Reconstruction of the optical potential from scattering data

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We propose a method for reconstructing the optical potential (OP) from scattering data. The algorithm is a two-step, procedure. In the first step, the real part of the potential is determined analytically via solution of the Marchenko equation. At this point, we use a rational function fit of the corresponding unitary S matrix. In the second step, the imaginary part of the potential is determined via the phase equation of the variable phase approach. We assume that the real and imaginary parts of the OP are proportional (the Lax-type interaction). We use the phase equation to calculate the proportionality coefficient. A numerical algorithm is developed for a single and for coupled partial waves. The developed procedure is applied to analyze the ${}^1S_0 NN, {}^3SD_1 NN$, and $P31\ \pi^- N$ data. For the NN states, we constructed partial potentials with forbidden states (Moscow potential). We examine the $\pi^- N$ and NN partial-wave analysis data and demonstrate that the Lax-type interaction may be valid for these systems at the considered energies.

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I. INTRODUCTION

A lot of developments and applications exist of the classical approaches of Gel'fand, Levitan [1], and Marchenko [2] for the solution of the inverse-scattering problem at fixed angular momentum, and we have several excellent reviews on the subject [3,4]. The direct application of these approaches to the construction of local two-body potentials which are phase equivalent to the effective potentials occurring in theories describing reactions of composite particles is impossible. For such potential must be complex in order to reproduce the loss of flux above the inelastic threshold. But it must reproduce the real phase shifts below the threshold and must be real itself. These requirements are incompatible for potentials being energy independent by construction. For a very low threshold, these approaches are applicable and produce energy-independent complex potentials [5,6]. In the general case, an empirical energy-dependent OP is usually inferred by fitting the parameters of an assumed analytic potential [7,8]. This approach has two major shortcomings: the complexity of fitting many nonlinear parameters and the lack of correlation of the parameters obtained at various energies.

In this paper, we develop an inversion method that is free of these shortcomings. The method is based on a fixed-l inverse scattering theory and on a special parametrization of the OP. The proposed procedure is a two-step process. In the first step, the partial-wave analysis (PWA) data are used to determine a real potential via solution of the Marchenko equation. At this point, we use a rational function fit of the corresponding unitary S matrix. In the second step, the imaginary part of the potential is determined. We assume the same radial shape for the real and imaginary parts of the OP. This form of the OP (Lax-type interaction [9]) is the optical limit of the Glauber approximation (OLGA) [10]. The Glauber approximation (GA) [11] was considered for high-energy hadron-hadron scattering [12–15]. The OP with the same radial shape for the real and imaginary parts was used for nuclear systems in phenomenological Woods-Saxon form [16,17] and for analysis of the NN scattering data up to 6 GeV [18]. Derived within the framework of the multiple scattering theory [19], this type of interaction provides a very satisfactory description of nucleon-nucleus [20], nucleus-nucleus [21,22], and antinucleon-nucleus scattering [23].

Thus, we assume that this approximation is valid for intermediate-energy hadron-hadron elastic scattering. In this case, the real and imaginary parts of the OP are proportional. The value of the proportionality coefficient is dependent on energy. We calculate this coefficient from the phase equation of the variable phase approach [24]. The calculated value may be refined by an iterative algorithm. We develop this method for a single and for coupled partial waves. The whole procedure is applied to analyze the ${}^1S_0$ NN data (up to $E_{\rm lab} = 3$ GeV), ${}^3SD_1$ NN data (up to $E_{\rm lab} = 1.1$ GeV), and the $P31\ \pi^- N$ data (up to $E_{\rm lab} = 2$ GeV). We demonstrate that prediction for the proportionality coefficient from the phase equation is close to a precise value that reproduces the experimental loss of flux. We examine the NN and $\pi^- N$ PWA data. Our analysis shows that at energies above 1.5 GeV (NN) and above 1.2 GeV ($\pi^- N$), the OLGA may be valid.

We do not take into account the Coulomb interaction which is essential only for low-energy scattering and is a minor part of the hadron-hadron interaction in the energy range where the data are analyzed.

We show that the modern PWA data of NN scattering are compatible with the concept of the Moscow potential (MP) (regular NN potential with forbidden states in S and P waves) [18,25]. Thus, our NN analysis is an alternative to inversion results that produce the repulsive core NN potentials [8].

The plan of the paper is as follows. In Sec. II, we describe the inverse scattering techniques based on the Marchenko integral equation. The used rational function fit of the corresponding S matrix allows an analytical solution of the Marchenko integral equation [2,4]. For a single partial wave, the general solution was presented in Refs. [5,26]. We present a solution for coupled partial waves. These techniques produce real local potentials from PWA data. In Sec. III, we consider the phase equation.

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We investigate how the S matrix is changing with a certain change of the potential. This consideration shows that we can calculate the proportionality coefficient between the real and imaginary parts of the OP in case of the Lax-type interaction. In Sec. IV, the feasibility of the method is shown in the examples of analyses of NN and $\pi^- N$ scattering data.

