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# Dyson boson mapping of effective bi-fermion Hamiltonians

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Implementation of Dyson boson mapping is discussed in connection with effective Hamiltonians. A feature of the mapping technique, when implemented in an ideal boson basis, is the possible appearance of spurious states. These spurious states typically signal the overcompleteness of the basis. Without truncation, no contamination of the physical states and spectrum takes place. However, in practice one may be required to select from the ideal boson basis the dominant components for a given interaction. It is shown that the correspondence between a perturbative expansion, *á la* Bloch-Horowitz, and Dyson boson mapping allows for the identification of spurious states. The proposed method is applied to the mapping of a bi-fermionic Hamiltonian.

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## I. INTRODUCTION

Dyson boson mapping (DBM) [1] is a powerful tool that has been used to explore several aspects of the solution of fermionic Hamiltonians. The DBM has proved its utility, particularly, in dealing with nuclear many-body Hamiltonians [2].

The microscopic foundation of the DBM has been discussed in the literature [2,3]. The technique, which has been applied mostly in connection with the algebraic structure of phenomenological nuclear Hamiltonians [4], was extended to the treatment of effective Hamiltonians in Ref. [5]. More recently, DBM was used in the study of bi-fermionic Hamiltonians [6]. Also, it was used in connection with QCD-inspired algebraic models [7]. In this context, the adequacy of the boson basis and the identification of spurious states has been discussed in Refs. [8–10]. As an alternative to the similarity transformation introduced in Ref. [10], which may be limited by dimensional reasons, we rephrase the method of Ref. [10] in terms of a perturbative approach [11]. Thus, in the present work, we focus on the correspondence existing between a perturbative approach and an effective operator theory when applied to a boson expansion.

The Bloch-Horowitz perturbative expansion [11] provides a convenient framework to treat non-Hermitian Hamiltonians, as well as to handle overcomplete bases. The consequences of the use of the effective operator theory in boson mappings, have been discussed in Ref. [10]. The role of spurious states in connection with approximate diagonalization and/or truncation of the basis was illustrated in Ref. [10] for the cases of schematic SDI and $QQ$ interactions. Spurious effects also arise from the use of physically inspired and complementary spaces in conjunction with the boson mapping. This feature, which should not be confused with the class of spurious states discussed in Ref. [10], is typical of situations where a basis and space are constructed with direct reference to degrees of freedom linked to dominance anticipated from interactions. In other words, we discuss the effects associated to the introduction of *physical* bosons, e.g., such as monopole-pairing or multipole one phonon excitations, which are coupled with fermions at minimal order. Because we are replacing pair of fermions, e.g., particle-hole or two-particle (two-hole) configurations (generally speaking: bi-fermions), by bosons using truncated boson expansions, spurious effects will therefore appear. The fact that one can start from a physical inspired interaction determines the identification of the fermion content of the proposed bosons with particle-hole or two-particle (or two-hole) states. Then, because the Hamiltonian is a scalar concerning the particle content of the bosons and bi-fermion operators, the spurious effects will indeed depend on the strength of the interactions that represent their couplings.

The Dyson boson mapping technique, which has been shown to be extremely useful to treat many-body Hamiltonians [5], may require, for practical applications, the definition of a physically inspired subspace, that is to say the subspace of bosons that are in direct correspondence with fermion pair operators. The chosen subspace is contained in the space of ideal bosons and it defines the basis where the Hamiltonian can be diagonalized, at the cost of potentially introducing spurious states. It is clear that the solutions of a fully mapped (i.e., without any truncation) Hamiltonian expressed in the (sometimes overcomplete) ideal boson space, do not exhibit spurious components, but this representation is not always feasible. Solutions of the Hamiltonian in a physically inspired boson subspace requires the consideration of spurious effects, as pointed out in Ref. [12]. In the spirit of the work of Ref. [12] we show in this work that the results of effective operator theory of Ref. [10] can be reinterpreted in a perturbative framework. We shall compare the results of the use of truncated ideal boson basis with the ones obtained from a microscopic

 

boson representation defined by the underlying fermionic structure. As we have said before, this comparison is needed to estimate the adequacy of the approximation based on physically inspired boson subspaces. In doing this we aim at establishing a connection between algebraic and physically inspired boson representations. As a first step we have taken a minimal bi-fermion Hamiltonian and explored the regime of couplings where any one of the interactions dominates, that is, one of the combinations of bi-fermion operators and bosons that are allowed by particle-number conservation.

