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# Exactly solvable model of low energy QCD

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Starting from the QCD Hamiltonian, we derive a schematic Hamiltonian for low energy quark dynamics with quarks restricted to the lowest s-level. The resulting eigenvalue problem can be solved analytically. Even though the Hamiltonian exhibits explicit chiral symmetry the severe restriction of the number of degrees of freedom breaks the pattern of chiral symmetry breaking for finite quark masses.

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## I. INTRODUCTION

As significant advances have been made over the past few years in lattice gauge QCD, progress in developing other, approximate solutions to low energy QCD has not been as impressive. Many models, such as the quark model, the bag model, the flux tube model, and many others have been utilized to capture selective feature of the theory. Traditional approaches to develop models with a more rigorous relation to QCD are based on the covariant representation of the theory. Recently, however, fixed gauge approaches have also been intensively pursued as they offer a bridge between QCD and the more traditional (nonrelativistic) many-body problems in nuclear and condensed-matter physics. The disadvantage of a fixed gauge approach is, however, that it is not manifestly Lorentz invariant and thus there are fewer restrictions on the dynamical operators resulting in general in complicated Hamiltonians. Fortunately experience gained form studies of other many-body systems can be helpful in identifying approximation schemes relevant for studies of particular aspects of the dynamics. Attempts to enlighten the nonperturbative structure of QCD using many-body techniques have recently been undertaken by several authors [1,2] and even the confinement scenario has been realized within the Coulomb gauge [2].

Though much progress has been made, schematic models are still very useful to shed some light onto the nonperturbative structure of QCD. For example, in Ref. [3] the gluon sector was investigated restricting the quark-gluon dynamics to an effective Hamiltonian for a fixed number of modes. The gluon spectrum was adjusted to reproduce lattice gauge calculations [4,5] and several other states have been predicted and confirmed by lattice gauge calculations. In Ref. [6] a Lipkin type model was introduced where the fermion sector consisted of two levels, one at positive and the other at negative energy, and a coupling to a boson level, occupied by color-spin zero gluon pairs, was considered. Only meson states were described. In Ref. [7] the model was extended to include baryons. The nucleon resonances (especially the Roper resonance) and $\Delta$ resonances were well described. One drawback of these models, however is that they are purely phenomenological and contain several parameters. Our long term goal is to investigate classes of schematic models which are *derived* from QCD. For example, one expects the high energy quark and gluon modes be largely irrelevant in determining the structure of the vacuum, and lowest excitations, e.g., the pion or the $\rho$ meson states, color confinement, etc. One advantage of such schematic models is that *they are QCD* and will depend on only one or few well determined coupling constant(s). Preferably, such models should allow analytic or alternatively nearly analytic solutions, the latter requiring at most a numerical diagonalization.

Since it is a good practice to start from the simple cases first, here we present exact solutions to a schematic model, which has all the most drastic approximations. The goal is to identify which of these play what role in the low energy dynamics. Hopefully by systematically relaxing these approximations an intuitive picture of QCD will emerge. The model presented here is derived from QCD, under the restriction to SU(2) in color and flavor. Only quarks and antiquarks will be considered. The interaction via gluons will be simulated via an effective, interaction reduced to the lowest momentum modes. These modes will correspond to the quarks and antiquarks restricted to be in a single spacial orbital level ($S$-state). Apart from this, the model will depend only on one parameter related to the energy of the color excitations thus ultimately irrelevant.

The paper is organized as follows. In Sec. II the derivation of the schematic model, starting from QCD, is given and the appropriate particle content is identified. In Sec. III the eigenvalues and the basis states will be constructed and the physical states are investigated in Sec. IV. In Sec. V conclusions will be drawn and future developments discussed.

## II. DERIVATION OF THE MODEL HAMILTONIAN

As discussed above, by choosing an appropriate gauge a set of degrees of freedom can be selected which appears most natural for description of certain features of QCD. In this case physical (gauge independent) quantities may be simpler to calculate when the "correct" gauge is chosen. For example, to compute various deep inelastic amplitudes it is advantageous to formulate QCD in the light-cone gauge, while to compute low energy spectra Coulomb gauge seems to be the natural choice. The Coulomb gauge has been extensively studied in

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Refs. [2,8]. The Gauss's law can be used to eliminate the longitudinal component of the electric field which leaves only the transverse gluons representing generalized coordinates and their conjugated momenta (given by the transverse electric fields). Schematically the Coulomb gauge Hamiltonian has the following structure [2,8]:

$$H = K_q + K_g + V_{qqg} + V_{g^3} + V_{g^4} + V_C. \quad (1)$$

Here $K_q$ and $K_g$ are the kinetic energies of the quarks, antiquarks, and gluons, respectively, and are given by the Dirac and Yang-Mills Hamiltonians. The next three terms have polynomial dependence on the canonical degrees of freedom and represent the local (anti)quark-gluon interaction, triple- and quartic-gluon coupling, respectively. Finally $V_C$ is the non-abelian generalization of the Coulomb potential. In an abelian case, $V_C = \alpha \int d\mathbf{x}d\mathbf{y}\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^{-1}\rho(\mathbf{y})$ represents the Coulomb energy between matter charges, which are described by the charge density $\rho(\mathbf{x})$. For simplicity we have already dropped the Faddeev-Popov determinant, which as shown in Ref. [9] can be accounted for by redefining the gluon wave functional.