II. THE REAL PART OF THE OP

The Marchenko inverse scattering theory is viewed in detail in Refs. [2–4], so we will only briefly describe this formalism. The input data of the Marchenko inversion are

$$\{S(q), (0 < q < +\infty), \tilde{q}_j, M_j, j = 1, \ldots, n_b\}, \tag{1}$$

where $S(q)$ is the scattering matrix dependant on the relative momentum q, $q^2 = Em$, $\tilde{q}_j^2 = mE_j \leqslant 0$. E_j is the energy of the j-th bound state so that $i\tilde{q}_j \geqslant 0$, and m is the reduced mass. The M_j matrix gives the asymptotic behavior of the j-th normalized bound state. We proceed from the Marchenko equation for a single channel

$$F(x, y) + L(x, y) + \int_x^{+\infty} L(x, t)F(t, y)\, dt = 0, \tag{2}$$

where the input kernel is given by

$$F(x, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h_l^+(qx)(I - S(q))h_l^+(qy)\, dq + \sum_{j=1}^{n_b} M_j^2 h_l^+(iq_j x)h_j^+(iq_j y), \tag{3}$$

and $h_l^+(z)$ is the Riccati-Hankel function. The output kernel $L(x, y)$ gives the reconstructed potential

$$V(r) = -2\frac{dL(r, r)}{dr}. \tag{4}$$

This local energy-independent operator $V(r)$ links the Marchenko equation (2) and the radial Schrödinger equation of a fixed angular momentum,

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)\right]\psi(r, q) = q^2\psi(r, q). \tag{5}$$

The scattering matrix $S(q)$, matrices M_j, and energies E_j are the output data of the direct scattering problem associated with the Schrödinger equation (5).

It has been known for several decades now that S matrices rational in q (ratio of polynomials) correspond to potentials known as Bargmann potentials expressible in terms of the elementary functions [2–4]. Such a fraction may have the same values at various points as the S matrix it represents. For a single partial wave, the general solution of the Marchenko equation via such rational function fit of the S matrix was presented in Ref. [5] and in Ref. [26]. We shall, therefore, only present it briefly and then turn to the case of coupled partial waves.

A rational function fit of the S matrix is given by

$$S(q) = e^{2i\delta} = \frac{f_2(q) + if_1(q)}{f_2(q) - if_1(q)}, \tag{6}$$

where $f_1(q)$ and $f_2(q)$ are odd and even polynomials of q, which do not turn to zero simultaneously. This approximant leads to the following expression for the phase shifts $\delta(q)$:

$$\tan\delta(q) = \frac{f_1(q)}{f_2(q)}. \tag{7}$$

Approximant (7) leads to a degenerate input kernel $F(x, y)$. We calculate the integral in Eq. (3) using the residue theorem. For approximant (6) the result of the integration is

$$\begin{aligned} F(x, y) &= i\sum_{i=1}^{n_{\text{pos}}} \text{Res}[h_l^+(qx)(I - S(q))h_l^+(qy)]\Big|_{q=\beta_i} \\ &\quad + \sum_{i=1}^{n_b} M_i^2 h_l^+(\tilde{q}_i x)\, h_l^+(\tilde{q}_i y) \qquad (8) \\ &= \sum_{i=1}^{n_{\text{pos}}} b_i h_l^+(\beta_i x)\, h_l^+(\beta_i y) \\ &\quad + \sum_{i=1}^{n_b} M_i^2 h_l^+(\tilde{q}_i x)\, h_l^+(\tilde{q}_i y) \\ &= \sum_{j=1}^{n} b_j h_l^+(\beta_j x) h_l^+(\beta_j y), \qquad (9) \end{aligned}$$

where β_i $(i = 1, \ldots, n_{\text{pos}})$ are all S-matrix poles with $\text{Im}[\beta_i] > 0$, $\beta = \{\beta_1, \ldots, \beta_{n_{\text{pos}}}, \tilde{q}_1, \ldots, \tilde{q_{n_b}}\}$, $n = n_{\text{pos}} + n_b$. We assume that all poles are of first order so that

$$\begin{aligned} &\text{Res}[h_l^+(qx)(I - S(q))h_l^+(qy)]|_{q=\beta_i} \\ &\quad = 2i\,\text{Res}\left[h_l^+(qx)\frac{f_1(q)}{f_2(q) - if_1(q)}h_l^+(qy)\right]\Big|_{q=\beta_i} \\ &\quad = 2i\frac{f_1(\beta_i)}{f_2'(\beta_i) - if_1'(\beta_i)}h_l^+(\beta_i x)\, h_l^+(\beta_i y) \\ &\quad = b_i h_l^+(\beta_i x)\, h_l^+(\beta_i y), \end{aligned} \tag{10}$$

here we denote $f_i'(q) = df_i(q)/dq$ $(i = 1, 2)$. The input kernel of Eq. (2) is degenerate

$$L(x, y) = \sum_{i=1}^{n} P_i(x)\, h_l^+(\beta_i y), \tag{11}$$

where $P_i(x)$ are unknown coefficients. Substitution of (9) and (11) into (2) yields

$$\sum_{i=1}^{n} h_l^+(\beta_i y)\left(b_i h_l^+(\beta_i x) + P_i(x) + b_i\sum_{k=1}^{n} P_k(x)\int_x^{+\infty} h_l^+(\beta_k t)\, h_l^+(\beta_i t)\, dt\right) = 0. \tag{12}$$