The article is organized as follows. Section II is devoted to the introduction of the formalism, starting with the boson representation of bi-fermion Hamiltonians (II A). In Sec. II B we demonstrate the analogy existing between the theory of effective operators of Ref. [10] and the perturbative treatment of Ref. [11], and in Sec. II C we perform the boson mapping of bi-fermion Hamiltonians, exploring the dependence of the solutions on the coupling constants. In Sec. III we present and discuss the results of the perturbative procedure in connection with the identification of spurious states. Finally, we present our conclusions in Sec. IV.

## II. FORMALISM

### A. Bosonization of bilinear Hamiltonians

The bosonization of bilinear Hamiltonians [8,13] can be performed by using the DBM method. There are three types of objects for which bosonlike algebras play a role: bi-fermion forms, the ideal bosons, and the boson images used to map fermion forms. We discuss the bosons of the mapping as those that transform the Hamiltonian in an algebra-preserving form. It means that a given Hamiltonian, written in terms of fermionic variables and its solutions, can be cast in a completely equivalent form by using boson variables. Thus, the relations between the generators of the algebra are fully preserved. The same is true for the matrix elements of the Hamiltonian, if it is transformed by using the bosons of the mapping. The ideal bosons belong to a subspace, chosen by convenience as a suitable basis to represent physical states. The transformation properties of the Hamiltonian are affected by this choice of the basis and spurious states may appear in the formalism.

Hereafter we distinguish between physical states, which are the states belonging to the physical subspace, and spurious states, which belong to the complement of the physical subspace. Both physical and spurious states are eigenstates of the Hamiltonian matrix in the complete space. However, after adopting a given subspace (inspired by physical considerations), spurious and physical states should ideally remain decoupled from each other. Although it may appear to be a trivial criterium, in realistic calculations the exact (complete space) solution is unknown and the use of a physical subspace is very convenient. In these cases, the identification of spurious states may be difficult. In this work we are discussing it from the point of view of the effective theories.

A possible way to identify spurious states, resulting from the use of ideal bosons, has been discussed in Refs. [8,10]. For the sake of completeness we summarize here the arguments of Ref. [8]. In Ref. [8] it was suggested that spurious states can be identified: (a) by constructing a matrix with the different terms of the interaction evaluated in a basis spanned by the eigenstates of the complete Hamiltonian calculated in the (possibly overcomplete) ideal boson basis and (b) by searching for vanishing blocks of this matrix. One feature of the formalism of Ref. [8] is that the eigenvalues associated with spurious states were obtained at both perturbed and unperturbed energies. Accidental cancellations can produce unperturbed eigenvalues even for nonvanishing interactions, as shown later.

### B. Theory of effective operators

In this subsection we discuss the analogy existing between the theory of effective operators of Ref. [10] and the perturbative treatment of Ref. [11]. Let us begin with the discussion of the results presented in Ref. [10]. Let $H$ be the Hamiltonian, $|\Psi\rangle$ the state vector and $E$ the corresponding eigenvalue. The Hilbert space is constructed by defining the subspaces of the projection operators $P$ (physical subspace) and $Q$ (the complement of the physical subspace), such that $P+Q=1$ and $\langle\Psi|PQ|\Psi\rangle = \langle\Psi|QP|\Psi\rangle = 0$. After some algebra one can show that the effective Hamiltonian

$$H_{\rm eff} = PHP + PHQ\frac{1}{E-QHQ}QHP, \tag{1}$$

satisfies the equation

$$H_{\rm eff}|\Psi\rangle = EP|\Psi\rangle. \tag{2}$$

Clearly, the expression of $H_{\rm eff}$ depends on the eigenvalue $E$. This dependence can be absorbed in an arbitrary operator $\omega$, which is taken as the generator of the similarity transformation

$$\mathcal{H} = \kappa H \kappa^{-1}. \tag{3}$$

At lower order in $\omega$ one has

$$\kappa = e^{-\omega} \approx 1-\omega. \tag{4}$$

The transformed eigenstate $|\Psi_\kappa\rangle = \kappa|\Psi\rangle$ obeys the equation

$$\mathcal{H}_{\rm eff}|\Psi_\kappa\rangle = EP|\Psi_\kappa\rangle. \tag{5}$$

We shall restrict $\mathcal{H}_{\rm eff}$ by the constraint $Q\mathcal{H}P=0$. With the choice

$$\begin{aligned}\omega &= Q\frac{1}{E-QHQ}QHP,\\ &= (1-P)\frac{1}{E-QHQ}QHP,\\ &= \frac{1}{E-QHQ}QHP,\end{aligned} \tag{6}$$