In a non-abelian theory like QCD the Coulomb potential depends not only on the relative separation between charges but also on the distribution of the gauge fields around them,

$$V_C = -g^2 \int d\mathbf{x}d\mathbf{y}\rho^a(\mathbf{x}) \left[\frac{1}{1-\lambda^\dagger}\frac{1}{\nabla^2}\frac{1}{1-\lambda}\right]_{a\mathbf{x};b\mathbf{y}} \rho^b(\mathbf{y}). \quad (2)$$

The matrix elements of $1-\lambda$ are given by

$$[1-\lambda]_{\mathbf{x},a;\mathbf{y},b} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) - g f_{acb}\nabla_y \frac{\mathbf{A}^c(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \quad (3)$$

and the color-charge density is given by $\rho^a(x) = \psi^\dagger(\mathbf{x})T^a\psi(\mathbf{x}) - f_{abc}\mathbf{A}^b(\mathbf{x})\cdot\mathbf{E}^c(\mathbf{x})$. $\mathbf{A}^a, -\mathbf{E}^a$ represent $a = 1,\ldots N_c^2-1$ transverse gluon coordinates and conjugate momenta, respectively, and $T^a = T^a_{ij}, i,j = 1\ldots N_C$, and $f_{abc}$ are the generators of the fundamental and adjoint representations of the color SU($N_c$) group. Thus, unlike QED, in QCD to define the potential between a state containing matter (quark, antiquark) sources it is necessary to know the gluon wave functional of the state. It was shown in Ref. [10] using a variational ansatz for the gluon wave functional of the vacuum that $V_C$ leads to a confining interaction between matter sources. The Coulomb potential has also been computed on the lattice [11] and found to be confining at large distances. The corresponding string tension was found to be larger then the string tension of the static quark-antiquark potential computed from the Wilson loop. This is to be expected since the quark-antiquark source polarizes the vacuum and the gluon distribution in a QCD eigenstate containing quark-antiquark sources is different from that of the vacuum [12]. An attractive interaction between quark-antiquark pairs destabilizes the vacuum and leads to formation of quark-antiquark condensates and chiral symmetry breaking [13]. The underlying mechanism is analogous to BCS superconductivity.

In the following we want to investigate the minimal requirements of a schematic model which yields the pattern of chiral symmetry breaking consistent with that expected in QCD. It should be noted, however that spontaneous symmetry breaking formally happens only for systems with an infinite number of degrees of freedom. Thus a schematic model with a finite number of degrees of freedom is by definition in odds with the expected pattern of symmetry breaking. A way out is to consider a Hamiltonian with explicit symmetry breaking terms,

$$H = H_0 + m_0 H' \quad (4)$$

with the symmetric limit corresponding to $m_0 \to 0$. If $|n\rangle$ label degenerate vacuua of $H_0$ with $n$ representing transformation properties under the symmetry group, then in the continuum limit it is expected that off-diagonal matrix elements $\langle n'|H|n\rangle$ are much smaller (exponentially suppressed with the volume of the system) compared to the diagonal matrix elements. Thus even a very small perturbation $H'$ will put the system in one of the $|n\rangle$ states, rather then in a linear combination of these. In our analysis of schematic models we will thus choose the ground state to correspond to that of $H'$ and then take the limit $m_0 \to 0$.

Since the necessary condition for the condensate is existence of an attractive interaction, in the schematic model considered here, we remove the gluon degrees of freedom (for example by fixing the gluon wave functional), and approximate the Coulomb kernel by its expectation value as in Refs. [10,11]. The system is then put in a finite volume, $\mathcal{V}$ with periodic boundary condition to mimic translational invariance, and the fermion fields are expanded in terms of the momentum normal modes. In the minimal model considered here only the zero-modes will be retained. In this case the interaction is given by the spacial average of the Coulomb potential,

$$V_C \to \frac{1}{\mathcal{V}}\int_{\mathcal{V}} d\mathbf{x} V_C(\mathbf{x}) \equiv \frac{g}{\mathcal{V}}. \quad (5)$$

While the low momentum modes are expected to be relevant for the nonperturbative aspects of hadron spectrum, restriction to zero-modes is certainly a severe truncation of the Fock space. However, by considering more momentum modes it is in principle possible to systematically approach the underlying Hamiltonian matrix. The main reason for studying the zero-modes is that the corresponding Hamiltonian can be diagonalized analytically and, as will be discussed later, already reproduces some aspects of low energy QCD dynamics.

Still in position space, Eq. (5) corresponds to a contact interaction between quark charge densities,

$$H = \int d\mathbf{x}\psi^\dagger(\mathbf{x})[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m_0]\psi(\mathbf{x}) + g\int d\mathbf{x}\rho^a(\mathbf{x})\rho^a(\mathbf{x}), \quad (6)$$

with the color charge density originating from quarks only, $\rho^a(\mathbf{x}) = \psi^\dagger(\mathbf{x})T^a\psi(\mathbf{x})$, and the coupling $g$ which has mass dimension $-2$ will be determined later. The quark fields $\psi(\mathbf{x})$ represent $N_C \times N_f$ degrees of freedom. The generators of the flavor axial rotations are

$$Q_5^\alpha = \int d\mathbf{x}\psi^\dagger(\mathbf{x})\gamma_5 T^\alpha\psi(\mathbf{x}), \quad (7)$$

with $T^\alpha$ being the generators of flavor, SU($N_f$). In the limit of vanishing quark mass, $m_0 = 0$, the Hamiltonian is invariant under flavor-axial rotations,

$$\lim_{m_0=0}\left[Q_5^\alpha, H\right] = 0, \tag{8}$$

while for a finite bare mass

$$\left[Q_5^\alpha, H\right] = -2m_0 P_5^\alpha, \tag{9}$$

with

$$P_5^\alpha = \int d\mathbf{x}\psi^\dagger(\mathbf{x})\gamma^0\gamma_5 T^\alpha\psi(\mathbf{x}). \tag{10}$$