Linear independence of the $h_l^+(\beta_i y)$ implies that

$$b_i h_l^+(\beta_i x) + P_i(x) + b_i \sum_{k=1}^{n} P_k(x) \times \int_x^{+\infty} h_l^+(\beta_k t)\, h_l^+(\beta_i t)\, dt = 0, \tag{13}$$

or

$$\sum_{k=1}^{n} A_{ik}(x)\, P_k(x) = D_i(x) \,(i = 1, \ldots, n), \tag{14}$$

where $D_i(x) = -b_i h_l^+(\beta_i x)$. Applying Riccati-Hankel integration formulas in (13), we have

$$A_{ik} = \begin{cases} 1 + b_i x[(h_l^+(\beta_i x))^2 - h_{l-1}^+(\beta_i x) h_{l+1}^+(\beta_i x)]/2, & \text{for } i = k, \\ -b_i \dfrac{\beta_i h_{l-1}^+(\beta_i x) h_l^+(\beta_k x) - \beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x)}{\beta_i^2 - \beta_k^2}, & \text{for } i \neq k. \end{cases} \tag{15}$$

The functional coefficients $P_k(x)$ are defined by (14) as

$$P_k(x) = (A^{-1} D)_k. \tag{16}$$

Finally we derive $V(r)$ from (11) and (4).

In the case of two coupled channels, we present only sketchy derivations because of their awkwardness. In this case, the system of the partial Schrödinger equations is

$$\left(\frac{d^2}{dr^2} + V(r) + \begin{pmatrix} \frac{l_1(l_1+1)}{r^2} & 0 \\ 0 & \frac{l_2(l_2+1)}{r^2} \end{pmatrix}\right)\begin{pmatrix} \chi_1(r) \\ \chi_2(r) \end{pmatrix} = q^2 \begin{pmatrix} \chi_1(r) \\ \chi_2(r) \end{pmatrix}, \tag{17}$$

$$V(r) = \begin{pmatrix} V_1(r) & V_T(r) \\ V_T(r) & V_2(r) \end{pmatrix}, \tag{18}$$

where $V_1(r)$, $V_2(r)$ are potentials in channels 1 and 2, $V_T(r)$ is the potential coupling the two channels, and $\chi_1(r)$ and $\chi_2(r)$ are channel wave functions.

By analogy with (6), we approximate the S matrix by the expression

$$S(x) = \begin{pmatrix} \exp(2i\tilde{\delta}_1)\cos 2\tilde{\varepsilon} & i\exp(i(\tilde{\delta}_1 + \tilde{\delta}_2))\sin 2\tilde{\varepsilon} \\ i\exp(i(\tilde{\delta}_1 + \tilde{\delta}_2))\sin 2\tilde{\varepsilon} & \exp(2i\tilde{\delta}_2)\cos 2\tilde{\varepsilon} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\dfrac{f_2^{(1)}(q) + i f_1^{(1)}(q)}{f_2^{(1)}(q) - i f_1^{(1)}(q)}\right)^2 \dfrac{\left(f_2^{(12)}(q)\right)^2 - \left(f_1^{(12)}(q)\right)^2}{\left(f_2^{(12)}(q)\right)^2 + \left(f_1^{(12)}(q)\right)^2} & -2i \dfrac{f_2^{(12)}(x)\, f_1^{(12)}(x)}{\left(f_2^{(12)}(q)\right)^2 + \left(f_1^{(12)}(q)\right)^2} \prod_{j=1,2} \dfrac{f_2^{(j)}(q) + i f_1^{(j)}(q)}{f_2^{(j)}(q) - i f_1^{(j)}(q)} \\ -2i \dfrac{f_2^{(12)}(q)\, f_1^{(12)}(q)}{\left(f_2^{(12)}(q)\right)^2 + \left(f_1^{(12)}(q)\right)^2} \prod_{j=1,2} \dfrac{f_2^{(j)}(q) + i f_1^{(j)}(q)}{f_2^{(j)}(q) - i f_1^{(j)}(q)} & \left(\dfrac{f_2^{(2)}(q) + i f_1^{(2)}(q)}{f_2^{(2)}(q) - i f_1^{(2)}(q)}\right)^2 \dfrac{\left(f_2^{(12)}(q)\right)^2 - \left(f_1^{(12)}(q)\right)^2}{\left(f_2^{(12)}(q)\right)^2 + \left(f_1^{(12)}(q)\right)^2} \end{pmatrix}. \tag{19}$$

This is again the most rational function fit for the S matrix. It was used in Ref. [27] in another form, but an analytical solution of the inverse scattering problem was not presented.