Eq. (5) is satisfied. In this way the original problem of Eq. (2) can be expressed as an eigenvalue problem in the subspace $P$, with energy-dependent norms. The factorization of the Hilbert space in the subspaces $P$ and $Q$ is driven by simplicity and it is constrained by the structure of $\mathcal{H}_{\rm eff}$. In the present context we define $P$ as the subspace that contains physical states (the states that can be written as linear combinations of the ideal bosons), whereas $Q$ corresponds to spurious states. These partitions are

not directly coupled, as required by the constraint $Q\mathcal{H}P = 0$ [10]. The presence of spurious states, in this theory, can be detected by the occurrence of eigenvectors with zero or imaginary norms. In the following, we show that this criteria can be met if the above formalism of effective operators is rephrased as a Bloch-Horowitz perturbative theory. If $|\alpha_i\rangle$ and $|a_i\rangle$ belong to the subspaces $\{P\}$ and $\{Q\}$, respectively, and $|n\rangle$ is an eigenstate of $H$, then

$$|n\rangle = \sum_i \langle \alpha_i | n \rangle |\alpha_i\rangle + \sum_i \langle a_i | n \rangle |a_i\rangle. \tag{7}$$

We can write, for the $n$-th eigenvalue $E_n$, the matrix equation

$$\begin{aligned} h_{\text{eff}}^{(n)} \xi^{(n)} &= E_n \xi^{(n)}, \\ \xi_i^{(n)} &= \langle \alpha_i | n \rangle, \end{aligned} \tag{8}$$

where

$$h_{\text{eff}}^{(n)} = h^{(1)} + h^{(12)} \frac{1}{E_n - h^{(2)}} h^{(21)}. \tag{9}$$

The block matrices $h^{(k)}$ are defined by

$$\begin{aligned} h_{ij}^{(1)} &= \langle \alpha_i | H | \alpha_j \rangle, \\ h_{ij}^{(12)} &= \langle \alpha_i | H | a_j \rangle, \\ h_{ij}^{(21)} &= \langle a_i | H | \alpha_j \rangle, \\ h_{ij}^{(2)} &= \langle a_i | H | a_j \rangle. \end{aligned} \tag{10}$$

The amplitudes $\{\xi_i^{(n)}\}$ are normalized in terms of the norm matrix [10]

$$M^{(n)} = \frac{1}{E_n - h^{(2)}} h^{(21)}. \tag{11}$$

Any eigenstate of the non-Hermitian Hamiltonian $h_{\text{eff}}^{(n)}$ is normalized by means of the relationship

$$\langle n | n' \rangle = \sum_{ij} \left( \delta_{ij} + \sum_k M_{ki}^{n\,*} M_{kj}^{n'} \right) \xi_i^{(n)*} \xi_j^{(n')}. \tag{12}$$

For a physical eigenstate, the norm $\langle n|n\rangle$ is positive defined. This is a condition on Eq. (12). By using the similarity transformation of Eq. (3) and the decomposition in the subspaces $\{P\}$ and $\{Q\}$ one can write a similar expression in the effective operator theory. It reads

$$\langle n | n' \rangle = \langle n | P(1 + \omega^\dagger Q \omega) P | n' \rangle. \tag{13}$$

This result implies that the operator $P(1 + \omega^\dagger Q\omega)P$ must be positive definite.

### C. Bi-fermion Hamiltonians

To illustrate the above presented formalism let us consider the bi-fermion Hamiltonian of Refs. [8,13]. The configuration space consists of two levels, each of them with $2\Omega$ substates. Bi-fermion operators can connect states in the same level or in different levels. The minimal form of the Hamiltonian is Refs. [8,13,16]

$$H = \epsilon n + \bar{\epsilon}\, \bar{n} - V S_+ S_- - \lambda L_+ L_- - \mu K_+ K_-, \tag{14}$$

where the operators that appear in $H$ are defined by

$$\begin{aligned} S_+ &= (S_-)^\dagger = \sum_m c^m c^{\overline{m}}, \\ S_z &= \tfrac{1}{2}(n + \overline{n} - 2\Omega), \\ L_+ &= (L_-)^\dagger = \sum_{m>0} c^m c^{-m}, \\ L_z &= \tfrac{1}{2}(n - \Omega), \\ K_+ &= (K_-)^\dagger = \sum_m c^{\overline{m}} c^{-\overline{m}}, \\ K_z &= \tfrac{1}{2}(\bar{n} - \Omega), \\ T_+ &= (T_-)^\dagger = \sum_m c^{-m} c_{\overline{m}} sgn(m), \\ T_z &= L_z - K_z. \end{aligned} \tag{15}$$