To obtain the particle content of the spectrum of this Hamiltonian we first rewrite it in a basis of massive quarks and anti-quarks defined by the operators $b(cf\lambda\mathbf{k})$ and $d(cf\lambda\mathbf{k})$, respectively with $c, f, \lambda, \mathbf{k}$ referring to color, flavor, spin component and momentum, and related to the fields in the standard way

$$\psi(\mathbf{x}) = \sum_{cf\lambda=\pm1/2}\int \frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{x}\cdot\mathbf{k}}[u(\lambda,\mathbf{k})b(cf\lambda\mathbf{k}) + v(\lambda,-\mathbf{k})d^\dagger(cf\lambda-\mathbf{k})]. \tag{11}$$

Here $u$ and $v$ are the eigenstates of the free Dirac Hamiltonian describing a fermion of mass $m$, which is not yet specified but is anticipated to be the constituent quark mass. In terms of these quark operators the Hamiltonian is given by

$$H = H_q + H_{q\bar{q}} + V. \tag{12}$$

Here $H_q$ contains operators proportional to $b^\dagger b$ and $d^\dagger d$, $H_{q\bar{q}}$ contains pair creation and annihilation operators proportional to $b^\dagger d^\dagger$ and $db$, and $v$ contains normal-ordered four-fermion operators. Since we are interested in studying the low energy phenomena we make the following simplification. First we confine quarks to a finite box of volume $\mathcal{V}$. The momentum states become discrete, with $\mathbf{k}\to\mathbf{n}$ and $\mathbf{k} = 2\pi\mathbf{n}/\mathcal{V}^{1/3}$, so that

$$\int\frac{d\mathbf{k}}{(2\pi)^3} \to \frac{1}{\mathcal{V}}\sum_{\mathbf{n}}. \tag{13}$$

In the finite volume it is also useful to rescale the particle operators,

$$b(cf\lambda\mathbf{k}) \to \tilde{b}(cf\lambda\mathbf{n}), \quad b(cf\lambda\mathbf{k}) = \mathcal{V}^{1/2}\tilde{b}(cf\lambda\mathbf{k}), \tag{14}$$

and the same of the antiquark operator $d$. The new operator are dimensionless and satisfy

$$\{\tilde{b}(cf\lambda\mathbf{n}), \tilde{b}^\dagger(c'f'\lambda'\mathbf{n}')\} = \delta_{cc'}\delta_{ff'}\delta_{\lambda\lambda'}\delta_{\mathbf{n}\mathbf{n}'}. \tag{15}$$

In the following we will rename $\tilde{b}, \tilde{d}$ back as $b$ and $d$, respectively. The final approximation is to retain only the lowest momentum states, e.g., $\mathbf{n} = 0$. Thus from now on we drop the momentum index on the quark operators. The next level of approximations would include the $P$- and higher waves. Within this approximation the Hamiltonian becomes

$$\begin{aligned} H = &\sum_1(\mathcal{E}+m_0)b_1^\dagger b_1 + \sum_1(\mathcal{E}+m_0)d_1^\dagger d_1 \\ &- \sum_{1234}V_{qq}(1234)b_1^\dagger b_2^\dagger b_3 b_4 - \sum_{1234}V_{\bar{q}\bar{q}}(1234)d_1^\dagger d_2^\dagger d_3 d_4 \\ &- 2\sum_{1234}V_{q\bar{q}}(1234)b_1^\dagger d_2^\dagger d_3 b_4 \end{aligned} \tag{16}$$

with

$$\mathcal{E} = \frac{gC_F}{\mathcal{V}}\sum_{\mathbf{n}}^{\mathbf{n}_{\max}}\delta_{\mathbf{n}0} = \frac{gC_F}{\mathcal{V}}, \tag{17}$$

and

$$\begin{aligned} V_{qq}(1234) &= \frac{g}{\mathcal{V}}T^a_{c_1c_3}T^a_{c_2c_4}\left[\delta_{f_1f_3}\delta_{\lambda_1\lambda_3}\right]\left[\delta_{f_2f_4}\delta_{\lambda_2\lambda_4}\right] \\ V_{\bar{q}\bar{q}}(1234) &= \frac{g}{\mathcal{V}}T^a_{c_3c_1}T^a_{c_4c_2}\left[\delta_{f_1f_3}\delta_{\lambda_1\lambda_3}\right]\left[\delta_{f_2f_4}\delta_{\lambda_2\lambda_4}\right] \\ W_{q\bar{q}}(1234) &= \frac{g}{\mathcal{V}}T^a_{c_1c_4}T^a_{c_3c_2}\left[\delta_{f_1f_4}\delta_{\lambda_1\lambda_4}\right]\left[\delta_{f_2f_3}\delta_{\lambda_2\lambda_3}\right]. \end{aligned} \tag{18}$$