The coefficients of this rational function fit are determined from equations

$$\tan\frac{\tilde{\delta}_i(q)}{2} = \frac{f_1^{(i)}(q)}{f_2^{(i)}(q)}, \quad i = 1, 2 \tag{20}$$

$$\tan\tilde{\varepsilon}(q) = \frac{f_1(q)}{f_2(q)}. \tag{21}$$

The Marchenko equation for coupled channels formally has the former view [28]

$$L(x, y) + F(x, y) + \int_x^{+\infty} L(x, t)\, F(t, y)\, dt = 0, \tag{22}$$

but functions involved are matrices (2×2)

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(qx)\,[I - S(q)]\, H(qy)\, dq + \sum_{i=1}^{n_b} H(\beta_i x)\, M_i\, H(\beta_i y), \tag{23}$$

where

$$H(x) = \begin{pmatrix} h^+_{l_1}(x) & 0 \\ 0 & h^+_{l_2}(x) \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

Inserting (19) into (23) and applying the residue theorem yields

$$\begin{aligned} F(x,y) &= i\sum_{i=1}^{n_{\rm pos}} \mathrm{Res}[H(qx)(I-S(q))H(qy)]|_{q=\beta_i} \\ &\quad + \sum_{i=1}^{n_b} H(\beta_i x)\, M_i^2 H(\beta_i y) \\ &= \sum_{i=1}^{n} H(\beta_i x) Q_i^1 H(\beta_i y) + \sum_{i=1}^{n^{(2)}_{\rm pos}} x H'(\beta_i x)\, Q_i^2 H(\beta_i y) \\ &\quad + \sum_{i=1}^{n^{(2)}_{\rm pos}} H(\beta_i x)\, Q_i^2 H'(\beta_i y)\, y, \end{aligned} \tag{25}$$

where β_i $(i = 1, \ldots, n_{\rm pos})$ are all S-matrix poles with $\mathrm{Im}[\beta_i] > 0$, β_i $(i = 1, \ldots, n^{(2)}_{\rm pos})$ are poles of the second order, $\beta = \{\beta_1, \ldots, \beta_{n^{(2)}_{\rm pos}}, \ldots, \beta_{n_{\rm pos}}, \tilde{q}_1, \ldots, \tilde{q}_{n_b}\}$, $n = n_{\rm pos} + n_b$, and

$$H'(x) = \begin{pmatrix} dh^+_{l_1}(x)/dx & 0 \\ 0 & dh^+_{l_2}(x)/dx \end{pmatrix}.$$

We note that poles of both the first and second orders are in the diagonal matrix elements, and poles of only the first order are in the off-diagonal matrix elements. Poles of the second order in the diagonal elements are poles of the first order in the off-diagonal matrix elements, and they are enumerated twice. Q_i^j $(j = 1, 2)$ are constant matrices. They are trivial but cumbersome; therefore, we do not give them.

We solve Eq. (22) using substitution

$$L(x,y) = \sum_{i=1}^{n} P_i(x) H(\beta_i y) + \sum_{i=1}^{n} N_i(x) y H'(\beta_i y), \tag{26}$$

where $P_i(x), N_i(x)$ are unknown functional (2×2) matrix coefficients. Linear independence of the $H(\beta_i y)$ and $yH'(\beta_i y)$ implies that

$$\begin{aligned} &\sum_i P_i(x) Q^3_{ij}(x) + \sum_i N_i(x) Q^5_{ij}(x) \\ &\quad = H(\beta_j x) Q^1_j + x H'(\beta_j x) Q^2_j, \\ &\sum_i N_i(x) Q^6_{ij}(x) + \sum_i P_i(x) Q^4_{ij}(x) \\ &\quad = H(\beta_j x) Q^2_j, \end{aligned} \tag{27}$$

where

$$\begin{aligned} Q^3_{ij}(x) &= I\delta_{ij} + \int_x^{+\infty} H(\beta_i t) H(\beta_j t)\, dt\, Q^1_j \\ &\quad + \int_x^{+\infty} t H(\beta_i t) H'(\beta_j t)\, dt\, Q^2_j, \\ Q^4_{ij}(x) &= \int_x^{+\infty} H(\beta_i t) H(\beta_j t)\, dt\, Q^2_j, \\ Q^5_{ij}(x) &= \int_x^{+\infty} t H'(\beta_i t) H(\beta_j t)\, dt\, Q^1_j \\ &\quad + \int_x^{+\infty} t^2 H'(\beta_i t) H'(\beta_j t)\, dt\, Q^2_j, \\ Q^6_{ij}(x) &= I\delta_{ij} + \int_x^{+\infty} t H'(\beta_i t) H(\beta_j t)\, dt\, Q^2_j. \end{aligned} \tag{28}$$

Integrals of expressions (28) can be derived from the recursion relations for the Riccati-Hankel functions and from known integrals. Matrix equations (27) are reducible to scalar linear equations. We solve this linear equation system and get the sought-for potential from (26) and (4).

The multichannel generalization is trivial.

III. THE IMAGINARY PART OF THE OP

The concept from the Lax-type interaction of using similar shapes for the real and imaginary parts of the OP motivated us to consider changes of the S matrix that are induced by a certain transformation of the real potential.

First we consider the one-channel problem.