These products of fermion creation ($c^m$) and annihilation ($c_m$) operators are the generators of Sp(4) [14]. Exact solutions of the Hamiltonian of Eq. (14) are expressed in the basis [15]

$$|mpk\rangle = N(m, p, k) L_+^m K_+^p O_+^k |\phi\rangle, \tag{16}$$

where

$$\begin{aligned} O_+ = {} & K_+(L_+ S_- + T_+(2L_0 - 1)) \\ & + [L_+ T_- - S_+(2L_0 - 1)](2K_0 - 1) \end{aligned} \tag{17}$$

and $L_-|\phi\rangle = K_-|\phi\rangle = 0$. The factor $N(m, p, k)$ is a normalization factor [15]. The matrix elements of the Hamiltonian [Eq. (14)] in the Eq. (16) are straightforwardly calculated, owned to the rules of the Sp(4) generators of Eq. (15).

The boson image of the bi-fermion Hamiltonian (14) is constructed by writing the operators of Eq. (15) as linear combinations of particle-hole, hole-hole, and particle-particle like-bosons and their products [8,13]. If $B_f^\dagger$, $B_h^\dagger$, and $B_p^\dagger$, and their Hermitian conjugates, represent particle-hole, hole-hole, and particle-particle bosons, the DBM images of the operators of Eq. (15) are written

$$\begin{aligned} S_+ &\to B_f^\dagger (2\Omega - N_f - 2N_p - 2N_h) - B_p^\dagger B_h^\dagger B_f, \\ S_- &\to B_f, \qquad S_z \to N_f + N_p + N_h - \Omega, \\ L_+ &\to B_p^\dagger (\Omega - N_p - N_f) - B_f^\dagger B_f^\dagger B_h, \\ L_- &\to B_p, \qquad L_z \to N_p + \tfrac{1}{2} N_f - \tfrac{1}{2}\Omega, \\ K_+ &\to B_h^\dagger (\Omega - N_h - N_f) - B_f^\dagger B_f^\dagger B_p, \\ K_- &\to B_h, \qquad K_z \to N_h + \tfrac{1}{2} N_f - \tfrac{1}{2}\Omega, \\ T_+ &\to 2B_f^\dagger B_h + B_p^\dagger B_f, \\ T_- &\to 2B_f^\dagger B_p + B_h^\dagger B_f, \qquad T_z \to N_p - N_h, \end{aligned} \tag{18}$$

in a similar way one can define boson number operators

$$N_f = B_f^\dagger B_f, \quad N_p = B_p^\dagger B_p, \quad N_h = B_h^\dagger B_h, \tag{19}$$

and with them, the boson image of the Hamiltonian of Eq. (14) is given by

$$\begin{aligned} H_B &= \omega_f N_f + \omega_p N_p + \omega_h N_h \\ &+ V(B_f^{\dagger\,2} B_f^2 + 2N_f(N_p + N_h) + B_p^\dagger B_h^\dagger B_f^2) \\ &+ \lambda(B_p^{\dagger\,2} B_p^2 + N_f N_p + B_f^{\dagger\,2} B_h B_p) \\ &+ \mu(B_h^{\dagger\,2} B_h^2 + N_f N_h + B_f^{\dagger\,2} B_h B_p). \end{aligned} \tag{20}$$

The quantities $\omega_i$ ($i = f, p, h$) are defined by

$$\omega_f = \epsilon + \bar{\epsilon} - 2\Omega V, \quad \omega_p = 2\epsilon - \Omega\lambda, \quad \omega_h = 2\bar{\epsilon} - \Omega\mu. \tag{21}$$

The boson image of $H$, Eq. (20), and the operators of Eq. (18) obey the original Sp(4) algebra, of Eq. (15). Working in the subspace of $2p$-$2h$ excitations one can introduced, as ideal basis, the states $|1\rangle$ and $|2\rangle$

$$\begin{aligned} |1\rangle &= \frac{1}{\sqrt{2}} B_f^{\dagger\,2} |0\rangle, \\ |2\rangle &= B_p^\dagger B_h^\dagger |0\rangle, \end{aligned} \tag{22}$$

where $\langle 1|2\rangle = 0$ holds by construction.