Here $1 = (f_1, c_1, \lambda_1)$, etc., denote all remaining (discrete) quantum numbers of the particle labeled by 1; $c_1$ denotes color, $f_1$ flavor and $\lambda_1$ spin projection. It is worth noting at this point that with $S$-orbitals only the pair creation part of the Hamiltonian vanishes. Scalar quark-antiquark pairs have quarks in relative spin-one coupled to one unit of orbital angular momentum which vanishes for $S$-waves. Within these approximations the flavor axial charge generators become

$$Q_5^\alpha = \sum_{12}\left(b_1^\dagger Q_{12}^\alpha d_2^\dagger + d_1 Q_{12}^\alpha b_2\right), \tag{19}$$

with

$$Q_{12}^\alpha = T^\alpha_{f_1f_2}\delta_{c_1c_2}\delta_{\lambda_1\lambda_2}, \tag{20}$$

and the pseudo-scalar charges $P_5^\alpha$ become

$$P_5^\alpha = \sum_{12}\left(b_1^\dagger Q_{12}^\alpha d_2^\dagger - d_1 Q_{12}^\alpha b_2\right). \tag{21}$$

We also note that

$$\left[P_5^a, H\right] = -2m_0 Q_5^a, \tag{22}$$

is still satisfied. For completeness, the vector flavor charges $V^\alpha$,

$$V^\alpha = \int d\mathbf{x}\psi^\dagger(\mathbf{x})T^\alpha\psi(\mathbf{x}), \tag{23}$$

become

$$V^\alpha = \sum_{12}\left(b_1^\dagger V_{12}^\alpha b_2 - d_1^\dagger V_{12}^\alpha d_2\right), \tag{24}$$

with

$$V_{12}^\alpha = T^\alpha_{f_1f_2}\delta_{c_1c_2}\delta_{\lambda_1\lambda_2}. \tag{25}$$

The Hamiltonian contains four parts. A noninteraction part, quark-quark, and antiquark-antiquark potentials and a quark-antiquark potential. We recall some basic properties of the particle operators. In the following we concentrate on the case of two colors and two flavors. Generalization to arbitrary $N_C$ and $N_f$ is straightforward. The creation and annihilation operators carry color, $c$, flavor, $f$ and spin, $\lambda$ indices and these all range from $-\frac{1}{2}$ to $+\frac{1}{2}$. We distinguish now between co- and contravariant indices in order to denote the different transformation properties of the fermion creation and annihilation operators. We denote the creation and annihilation operators for quarks by $b_\alpha^\dagger$ and $b^\alpha$, respectively, where $\alpha$ is a shorthand notation for $(cf\lambda)$. Subsequently representation of SU(2)-color, flavor, spin will be similarly denoted by three numbers $(S_c S_f S)$, where $S_c$ is the color angular momentum and similar for flavor, $S_f$, and spin, $S$. Similarly for the antiquark operators we have $d^{\dagger\alpha}$ for the creation and $d_\alpha$ for the annihilation operators. The anticommutation relations are now given by $\{b^\beta, b^\dagger_\alpha\} = \{d_\beta, d^{\dagger\alpha}\} = \delta^\alpha_\beta$. The indices are lowered according to the following convention. If $a^\alpha$ denotes any of the four operators ($b^\dagger$, $b$, $d$, or $d^\dagger$) with an upper index, lowering this index corresponds to

$$a^{cf\lambda} = (-1)^{\frac{1}{2}-c}(-1)^{\frac{1}{2}-f}(-1)^{\frac{1}{2}-\lambda}a_{-c-f-\lambda}. \tag{26}$$

We can now rewrite the Hamiltonian. The noninteracting part is trivial and given by

$$H_q = (\mathcal{E} + m_0)(\hat{n}_q + \hat{n}_{\bar{q}}) \tag{27}$$

with $\hat{n}_q = b^\dagger_\alpha b^\alpha$ and $\hat{n}_{\bar{q}} = d^{\dagger\alpha} d_\alpha$ being the quark and antiquark number operators, respectively. The quark-quark interaction is given by

$$V_{qq} = -\sum_{c's f's \lambda's} \frac{g}{\mathcal{V}} T^a_{c_1c_3} T^a_{c_2c_4} [\delta_{f_1f_3}][\delta_{f_2f_4}] \times b^\dagger_{c_1f_1\lambda_1} b^\dagger_{c_2f_2\lambda_2} b^{c_3f_3\lambda_3} b^{c_4f_4\lambda_4}. \tag{28}$$

Using

$$T^a_{c_1c_3} T^a_{c_2c_4} = \frac{1}{2}\left(\delta_{c_1c_4}\delta_{c_3c_2} - \frac{1}{2}\delta_{c_1c_3}\delta_{c_2c_4}\right) \tag{29}$$

and joining operators with common indices through the anticommutation relations, we obtain in an intermediate step

$$V_{qq} = -\frac{3g}{4\mathcal{V}}\hat{n}_q - \frac{g}{4\mathcal{V}}\hat{n}_q^2 + \frac{g}{2\mathcal{V}}\sum_{c_1c_2}\left(\sum_{f_1\lambda_1} b^\dagger_{c_1f_1\lambda_1} b^{c_2f_1\lambda_1}\right)\left(\sum_{f_2\lambda_2} b^\dagger_{c_2f_2\lambda_2} b^{c_1f_2\lambda_2}\right). \tag{30}$$