The phase equation [24] for the initial potential $V^{(0)}(r)$ obtained by some inversion procedure (from the Marchenko equation in our calculations) is

$$\delta_l^{(0)} = -\frac{1}{q}\int_0^{\infty} V^{(0)}(r)\, \hat{D}_l^2(qr) \sin^2(\hat{\delta}_l(qr) + \delta^{(0)}(r))\, dr, \tag{29}$$

where $\hat{D}_l(z)$ and $\hat{\delta}_l(z)$ are a Riccati-Bessel amplitude and phase, correspondingly [24],

$$\begin{aligned} \hat{D}_l(x) &= \sqrt{j_l^2(x) + n_l^2(x)}, \\ \hat{\delta}_l(x) &= -\arctan(j_l(x)/n_l(x)). \end{aligned}$$

Let us consider the complex-valued potential $V^{(1)}(r)$ obtained from $V^0(r)$ by transformation

$$V^{(1)}(r) = (1 + i\alpha) V^{(0)}(r), \tag{30}$$

where α is some real parameter. Such parametrization was used in Ref. [18] but without analysis (α was fitted). Evidently the phase equation for this potential is

$$\begin{aligned} \delta^{(1)} = -\frac{1}{q}(1 + i\alpha)\int_0^{\infty} & V^{(0)}(r)\, \hat{D}_l^2(qr) \\ & \times \sin^2(\hat{\delta}_l(qr) + \delta^{(1)}(r))\, dr. \end{aligned} \tag{31}$$

From Eqs. (29) and (31) we get

$$\begin{aligned} \delta^{(1)} - (1 + i\alpha)\delta^{(0)} = -\frac{1 + i\alpha}{q}\int_0^{\infty} & V^{(0)}(r) \hat{D}_l^2(qr) \\ & \times \underline{\sin(2\hat{\delta}_l(qr) + \delta^{(1)}(r) + \delta^{(0)}(r))} \\ & \times \sin(\delta^{(1)}(r) - \delta^{(0)}(r))\, dr. \end{aligned} \tag{32}$$

For smooth enough potentials, the right side of Eq. (32) rapidly decreases comparing with $\delta^{(0)}$ and $\delta^{(1)}$, because there

is a rapidly oscillating around zero function under the integral in (32) (underlined). Its frequency behaves as $2q$ for big q [see (30)]. Then as the first approximation, we may take

$$\delta^{(1)} \approx (1 + i\alpha)\,\delta^{(0)} = \mathrm{Re}\,\delta + i\,\mathrm{Im}\,\delta. \tag{33}$$

For inelastic scattering, the S matrix is expressed through the real inelasticity parameter ρ and the real phase shift δ as

$$S = \cos^2(\rho) e^{2i\delta} = \exp 2i(\mathrm{Re}\,\delta + i\ \mathrm{Im}\,\delta), \tag{34}$$

so we easily arrive at

$$\mathrm{Re}\,\delta^{(1)} \approx \delta^{(0)}, \qquad \cos^2\rho \approx e^{-2\alpha\delta^{(0)}}, \tag{35}$$

from whence it follows that $\alpha\delta \geqslant 0$. Equation (35) allows us to calculate the parameter α from the known values ρ and $\delta^{(0)} \approx \delta$.

Consideration of the coupled partial waves is more complicated. The initial real potential is

$$V^{(0)}(r) = \begin{pmatrix} V_1^{(0)} & V_T^{(0)} \\ V_T^{(0)} & V_2^{(0)} \end{pmatrix}. \tag{36}$$

The equations for eigenphases and mixing parameters of potential (36) are [24]

$$\delta_1^{(1)} = I_{11}^{(0)} + I_{12}^{(0)} + I_{13}^{(0)}, \tag{37}$$

$$\delta_2^{(1)} = I_{21}^{(0)} + I_{22}^{(0)} + I_{23}^{(0)}, \tag{38}$$

$$\epsilon^{(1)} = I_{31}^{(0)} + I_{32}^{(0)} + I_{33}^{(0)}, \tag{39}$$

where

$$\begin{aligned}
I_{11}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_1^{(0)}(r)\cos^2\epsilon^{(0)}(r)\hat{D}_{l_1}^2(qr) \\
&\quad\times \sin^2\left(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)\right), \\
I_{12}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_2^{(0)}(r)\sin^2\epsilon^{(0)}(r)\hat{D}_{l_2}^2(qr) \\
&\quad\times \sin^2\left(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)\right), \\
I_{13}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_T^{(0)}(r)\sin 2\epsilon^{(0)}(r)\hat{D}_{l_2}(qr) \\
&\quad\times \sin\left(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)\right)\hat{D}_{l_1}(qr)\sin\left(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)\right),
\end{aligned} \tag{40}$$

$$\begin{aligned}
I_{21}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_1^{(0)}(r)\sin^2\epsilon^{(0)}(r)\hat{D}_{l_1}^2(qr) \\
&\quad\times \sin^2\left(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)\right), \\
I_{22}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_2^{(0)}(r)\cos^2\epsilon^{(0)}(r)\hat{D}_{l_2}^2(qr) \\
&\quad\times \sin^2\left(\hat{\delta}_{l_2}(qr) + \delta_2^{(0)}(r)\right), \\
I_{23}^{(0)} &= -\frac{1}{q}\int_0^\infty dr\, V_T^{(0)}(r)\sin 2\epsilon^{(0)}(r)\hat{D}_{l_2}(qr) \\
&\quad\times \sin\left(\hat{\delta}_{l_2}(qr) + \delta_2^{(0)}(r)\right)\hat{D}_{l_1}(qr)\sin\left(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)\right),
\end{aligned} \tag{41}$$