This choice of the ideal basis is arbitrary, because, depending on the relative magnitude of the couplings $V$, $\lambda$, and $\mu$, the meaning of the $\{P\}$ and $\{Q\}$ subspaces can be interchanged. To fix the ideas, let us mention that the arbitrariness, in the choice of the ideal basis, is a direct consequence of the fact that although the Hamiltonian [Eq. (20)] has been Dyson mapped without truncation, the ideal basis [Eq. (22)] has been constructed without exact mapping. This basis is at least complete, but possibly overcomplete, because the boson images of the operators in Eq. (18) are expressed at the lowest order. This is easily seen by noticing that the state $|m = 1, p = 1, k = 0\rangle$, of Eq. (16), coincides with $|2\rangle$, up to a normalization factor and at dominant order in $1/\Omega$, whereas $|1\rangle$ cannot be expressed in a similar manner if terms proportional to the number operators are neglected in $S_+$ of Eq. (18). In other words, some of the states that belong to the ideal boson basis may not be expressed in terms of the exact basis and/or may overlap only partially with the states of the exact basis. It means that the orthogonality and completeness of the exact basis are not obeyed when the states are written in the ideal boson basis rather than in terms of the generators of the algebra. Therefore, the obtained eigenvectors may indeed be nonorthogonal, as we shall see below. Therefore, the matrix equation that is Hermitian (if the mapping is performed for both the Hamiltonian and a chosen complete fermion basis) becomes non-Hermitian in general.

At this point, we can advance the notion that spurious states could appear as a consequence of truncation. Formally speaking, the boson image of the Hamiltonian, given by Eq. (20) is the result of the complete mapping [Eq. (18)], whereas in defining the ideal basis [Eq. (22)] one keeps dominant components of $S_+^2$ and $L_+K_+$. It is easy to see that the algebraic structure of $H$ is preserved by the mapping but only some subalgebras are obeyed by the operators entering in the definition of the ideal states.

The matrix elements of $H_B$, in this basis, are given by

$$\begin{pmatrix} 2\omega_f + 2V & \sqrt{2}(\lambda + \mu) \\ \sqrt{2}V & \omega_p + \omega_h \end{pmatrix}, \tag{23}$$

and the eigenvectors are written as

$$|n\rangle = N^{(n)}(|1\rangle + x^{(n)}|2\rangle), \tag{24}$$

where $N^{(n)}$ is a normalization factor and $x^{(n)}$ is the ratio between the amplitudes. In the following, we shall discuss two trivial model situations, namely $\lambda = \mu = 0$, $V \neq 0$, and vice versa.

For the case $\lambda = \mu = 0$, the solutions of the eigenvalue equation are

$$\begin{aligned} E_{n=1} &= 2E_0 - 2V(2\Omega - 1), \\ E_{n=2} &= 2E_0, \end{aligned} \tag{25}$$

where

$$E_0 = \epsilon + \bar{\epsilon}, \tag{26}$$

with eigenvectors

$$\begin{aligned} |n = 1\rangle &= N^{(1)}(|1\rangle - \frac{1}{\sqrt{2}(2\Omega - 1)}|2\rangle), \\ |n = 2\rangle &= |2\rangle, \end{aligned} \tag{27}$$

respectively. These two solutions are not orthogonal. We can show their linear dependence by choosing, in Eqs. (8)–(10), the block structure

$$\begin{aligned} h^{(1)} &= 2E_0 - 2V(2\Omega - 1), \\ h^{(12)} &= 0, \\ h^{(21)} &= \sqrt{2}V, \\ h^{(2)} &= 2E_0. \end{aligned} \tag{28}$$

Therefore, for the eigenvalue $E_{n=1}$, one obtains

$$E_{n=1}I - H_B = \begin{pmatrix} 0 & 0 \\ -\sqrt{2}V & -2V(2\Omega - 1) \end{pmatrix}. \tag{29}$$

The eigenvector corresponding to this solution has the amplitude

$$x^{(1)} = -\frac{1}{\sqrt{2}(2\Omega - 1)}. \tag{30}$$

For the eigenvalue $E_{n=2}$ one gets

$$E_{n=2}I - H_B = \begin{pmatrix} 2V(2\Omega - 1) & 0 \\ -\sqrt{2}V & 0 \end{pmatrix}, \tag{31}$$

and $x^{(2)}$ remains undefined, which means that $|n = 2\rangle$ is an spurious state. This is a case where one gets a solution at the unperturbed energy even for nonvanishing interactions.