Finally using Eq. (26) and coupling to definite color, flavor and spin, we arrive at

$$V_{qq} = -\frac{3}{4}\frac{g}{\mathcal{V}}\hat{n}_q - \frac{2g}{\mathcal{V}}\sqrt{3}[[b^\dagger \otimes b]^{100} \otimes [b^\dagger \otimes b]^{100}]^{000}_{000}, \tag{31}$$

where $[A^{\Gamma_1} \otimes B^{\Gamma_2}]^\Gamma_\mu$ with $\Gamma = S_c S_f S$ and $\mu = cf\lambda$ denotes the coupling of $A$ and $B$ in color, flavor and spin,

$$[A^{\Gamma_1} \otimes B^{\Gamma_2}]^\Gamma_\mu = \sum_{\mu_1\mu_2}\langle\Gamma_1\mu_1, \Gamma_2\mu_2|\Gamma\mu\rangle A^{\Gamma_1}_{\mu_1} B^{\Gamma_2}_{\mu_2}, \tag{32}$$

and $\langle\Gamma_1\mu_1, \Gamma_2\mu_2|\Gamma\mu\rangle$ is the product of three Clebsch-Gordan coefficients in color, flavor, and spin.

Note, that the quadratic dependence on the quark number operator is canceled and only the linear dependence remains. The last term in Eq. (31) represents the color angular momentum squared, whose components in spherical basis are given by

$$S^c_{q,m} = \sqrt{2}[b^\dagger \otimes b]^{100}_{m00}$$
$$S^c_{\bar{q},m} = -\sqrt{2}[d^\dagger \otimes d]^{100}_{m00}, \tag{33}$$

for the quark and antiquark part, respectively. With this, the final form of $V_{qq}$ is

$$V_{qq} = -\frac{3}{4}\frac{g}{\mathcal{V}}\hat{n}_q + \frac{g}{\mathcal{V}}(S^c_q \cdot S^c_q), \tag{34}$$

and we used $[S^c_q \otimes S^c_q] = -(\mathbf{S}^c_q \cdot \mathbf{S}^c_q)$. In a complete analogy one can show that the antiquark-antiquark part is found to be

$$V_{\bar{q}\bar{q}} = -\frac{3}{4}\frac{g}{\mathcal{V}}\hat{n}_{\bar{q}} + \frac{g}{\mathcal{V}}(\mathbf{S}^c_{\bar{q}} \cdot \mathbf{S}^c_{\bar{q}}). \tag{35}$$

And finally for the quark-antiquark interaction is given by

$$V_{q\bar{q}} = \frac{2g}{\mathcal{V}}(\mathbf{S}^c_q \cdot \mathbf{S}^c_{\bar{q}}). \tag{36}$$

Summing all terms leaves us with a surprisingly simple Hamiltonian whose interactions are easily identified,

$$H = \left(\mathcal{E} + m_0 - \frac{3}{4}\frac{g}{\mathcal{V}}\right)(\hat{n}_q + \hat{n}_{\bar{q}}) + \frac{g}{\mathcal{V}}\mathbf{S}^2_c, \tag{37}$$

where $\mathbf{S}^2_c = (\mathbf{S}^c_q + \mathbf{S}^c_{\bar{q}})^2$ is the total color angular momentum squared.

## III. THE SPECTRUM

The basis used to diagonalize $H$ is determined by the number of degrees of freedom each quark (antiquark) carries. There are eight degrees of freedom: two spin times two flavor and times two color components. The Fock space is thus finite and contains maximally eight quarks and eight antiquarks. The group structure for each sector is given by [14]

$$\begin{array}{ccc} U(8) \supset & U_c(2) \otimes & U_{fS}(4) \\ [1^{n_q}] & [h_1h_2] & [2^{h_2}1^{h_1-h_2}], \\ U_{fS}(4) \supset & U_f(2) \otimes & U_S(2) \\ [2^{h_2}1^{h_1-h_2}] & S_f & S. \end{array} \tag{38}$$

The notation $[p_1p_2\ldots p_n]$ refers to the Young diagrams [14], which describes the symmetry under permutation of a given irreducible representation (irrep) of a unitary group. In Eq. (38) we have $h_1 + h_2 = n_q$ and the reduction of the flavor-spin group $U_{fS}(4)$ is given in Ref. [14]. In Table I we give a list of the color-flavor-spin content as a result of Eq. (38).

We now consider meson-like excitations, i.e., the Fock sector with equal number of quarks and antiquarks. For this case, the basis can be labeled by the following set of quantum numbers,

$$|n_{\bar{q}} = n_q; (S^c_{\bar{q}}, S^c_q)S^c m^c; (S^f_{\bar{q}}, S^f_q)S^f m^f; (S_{\bar{q}}, S_q)Sm\rangle, \tag{39}$$

where $m^c, m^f$ and $m$ refer to the magnetic color, flavor, and spin projection. The eigenvalue of the Hamiltonian with respect to such states is given by

$$E = \left(\mathcal{E} + m_0 - \frac{3}{4}\frac{g}{\mathcal{V}}\right)(n_q + n_{\bar{q}}) + \frac{g}{\mathcal{V}}S_c(S_c + 1). \tag{40}$$

For physical states with no net color only the first term contributes, and using Eq. (17) we find $E = m_0$ and the spectrum is degenerate with respect to flavor and spin. Color excitations are separated by a finite energy gap which is an artifact of the contact approximation for the quark interactions. In full QCD the splitting is expected to be infinite as the potential between the quarks grows with the relative separation. Nevertheless, one can investigate the structure of colored excitations in the model, which might play a role in models like the quark-gluon glass condensate [15], important at high densities.

