Eq. (15).

In Table I we list the portion of transition strength removed from the total contaminated 1^- transition strength S_{a1} (denoted $s^{\rm sp}_{\rm tot}$), as well as the portion of the strength removed from above 6 MeV excitation energy $s^{\rm sp}_{>6{\rm MeV}}$ (with respect to the uncorrected strength above 6 MeV), for SkM* and for the three examined channels. For all three mechanisms, about 90% of the total transition rate was spurious. We expect this result to be independent of the interaction used, because the spurious state at low energy always dominates the isoscalar dipole strength distribution (which contributes in all three mechanisms) and because the corrected operators are, by construction, most effective for this state (thus removing practically all its strength). We were not able to demonstrate this in the particular cases of SGII and MSk7, because the energy of the spurious state in these cases was found imaginary, within the present calculation. In other words, we were not able to evaluate and take into account properly the strength of the spurious state, before or after the correction.

TABLE I. Percentage of the total 1^- transition strength S_{a1} ($s^{\rm sp}_{\rm tot}$) and of the strength above 6 MeV ($s^{\rm sp}_{>6{\rm MeV}}$) consumed by spurious transitions, for the interaction SkM*, and for the three channels γ, W, Z.

$\mu - e$ mechanism	γ	W	Z
$s^{\rm sp}_{\rm tot}(\%)$	86.9	96.3	90.5
$s^{\rm sp}_{>6\ {\rm MeV}}(\%)$	−1.3	7.8	6.1

From Table I we notice that for $s^{\rm sp}_{>6{\rm MeV}}$ and for $a = \gamma$ the result is small in absolute value, but negative. It represents the numerical accuracy of our calculation and, being small, it indicates that the degree of self consistency reached by our HF + CRPA model is sufficient to achieve a satisfactory separation of the spurious transition. For the W and Z cases, however, spurious admixtures of more than 6% are found above 6 MeV. These numbers vary when different Skyrme interactions are used, with their values remaining below 10%. (As mentioned before, they are not free of numerical inaccuracies.) One should apply the same treatment in the case of other, not self-consistent RPA methods, where, e.g., the lowest 1^- state is shifted artificially, by means of additional parameters, to zero energy. It is possible that larger corrections would be obtained, above 6 MeV. Such a result would mean that just excluding the lowest 1^- state from the calculation of the incoherent rate, as was done on Ref. [10], would not be an adequate treatment. As can be seen from Fig. 2 of Ref. [10], a considerable amount of spurious strength is distributed at higher excitation energies in the case of the QRPA calculations reported there.

Within the QRPA calculations of Ref. [10], for six nuclei (including ^{208}Pb), a more moderate correction (less than 60%) on S_{a1} was achieved by considering the lowest 1^- state as purely spurious and simply excluding it. If we follow an equivalent procedure, namely if we use the uncorrected transition operators and simply exclude the spurious state (the 1^- strength below 6 MeV) from the calculation of the 1^- rate, we would remove 88% from $S_{\gamma 1}$ (photonic case) and even more from S_{W1}, S_{Z1}. One should keep in mind the differences between the models used in Ref. [10] (QRPA calculations with a renormalized G-matrix interaction) and the ones used here. First of all, priority was given in Ref. [10] to the best possible reproduction of experimental spectra rather than controlling the self-consistency. In addition, a truncated model space was used, contrary to the method used in the present work. The various approximations entering may have influenced the spuriosity results in an unpredictable way. Both consistency and completeness of the space are important to move as much spurious strength as possible close to zero energy. Of course, the lowest 1^- state appeared very close to zero energy. It was found, however, that it was spurious by 60%–80% (except for ^{126}Yb, for which it was about 90% spurious). Thus, the approximation of considering it as purely spurious is not a very good one. The numbers mentioned above refer to the overlap of the lowest 1^- state with the " purely spurious " RPA state. Approximations were inevitable when constructing the purely spurious state to normalize it.

More appropriate for removing spurious contributions from (Q)RPA calculations in configuration space are the more rigorous methods developed in Refs. [12,14]. The idea is to calculate, for a given Hamiltonian and within a given model space, interaction counterterms that when added to the Hamiltonian, will eliminate the coupling between the intrinsic and c.m. degrees of freedom. Then the intrinsic eigenstates

of the new (Q)RPA matrix will separate exactly from the spurious c.m. excitation. The latter will in general appear at finite energy, but it will be purely spurious.

IV. SUMMARY AND CONCLUSIONS

In the present article we have focused on the investigation of the incoherent rate of the exotic $\mu^- - e^-$ conversion in the heavy nuclear target ^{208}Pb. We employed, for the first time in this process, the CRPA method that is appropriate for explicit construction of the excited states lying in the continuum spectrum of the nuclear target and we focused on the distribution of the transition strength as a function of the excitation energy of the target.

We have investigated in detail the various transition strength distributions coming from natural-parity *ph* excitations up to $L = 4$ by using two different Skyrme interactions. We found that a significant portion of the incoherent $\mu^- - e^-$ rate comes from high-lying nuclear excitations. A similar study could be done for unnatural-parity transitions.

The spurious 1^- admixture was eliminated by constructing the purified dipole operators of the $\mu^- - e^-$ conversion within the collective model. This study provided us with the interesting result that the greatest portion of the 1^- transition strength is because of the spurious c.m. excitation, a result in agreement with that of an exact method constructed recently [14] for removing spurious contaminations. The latter is a significant result for the $\mu^- - e^-$ conversion experiments searching for the coherent rate.

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