For the case $V = 0$, the structure of the states $|1\rangle$ and $|2\rangle$, of Eq. (22), is exchanged and the solutions can be written as in the previous case. It is then clear that, with the construction [Eq. (22)], only one of the eigenstates belongs to the physical sector, whereas the other is fully spurious. This is a common feature of the mapping procedure, and it is independent of the coupling, of course. Now, let us consider the general case with

the couplings $V$, $\lambda$, and $\mu$ different from zero. The eigenvalues of $H_B$ are

$$\begin{aligned} E_1 &= 2E_0 - \frac{\Delta}{2} - \frac{r}{2}, \\ E_2 &= 2E_0 - \frac{\Delta}{2} + \frac{r}{2}, \end{aligned} \tag{32}$$

where

$$\Delta = (4\Omega - 2)V + \Omega L \tag{33}$$

and

$$r = \sqrt{\Delta^2 - 8(2\Omega + 1)(\Omega - 1)VL}, \tag{34}$$

with $L = \lambda + \mu$. For this case, Eqs. (8)–(10) leads to the amplitudes:

$$\begin{aligned} x^{(1)} &= \frac{\sqrt{2}V}{\Omega L - \frac{\Delta + r}{2}}, \\ x^{(2)} &= \frac{\sqrt{2}V}{\Omega L - \frac{\Delta - r}{2}}. \end{aligned} \tag{35}$$

To understand the structure of the solutions given above we shall consider the nontrivial case $\Omega \gg 1$ (because the geometrical restrictions are self-evident for $\Omega = 1$). In this limit one gets

$$\begin{aligned} \Delta &= \Omega(4V + L) \\ r &= \Omega(4V - L) \\ E_1 &= 2E_0 - 4\Omega V \\ E_2 &= 2E_0 - \Omega L \end{aligned} \tag{36}$$

and, in consequence,

$$\begin{aligned} x^{(1)} &= -\frac{\sqrt{2}V}{\Omega(4V - L)} \to 0 \\ x^{(2)} &\to \infty \end{aligned} \tag{37}$$

because $h^{(2)} - E_2 = 0$ for $4V > L$. Notice that if $4V < L$ the first and second states of the ideal basis, Eq. (22), are interchanged, as well as the eigenvalues and eigenvectors. At the point $4V = L$ the solutions display an avoided crossing. It is then obvious that for the case $\Omega \gg 1$ and $4V > L$ the eigenstates in the ideal space are normalized like $N^{(1)} \to 1$ and $N^{(2)} \to 0$, as $\Omega \to \infty$. (and the other way around if $4V < L$). These features are confirmed by the exact solutions, as we shall show in the following.

For the case of 2p-2h the exact basis is given by [see Eq. (16)]

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{\Omega} L_+ K_+ |\phi\rangle, \\ |\phi_2\rangle &= \frac{1}{2\sqrt{(\Omega - 1)(2\Omega + 1)}} \left( S_+^2 + \frac{2}{\Omega} L_+ K_+ \right) |\phi\rangle, \end{aligned} \tag{38}$$

and the Hamiltonian matrix of Eq. (14) reads

$$\begin{pmatrix} 2E_0 - \Omega L - \frac{2V}{\Omega} & \frac{2V}{\Omega}\sqrt{(\Omega - 1)(2\Omega + 1)} \\ \frac{2V}{\Omega}\sqrt{(\Omega - 1)(2\Omega + 1)} & 2E_0 - \frac{2V}{\Omega}(\Omega - 1)(2\Omega + 1) \end{pmatrix}. \tag{39}$$

Note that the basis of Eq. (22) may be viewed as the lowest-order terms (in powers of $1/\Omega$) of the one obtained by mapping from the complete fermion basis, Eq. (38). However, one should be cautious in writing the correspondence between both basis, because of the appearance of divergent normalization factors. This is the case of the state $|\phi_2\rangle$ of Eq. (38), which is nonrenormalizable if $\Omega = 1$. One may also think of this as a result of the violation of the occupation number symmetry.

The eigenvalues corresponding to the complete Hamiltonian matrix above are

$$\begin{aligned} E_1 = 2E_0 &- V(2\Omega - 1) - \tfrac{1}{2}L\Omega \\ &- \sqrt{2LV + \left(\tfrac{1}{2}L\Omega - V(2\Omega - 1)\right)^2}, \end{aligned} \tag{40}$$

$$\begin{aligned} E_2 = 2E_0 &- V(2\Omega - 1) - \tfrac{1}{2}L\Omega \\ &+ \sqrt{2LV + \left(\tfrac{1}{2}L\Omega - V(2\Omega - 1)\right)^2}, \end{aligned} \tag{41}$$

which for $\Omega = 1$ reduces to

$$\begin{aligned} E_1 &= 2E_0 - 2V - L, \\ E_2 &= 2E_0. \end{aligned} \tag{42}$$

These exact eigenstates coincide with the results [Eq. (32)]. Note that they are finite even for $\Omega = 1$, although there is a nonrenormalizable vector in Eq. (38). Naturally, one should choose only one state of Eq. (38) for the definition of the collective subspace, because the other belongs to the (unperturbed) complement subspace.