The energy solutions are simple and degenerate for all states with the same color. At a first glance one might think that the physical states should be a certain sum over all basis states [Eq. (39)] with the same color. As a consequence, in our schematic model one would look for adequate superposition of the degenerate states in order to construct, e.g., the physical vacuum state. One criterion used can be to reproduce the quark condensate (see next section). However, as we will show further below, arguments of coninuity require that the lowest state has to be the vacuum state $|0\rangle$. To get more physical solutions, it will be necessary to introduce an interaction which lifts the large degeneracy of the Hamiltonian.

## IV. PHYSICAL STATES AND THE CHIRAL LIMIT

In the chiral limit, $m_0 = 0$ all color singlet states have zero energy. The vacuum state should be identified as a state with all scalar quantum numbers. Since the single quark-antiquark pair in the $S$-wave has pseudoscalar quantum numbers, the vacuum will be given by a superposition of states with an even number of quark-antiquark pairs with total color, flavor and spin zero. The most general (unnormalized) vacuum state can be schematically written as

$$|\mathbf{z}\rangle = |0\rangle + \sum_{n=1}^{4} z_n (b^\dagger b^\dagger d^\dagger d^\dagger)^n |0\rangle. \tag{41}$$

Since in the chiral limit all $J^{PC} = 0^{++}$ states are degenerate in this model we cannot distinguish between the true vacuum and, for example the $\sigma$ meson. Thus we take for the vacuum a state given by the sum of the perturbative vacuum $|0\rangle$ and the state with the lowest number (two) of the quark-antiquark pairs coupled to definite color, flavor and spin. Each pair can be written in the following equivalent form:

$$|S_c S_f S\rangle = \frac{1}{\sqrt{2}} [[b^\dagger \otimes b^\dagger]^{S_c S_f S} \otimes [d^\dagger \otimes d^\dagger]^{S_c S_f S}]^{000}_{000} |0\rangle. \tag{42}$$

Because the two-quark state has to be antisymmetric (the same for the antiquarks) the only allowed color, flavor, and spin values are $(S_c S_f S) = (000)$, $(110)$, $(101)$, and $(011)$. The coupling of first two quarks and then two antiquarks to a total color, flavor, and spin zero can be reexpressed easily in terms of the coupling of two quark-antiquark pairs as follows:

$$[[b^\dagger \otimes b^\dagger]^{S_c S_f S} \otimes [d^\dagger \otimes d^\dagger]^{S_c S_f S}]^{000}_{000} = -\sum_{S_c' S_f' S'} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S_c \\ \frac{1}{2} & \frac{1}{2} & S_c \\ S_c' & S_c' & 0 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S_f \\ \frac{1}{2} & \frac{1}{2} & S_f \\ S_f' & S_f' & 0 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & S \\ S' & S' & 0 \end{Bmatrix} \times [[b^\dagger \otimes d^\dagger]^{S_c' S_f' S'} \otimes [b^\dagger \otimes d^\dagger]^{S_c' S_f' S'}]^{000}_{000}, \tag{43}$$

where the symbols $\{\ldots\}$ refer to the usual 9-$j$ symbols [16]. For the vacuum we thus take the normalized state in the form

$$|z_0 z_1\rangle = \frac{1}{\sqrt{1+2\rho^2}} \left( |0\rangle + \sum_{S_c} z_{S_c} \sum_{S_f S} |S_c S_f S\rangle \right), \tag{44}$$

with $|S_c S_f S\rangle$ given in Eq. (42). Here we assumed that due to the degeneracy of the states with the same color, there is no dependence of the trial state parameters $z_{S_c}$, $S_c = 0, 1$ on flavor and spin. In general the $z$-values are complex and can be written as $z_{S_c} = \rho_{S_c} e^{i\phi_{S_c}}$, with $\rho_0 = \rho\cos(\phi)$ and $\rho_1 = \rho\sin(\phi)$, and $\rho = |z_0|^2 + |z_1|^2$ being the total radius. In such a vacuum expectation values of $\hat{n}_q$ and $\hat{n}_{\bar{q}}$, which determine the quark condensate, are given by

$$\langle z_0 z_1 | \hat{n}_q | z_0 z_1\rangle = \langle z_0 z_1 | \hat{n}_{\bar{q}} | z_0 z_1\rangle = \frac{4\rho^2}{1+2\rho^2}. \tag{45}$$

In the limit $z_{S_c} = 0$, $\hat{n}_q = \hat{n}_{\bar{q}} = 0$ as expected for the perturbative vacuum. For large values of $\rho$, each expectation value approaches 2, as it has to be, because then the main contribution comes from the two quark-antiquark pairs. Using Eq. (45) it is possible to define the collective potential as the expectation value of the Hamiltonian, the result is

$$V(z_0, z_1) = \langle z_0, z_1 | H | z_0, z_1\rangle = \left( \mathcal{E} - \frac{3}{4}\frac{g}{\mathcal{V}} \right) \frac{8\rho^2}{1+2\rho^2}, \tag{46}$$

which corresponds near $\rho = 0$ to a harmonic oscillator and the potential saturates for $\rho \to \infty$ at $4(\mathcal{E} - \frac{3}{4}\frac{g}{\mathcal{V}})$. The use of such trial states played primordial role in nuclear physics to help understand the structure of a complicated many body problem [18] and might be here also of great value when a more sofisticated Hamiltonian is used. Because, as we showed above, the factor which contains $\mathcal{E}$ is zero one obtains a flat potential which reflects the complete degeneracy of color zero states. As already mentioned, the $z$ parameters are complex, but the expectation value above depends only on the total radius $\rho$. This implies that equipotential lines flow along constant $\rho$ with arbitrary angles $\phi_{S_c}$ and $\sqrt{\rho_0^2 + \rho_1^2} = \rho$.