$$\begin{aligned}
I_{31}^{(0)} &= \frac{1}{2q}\int_0^\infty \frac{\sin 2\epsilon^{(0)}(r)\,dr}{\sin\left(\delta_1^{(0)}(r) - \delta_2^{(0)}(r)\right)} V_1^{(0)}(r) \\
&\quad\times \hat{D}_{l_1}^2(qr)\sin\left(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)\right)\sin\left(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)\right), \\
I_{32}^{(0)} &= -\frac{1}{2q}\int_0^\infty \frac{\sin 2\epsilon^{(0)}(r)\,dr}{\sin\left(\delta_1^{(0)}(r) - \delta_2^{(0)}(r)\right)} V_2^{(0)}(r) \\
&\quad\times \hat{D}_{l_2}^2(qr)\sin\left(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)\right)\sin\left(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)\right), \\
I_{33}^{(0)} &= -\frac{1}{2q}\int_0^\infty \frac{V_T^{(0)}(r)\hat{D}_{l_1}(qr)\hat{D}_{l_2}(qr)\,dr}{\sin\left(\delta_1^{(0)}(r) - \delta_2^{(0)}(r)\right)} \\
&\quad\times \Bigg[\cos 2\epsilon^{(0)}(r)\sin\left(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)\right)\sin\left(\hat{\delta}_{l_2}(qr)\right. \\
&\quad\left. + \delta_2^{(0)}(r)\right) - \frac{1}{2}(\cos 2\epsilon^{(0)}(r) - 1)\sin\left(\delta_1^{(0)}(r)\right. \\
&\quad\left. - \delta_2^{(0)}(r)\right)\sin(\hat{\delta}_{l_1}(qr) - \hat{\delta}_{l_2}(qr))\Bigg].
\end{aligned} \tag{42}$$

By analogy with the one-channel case, the following generalization for the OP is derived:

$$V^{(1)}(r) = \begin{pmatrix} (1 + i\alpha_1)V_1^{(0)} & (1 + i\alpha_3)\,V_T^{(0)} \\ (1 + i\alpha_3)V_T^{(0)} & (1 + i\alpha_2)V_2^{(0)} \end{pmatrix}. \tag{43}$$

Evidently, the phase equations for this potential are

$$\delta_1^{(1)} = (1 + i\alpha_1)I_{11}^{(1)} + (1 + i\alpha_2)I_{12}^{(1)} + (1 + i\alpha_3)I_{13}^{(1)}, \tag{44}$$

$$\delta_2^{(1)} = (1 + i\alpha_1)I_{21}^{(1)} + (1 + i\alpha_2)I_{22}^{(1)} + (1 + i\alpha_3)I_{23}^{(1)}, \tag{45}$$

$$\epsilon^{(1)} = (1 + i\alpha_1)I_{31}^{(1)} + (1 + i\alpha_2)I_{32}^{(1)} + (1 + i\alpha_3)I_{33}^{(1)}. \tag{46}$$

Integrals $I_{ij}^{(1)}$ are defined as $I_{ij}^{(0)}$ in (40)–(42) but through $\delta_1^{(1)}(r)$, $\delta_2^{(1)}(r)$, and $\epsilon^{(1)}(r)$ instead of $\delta_1^{(0)}(r)$, $\delta_2^{(0)}(r)$, and $\epsilon^{(0)}(r)$.

We consider (37)–(46) in a manner like (32) and come to the following equations:

$$\Sigma^{(1)} + X^{(1)} \approx (1 + i\alpha_1)(\Sigma^{(0)} + X^{(0)}), \tag{47}$$

$$\Sigma^{(1)} - X^{(1)} \approx (1 + i\alpha_2)(\Sigma^{(0)} - X^{(0)}), \tag{48}$$

$$Y^{(1)} \approx (1 + i\alpha_3)Y^{(0)}, \tag{49}$$

where

$$\Delta^{(i)} = \delta_1^{(i)} - \delta_2^{(i)}, \qquad \Sigma^{(i)} = \delta_1^{(i)} + \delta_2^{(i)}, \tag{50}$$

$$X^{(i)} = (\tan\Delta^{(i)} + \Delta^{(i)})\cos 2\epsilon^{(i)*} - \tan\Delta^{(i)}\cos 2\epsilon^{(i)}, \tag{51}$$

$$Y^{(i)} = (\tan\Delta^{(i)} + \Delta^{(i)})\sin 2\epsilon^{(i)**} - \tan\Delta^{(i)}\sin 2\epsilon^{(i)}, \tag{52}$$

$$\cos 2\epsilon^{(i)*} = \frac{\int_0^\infty \cos 2\epsilon^{(i)}(r)(1 + \cos^{-2}\Delta^{(i)}(r))\frac{d\Delta^{(i)}(r)}{dr}\,dr}{\Delta^{(i)} + \tan\Delta^{(i)}}, \tag{53}$$

$$\sin 2\epsilon^{(i)**} = \frac{\int_0^\infty \sin 2\epsilon^{(i)}(r)(1 + \cos^{-2}\Delta^{(i)}(r))\frac{d\Delta^{(i)}(r)}{dr}\,dr}{\Delta^{(i)} + \tan\Delta^{(i)}}, \quad \text{for } i = 0, 1. \tag{54}$$