Another good example of the geometric restriction is given by the case of $4p$-$4h$ states. The exact basis is spanned by the states

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2\Omega(\Omega - 1)} L_+^2 K_+^2 |\phi\rangle, \\ |\phi_2\rangle &= \frac{1}{2(\Omega - 2)\sqrt{(\Omega - 1)(2\Omega + 1)}} \\ &\quad \times \left( L_+ K_+ S_+^2 |\phi\rangle + 4(\Omega - 1)|\phi_1\rangle \right), \\ |\phi_3\rangle &= \frac{1}{4\sqrt{6\Omega(\Omega - 3)(2\Omega - 1)(2\Omega + 1)}} \\ &\quad \times (S_+^4 |\phi\rangle + 24\sqrt{(\Omega - 1)(2\Omega + 1)}|\phi_2\rangle - 24|\phi_1\rangle). \end{aligned} \tag{43}$$

The matrix elements of the Hamiltonian of Eq. (14) are the following:

$$\begin{aligned} h_{11} &= 4E_0 - 2(\Omega - 1)L - \frac{8V}{\Omega}, \\ h_{12} = h_{21} &= \frac{4V(\Omega - 2)(2\Omega + 1)}{\Omega\sqrt{(\Omega - 1)(2\Omega + 1)}}, \\ h_{13} = h_{31} &= 0, \\ h_{22} &= 4E_0 - (\Omega - 2)L - 2V(2\Omega - 5) - \frac{4V(\Omega + 4)}{\Omega(\Omega - 2)}, \\ h_{23} = h_{32} &= \frac{V\sqrt{24\Omega(\Omega - 3)(2\Omega - 1)}}{\sqrt{(\Omega - 1)(\Omega - 2)}}, \\ h_{33} &= 4E_0 - 4V\left(2\Omega - 3 - \frac{3}{\Omega - 2}\right). \end{aligned} \tag{44}$$

As shown in Eq. (43), the geometric restrictions are evident for $\Omega \leqslant 3$. Moreover, for the case $\Omega \leqslant 2$ the determinant of the matrix in Eq. (44) is undetermined. For $\Omega = 3$, that matrix has one eigenvalue $E_n = 4E_0$, which is the unperturbed energy of the 4p-4h configuration. These features have been interpreted earlier in connection with spurious states [8]. In the context of the present work, we refer to them as geometric restrictions. Notice that the leading-order terms of Eq. (43) are the three vectors that appear in Ref. [8] for the 4p-4h case.

As done in the previous example of the 2p-2h case, these spurious states can be isolated by an adequate definition of the collective space.

## III. DISCUSSION OF RESULTS

In this section we discuss the results of the calculations we have performed by using the formalism of the previous sections.

To start with we have diagonalized the Hamiltonian of Eq. (20) in the ideal basis of Eq. (22). We have followed the dependence of the solutions on $\Omega$, $V$, and $L$. Next, we have compared the solutions of the Eqs. (6) and (13) with the amplitudes $x^{(j)}$ of Eq. (35). The results are shown in Figs. 1 and 2. In these figures we show the functional structure of the eigenvalues $E_{n=1}$ and $E_{n=2}$ and the values of $h^{(2)}$ for different values of $\Omega$, $V$, and $L = \lambda + \mu$. Following the arguments of Sec. II B, the normalization factor of

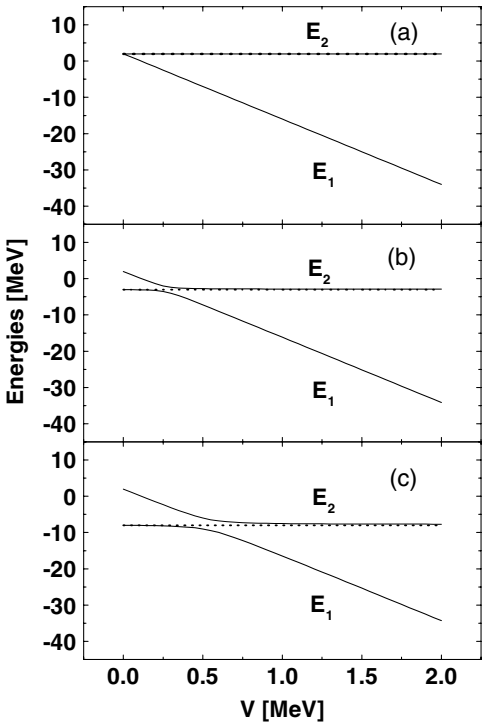


FIG. 1. Eigenvalues $E_{n=1}$ and $E_{n=2}$, solid lines, and expectation values of $h^{(2)}$, dotted lines. The values are display as a functions of the strength $V$, for different values of $L = \lambda + \mu$. Cases (a), (b), and (c) correspond to $L = 0$, 1, and 2 MeV, respectively. All results correspond to $\Omega = 5$, the value of $E_0$ of Eq. (26) is fixed at $E_0 = 1$, and all values are given in units of mega-electron volts.