To further determine parameters of the vacuum one can consider the quark condensate $\langle\bar{q}q\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{\psi}(0)\psi(0)\rangle/2 \sim -(225\text{MeV})^3 \sim -1\,\text{fm}^{-3}$ [17],

$$\langle\bar{q}q\rangle = -\frac{1}{\mathcal{V}} \left[ N_C N_S - \frac{1}{2}(\langle\hat{n}_q\rangle + \langle\hat{n}_{\bar{q}}\rangle) \right] = -\frac{4}{\mathcal{V}} \frac{1+\rho^2}{1+2\rho^2}. \tag{47}$$

It should be noticed that the state $|0\rangle$ is not the zero-particle state of perturbation theory, ($g = 0$) of mass-less fermions (or in general with mass $m_0$). Instead, the fermions have effective, constituent masses, given by $\mathcal{E}$ [cf. Eq. (16)]. It is therefore not surprising that there is a non-vanishing quark condensate in the state $|0\rangle$. One might be tempted to use this to determine $\rho$ for given volume (e.g., taking as a volume of sphere of radius of 0.8 fm would yield $\rho = 0.67$). This is however not correct since there are further constraints from the spontaneous realization of chiral symmetry breaking. As expected (see discussion in Sec. II) with a finite number of degrees of freedom, in the chiral limit, the ground state is degenerate, and in general given by a coherent sum of states as in Eq. (41). In the continuum limit some of these will move to the continuum part of the spectrum, and would be identified with physical states (with scalar quantum numbers); other will stay in the discrete part of the spectrum and will represent equivalent vacuua. The Hamiltonian in these vacuua is expected to be diagonal (in the infinite volume limit) thus a small symmetry breaking term will break the degeneracy and select a single vacuum state. Thus an alternative way to identify the physical ground state, which is in accord with expectations from the continuum limit, is to study the spectrum for $m_0 \neq 0$ and define the ground state as the lowest energy state when $m_0 \to 0$. Away from the chiral limit, $m_0 \neq 0$ each additional quark-antiquark pair raises the energy by $2m_0$. Thus, for $m_0 \neq 0$ the vacuum has to be given by the single state $|0\rangle$, so $\rho = 0$. The quark condensate is then entirely determined by the volume and the total number of degrees of freedom,

$$\mathcal{V} = -N_C N_S \langle \bar{q} q\rangle^{-1} = 2.7\,\text{fm}^3. \qquad (48)$$

As expected for spontaneous breaking the generators of chiral symmetry, Eq. (19), which can be also written as

$$Q_f^5 = \frac{\sqrt{N_C N_f N_S}}{2}\left([b^\dagger \otimes d^\dagger]^{010}_{0f0} + [d \otimes b]^{010}_{0f0}\right), \qquad (49)$$

do not annihilate the vacuum, instead they mix the vacuum with the single pion state,

$$\langle \pi, f'|Q_f^5|0\rangle = \delta_{f'f} f_\pi m_\pi \mathcal{V} \qquad (50)$$

with $f_\pi = 93$ MeV being the pion decay constant. Chiral symmetry, and relativistic normalization of single particle states,

$$\langle \pi, f'|\pi f\rangle = 2m_\pi \mathcal{V}, \qquad (51)$$

implies that in the chiral limit $m_0 \to 0$, $m_\pi = O(m_0^2)$ and $f_\pi = O(1)$. Since pion has $J^{PC} = 0^{-+}$ quantum numbers and is generated by the axial rotation from the vacuum the most general (unnormalized) pion state is given by

$$|\pi\rangle \sim b^\dagger d^\dagger \left[|0\rangle + \sum_{n=1,4} w_n (b^\dagger b^\dagger d^\dagger d^\dagger)^n |0\rangle\right]. \qquad (52)$$

Mixing with the vacuum through the axial charge, as given by Eq. (50), constraints the quark-antiquark component to

$$|\pi, f\rangle = f_\pi m_\pi \mathcal{V} \frac{2}{\sqrt{N_C N_f N_S}} \times [b^\dagger \otimes d^\dagger]^{010}_{0f0}\left(|0\rangle + \sum_{n=1}^{4} w_n (b^\dagger b^\dagger d^\dagger d^\dagger)^n |0\rangle\right). \qquad (53)$$

However, all states in the expansion in Eq. (53) are eigenstates of the Hamiltonian with increasing eigenvalues and physical states cannot be given such a linear combination. We thus conclude that the single pion state should be identified with the valence component alone,

$$|\pi, f\rangle = f_\pi m_\pi \mathcal{V} \frac{2}{\sqrt{N_C N_f N_S}} [b^\dagger \otimes d^\dagger]^{010}_{0f0}|0\rangle. \qquad (54)$$

TABLE I. Color-flavor-spin content as a function on the number $n_q$ of quarks. The list is equivalent for the antiquarks.