Eigenphases $\hat{\delta}_i^{(0)}$, $i = 1, 2$, and mixing parameter $\hat{\epsilon}^{(0)}$ are real, and they define a unitary $S^{(0)}$ matrix,

$$S^{(0)} = S\left(\delta_1^{(0)}, \delta_2^{(0)}, \epsilon^{(0)}\right) = \begin{pmatrix} \cos^2\epsilon^{(0)} e^{2i\delta_1{}^{(0)}} + \sin^2\epsilon^{(0)} e^{2i\delta_2{}^{(0)}} & \cos\epsilon^{(0)}\sin\epsilon^{(0)}\left(e^{2i\delta_1{}^{(0)}} - e^{2i\delta_2{}^{(0)}}\right) \\ \cos\epsilon^{(0)}\sin\epsilon^{(0)}\left(e^{2i\delta_1{}^{(0)}} - e^{2i\delta_2{}^{(0)}}\right) & \sin^2\epsilon^{(0)} e^{2i\delta_1{}^{(0)}} + \cos^2\epsilon^{(0)} e^{2i\delta_2{}^{(0)}} \end{pmatrix}. \quad (55)$$

Eigenphases $\hat{\delta}_i^{(1)}$ and mixing parameter $\hat{\epsilon}^{(1)}$ are complex, but they define the $S^{(1)}$ matrix in the regular way

$$S^{(1)} = S\left(\delta_1^{(1)}, \delta_2^{(1)}, \epsilon^{(1)}\right). \quad (56)$$

The complex eigenphases $\delta_i^{(1)}$ and mixing parameter $\epsilon^{(1)}$ are defined by the experimental S matrix ($S \equiv S^{(1)}$). For $|\mathrm{Im}\,\epsilon^{(1)}| \ll |\mathrm{Re}\,\epsilon^{(1)}| \ll 1$,

$$\mathrm{Re}\,\delta_i^{(1)} \approx \delta_i^{(0)}, \qquad i = 1, 2; \quad \mathrm{Re}\,\epsilon^{(1)} \approx \epsilon^{(0)}. \quad (57)$$

$$\mathrm{Im}\,\delta_i^{(1)} \approx \alpha_i \delta_i^{(0)}, \quad i = 1, 2;$$

$$\mathrm{Im}\,\epsilon^{(1)} \approx \left(\frac{\alpha_1\delta_1^{(0)}}{\delta_2^{(0)} - \delta_1^{(0)}} - \frac{\alpha_2\delta_2^{(0)}}{\delta_2^{(0)} - \delta_1^{(0)}} + \alpha_3\right)\epsilon^{(0)}. \quad (58)$$

We calculate α_i, $i = 1, 2, 3$ from (57) and (58) and thereafter improve them using (47)–(54).

The above derivation can be easily extended to the general Lax-type interaction. Suppose that the scattering composite particles consist of some q particles. The Lax OP is the zero-range double-folding (DF) potential,

$$V_{\mathrm{Lax}}(r) = -\frac{v}{2}(\xi + i)\sigma_T^{qq}\int \rho_T(\vec{r}')\rho_P(\vec{r} - \vec{r}')d\vec{r}', \quad (59)$$

where v is the relative velocity between two scattering composite particles, σ_T^{qq} is a total qq cross section averaged over possible q states of the target q_T and over possible q states of the projectile q_P. ρ_P and ρ_T are the projectile and target densities, and $\xi = \mathrm{Re}\,[f]\,/\mathrm{Im}\,[f]$. $f = f^{qq}(E, 0)$ is the $q_T q_P$ forward elastic amplitude evaluated at the same velocity. In the case of hadron-hadron scattering, the q particles are quarks. Therefore, values of σ_T^{qq} and ξ cannot be experimentally measured. Then the Lax OP is

$$V(r) = (\beta + i\gamma)\, V_0(r), \quad (60)$$

where both coefficients β and γ are energy dependent. It is generally assumed that at low energies the elastic scattering is described by a local energy-independent potential. Therefore, we suppose that $\beta \equiv 1$ and calculate $\gamma \equiv \alpha$ from (35).