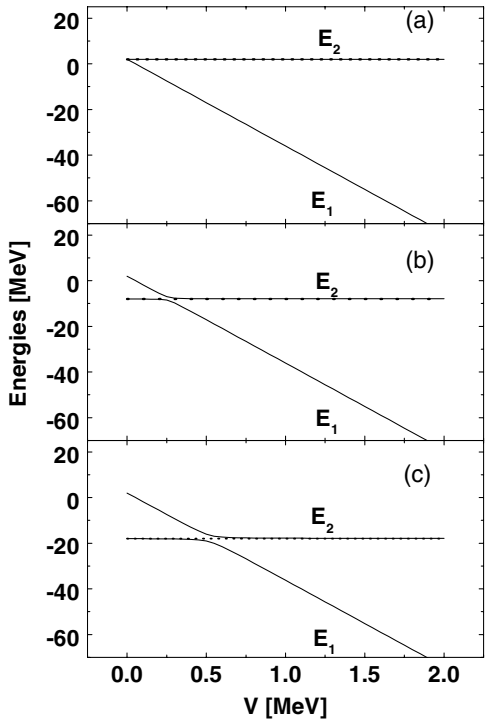


FIG. 2. Same as Fig. 1 for $\Omega = 10$.

Eq. (12) vanishes or becomes negative if $E_n$ is the eigenvalue corresponding to a spurious state (i.e., a state belonging to the complementary space $Q$). This condition is met when the denominator $E - h^{(2)}$ vanishes. Figure 1 shows the results corresponding to $\Omega = 5$. Except for the trivial singularity at $V = L = 0$, the branch $E_2$, at $L = 0$, represents a spurious state for all values of $V$. This identifies $E_{n=1}$ as the eigenvalue of the collective subspace and $E_{n=2}$ as the eigenvalue of the unphysical subspace. Both solutions are, nevertheless, present in the exact diagonalization.

The same feature appears by changing $\Omega$ and $V$, for $L = 0$, as shown in Fig. 2. The results of both figures (cases with different values of $L$) exhibit an avoided crossing between levels at $4V = L$. As stated previously (see Sec. II C), if $4V < L$ the first and second states of the ideal basis, Eq. (22), are exchanged, with $N^{(1)} \to 0$ and $N^{(2)} \to 1$, as $\Omega \to \infty$. If $4V > L$ the eigenstates in the ideal space are normalized like $N^{(1)} \to 1$ and $N^{(2)} \to 0$, in the same limit $\Omega \to \infty$.

The distance between levels at the crossing point, which is a function of $V$ decreases as $\Omega$ increases. Also the crossing point shifts to higher values of $V$ as $\Omega$ and $L$ increase. In all cases the crossing indicates the point where the collective and complement spaces are exchanged. This is a well-known fact of perturbative treatments and it appears also here in the DBM bosonization. To be more precise, in performing the Dyson boson expansion and choosing a certain physical space (of boson operators) one is splitting the Hilbert space in the subspaces $P$ and $Q$ of Eq. (1). The degree of mixing between the physical and spurious states will certainly depend on the chosen subspaces. Alternatively, the same mixing can be view as a result of the renormalization of the interactions in the physical space, as done in Eq. (7), an effect that is achieved by

working in the subspace of $h^{(1)}$. Thus, the normalization factor in Eq. (12) will indicate the effect because of the configurations removed from the physical space.

## IV. CONCLUSIONS

We have discussed the appearance of spurious states in the treatment of bi-fermion boson Hamiltonians in ideal terms. Particularly, we have focused on the effect of truncation of the basis. We have compared the formalism obtained by using similarity transformations with the one obtained from the use of a perturbative scheme.

The above results indicate that, as a general prescription to isolate spurious states, one can search for singularities of the similarity transformation, after performing Dyson boson mapping.

In situations where the symmetries of the Hamiltonian are preserved by boson mapping, the ideal basis can be labeled by the quantum numbers of the symmetry operators, and spurious states can approximately be removed by performing variations around values fixed by the symmetry. Examples of this procedure can be found in Ref. [7].

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