| $n_q$ | $[h_1 h_2]$ | $S_c$ | $\sum (S_f, S)$ |
|---|---|---|---|
| 0 | [0] | 0 | (0,0) |
| 1 | [1] | $\frac{1}{2}$ | $(\frac{1}{2}, \frac{1}{2})$ |
| 2 | [2] | 0 | (0,0) + (1,1) |
| 2 | $[1^2]$ | 1 | (1,0) + (0,1) |
| 3 | [21] | $\frac{1}{2}$ | $(\frac{1}{2}, \frac{1}{2}) + (\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ |
| 3 | $[1^3]$ | $\frac{3}{2}$ | $(\frac{1}{2}, \frac{1}{2})$ |
| 4 | $[2^2]$ | 0 | (0,0) + (1,1) + (2,0) + (0,2) |
| 4 | $[21^2]$ | 1 | (1,0) + (0,1) + (1,1) |
| 4 | $[1^4]$ | 2 | (0,0) |
| 5 | $[2^2 1]$ | $\frac{1}{2}$ | $(\frac{1}{2}, \frac{1}{2}) + (\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ |
| 5 | $[21^3]$ | $\frac{3}{2}$ | $(\frac{1}{2}, \frac{1}{2})$ |
| 6 | $[2^3]$ | 0 | (0,0) + (1,1) |
| 6 | $[2^2 1^2]$ | 1 | (1,0) + (0,1) |
| 7 | $[2^3 1]$ | $\frac{1}{2}$ | $(\frac{1}{2}, \frac{1}{2})$ |
| 8 | $[2^4]$ | 0 | (0,0) |

With the pion mass related to the bare quark mass by $m_\pi = 2m_0$. The normalization condition of Eq. (51) then leads to

$$f_\pi = \sqrt{\frac{N_C N_f N_S}{2m_\pi \mathcal{V}}} = \sqrt{-\frac{N_f \langle \bar{q} q\rangle}{2m_\pi}} = 200\sqrt{N_f}\ \text{MeV}.$$

The identification of other physical states with the spectrum given in Table I is now straightforward. Since the number of quarks and antiquarks are well defined and each additional (anti)quark raises energy by $m_0$ the spectrum of single meson and baryon states would correspond to stated with a single $q\bar{q}$ pair and three quarks, respectively. States with other numbers of quarks or antiquark should be identified with multiparticle states e.g., $qqqq\bar{q}$ with a meson-baryon state. Colored states are split from the physical color singlet states by $gS_c(S_c+1)/\mathcal{V}$ where $S_c$ is a half-integer or integer total color for an odd or even number of quarks and antiquarks in the state, respectively, and $g$ is the effective strength of the colored interactions. We thus see that it is now possible to take the limit $g \to \infty$ which is expected for the zero-mode component of a confining interactions without affecting the physical spectrum.

## V. SUMMARY

Models play an important role in understanding complicated dynamical structures. Our goal here was not to build the most sophisticated model of low energy QCD, but on the contrary to identify the most basic starting point for such an endeavor. Starting from the underlying QCD interactions in the Coulomb gauge we have defined an approximations scheme which gave us a model for the interactions of the quark zero modes. The model is exactly solvable and physical states can be identified with using the symmetry patterns observed in the physical spectrum. In particular spontaneous breaking of chiral symmetry enables to identify the vacuum state and the single pion state and conservation of the particle number by our model Hamiltonian then leads to mapping between the representations of the underlying $U(N_C \times N_f \times N_S)$ symmetry and the physical states. We worked with the $N_C = 2$ number of colors, but extension to $N_C = 3$ is straightforward since the coupling and recoupling methods in SU(2) can be readily extended to SU(3) (see, for example the appendix of Ref. [20]). For the basis, instead of $U(8)$ we would start from $U(12)$ if flavor is still SU(2) or $U(18)$ if flavor is also SU(3). The reductions are known (see Refs. [6,7,21]). In the model we find splitting between physical states to be proportional to the total bare mass of the quarks and anti-quarks independently of the strength of the color or confining interaction. The quark basis itself, however, corresponds to constituent quarks with constituent masses generated from self-interactions [13]. The spectrum of physical states is rather trivial since the interaction is independent from spin, flavor and, for zero-modes, also spacial degrees of freedom. The color interaction is responsible for lifting the energy of the color nonsinglet states. In the continuum limit this splitting is expected to be infinite (since the spacial average of a long-ranged confining interaction diverges) thus the model naturally removes color non-singlet states from the physical spectrum. This aspect of many-body dynamics is typically missing in quark potential models [22]. The model respects the pattern of chiral symmetry breaking in the sense that the vacuum can be selected as a state that is noninvariant under chiral rotations. However, the chiral behavior of the physical constants, e.g., the pion mass and the pion decay constant is not as expected. This is because in the model with a finite number of degrees of freedom there is no spontaneous chiral symmetry breaking in the usual sense, with an infinite number of degenerate vacuua. If we identify the pion with the lowest energy state with pseudosclar quantum numbers, its mass turns out to be a linear and not quadratic function of the symmetry breaking parameter, $m_0$, and the decay constant depends on $m_0$. This is an expected behavior for large values of $m_0$, or the nonrelativistic quark model. It is not surprising that our schematic model away from the exact chiral limit of $m_0 = 0$ immediately follows the pattern of a heavy quark theory since the model conserves the quark number. This in turn is the consequence of reduction of the quark degrees of freedom. With the gauge degrees of freedom integrated out and quark Fock space reduced to the zero modes there are no pair production interactions in the Coulomb gauge. This suggests that by extending the Fock space to include a limited number of nonzero momentum modes and/or adding gluon degrees of freedom it may be possible to address the low energy phenomena in a model with a finite number of degrees of freedom.

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