The simple Lax interaction (59) may be complicated for q particles with spin [19,20]. It may contain all terms corresponding to the possible terms of the averaged $q_T q_P$ amplitude. At the considered energies, the q particles (partons) for hadron are the constituent quarks. The average includes averages over spins, colors, and flavors of q_T and over spins, colors, and flavors of q_P separately. Then we approximate the amplitude by setting equal to zero all terms linear in color operators of q_T and linear in color operators of q_P. Thus the confinement term vanishes for a hadron-hadron scattering. Similarly, we set equal to zero all terms linear in target (projectile) spin operator for target (projectile) with zero spin. The OLGA interaction may include the central V_C and spin-orbit V_{SL} terms in case of $\pi^- N$ scattering. The DF nucleon-nucleus interactions [29] and deuteron-nucleus interactions [30] usually include these terms. There are tensor V_T and other possible terms in the case of NN interaction. Therefore, the DF potential may appear as

$$V_{\mathrm{DF}}(E_{\mathrm{lab}}, L, S, J, r) = A v_C(r) + B v_{ss}(r)(s_1 \cdot s_2) + C v_{\mathrm{SL}}(r)(S \cdot L) + D v_T(r)\hat{S}_{12} + \cdots. \quad (61)$$

The multipliers $A, B, C, \ldots$ are proportional to sums of various $q_T q_P$ amplitudes. The radial dependence of all terms is determined by the central part

$$v_{ss}(r) \sim v_C(r), \qquad v_{\mathrm{SL}}(r) \sim \frac{1}{p^2}\frac{dv_C(r)}{rdr}, \qquad \ldots. \quad (62)$$

The first two terms are of uniform radial dependence in (61), other terms may spoil the proportionality of the real and imaginary parts of the DF potential. All noncentral terms decrease with increase of the energy, and the L independence of the Lax interaction is a result of the zero-range approximation of the $q_T q_P$ interaction, in which the angular dependence of the $f_{qq}(\vec{q})$ is neglected [31].

The commonly accepted model of the low-energy NN interaction is a "minimal" nonlocality, i.e., L, S, J dependent and energy-independent real r-space potential $V(L, S, J, r)$. This concept is applicable to other hadron-hadron interactions. Therefore, the following model combines the low-energy,

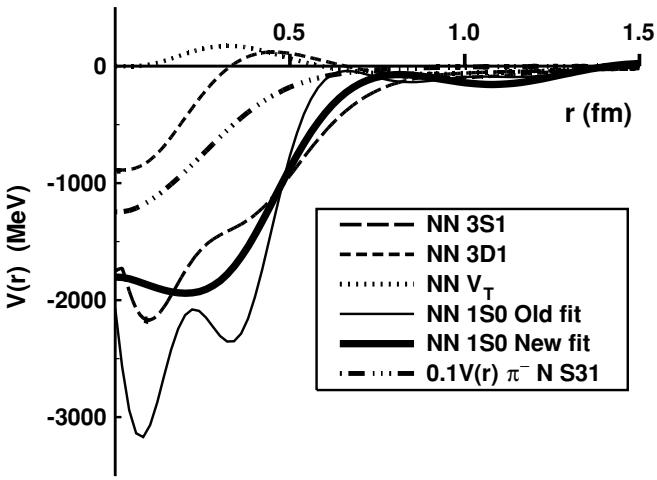

FIG. 1. Real parts of NN and $\pi^- N$ potentials. ${}^1S_0$ NN: thick solid line, inversion from new fit of Fig. 2; thin solid line, inversion from old fit of Fig. 2. ${}^3SD_1$ NN: long dashed line, $V_{\mathrm{CS}}(r)$ (${}^3S_1$); short dashed line, $V_{\mathrm{CD}}(r)$ (${}^3D_1$); dotted line, $V_{\mathrm{tens}}(r)$ potential. $\pi^- N$ $S31$ potential: dashed-three-points, line.

TABLE I. Deuteron properties.

	Exp.[a]	Calculation with MP
Energy (MeV)	2.22458900(22)	2.2246[b]
Q (fm^2)	0.2859(3)	0.277[c]
A_S (fm$^{-1/2}$)	0.8802(20)	0.8802
r_d (fm)	1.9627(38)	1.956
$\eta_{d/s}$	0.02714	0.02714
μ_d	0.857406(1)	0.859[c]

[a]From Ref. [37].
[b]Relativistic correction included.
[c]Meson exchange currents are not included.

general DF model (61) and simplest DF model (59):

$$V = \begin{cases} V(L,S,J,r), & \text{for } E_{\rm lab} < E_{\rm local} \\ V_{\rm DF}(E_{\rm lab},L,S,J,r), & \text{for } E_{\rm DF} < E_{\rm lab} < E_{\rm Lax} \\ V_{\rm Lax}(E_{\rm lab},r), & \text{for } E_{\rm Lax} < E_{\rm lab}. \end{cases} \quad (63)$$

Is there a gap between $E_{\rm local}$ and $E_{\rm DF}$? It depends on the required precision of the description. The continuity of $q_T q_P$ amplitudes demands independence of $\mathrm{Re}\, V_{\rm DF}(L,S,J,r)$ on the energy $E_{\rm lab}$ in the case when there is no gap.

We apply the developed method of inversion to the analysis of NN and $\pi^- N$ data up to energies at which relativistic effects are essential. We take into account these effects in the frames of the relativistic quantum mechanics of systems with a fixed number of particles [32]. A system of two particles is described by the wave function, which is an eigenfunction of the mass operator. In this case, we may represent this wave function as a product of the external and internal wave functions [33,34]. The internal wave function χ is also an eigenfunction of the mass operator and satisfies the equation

$$\left[\sqrt{\hat{q}^2+m_1^2}+\sqrt{\hat{q}^2+m_2^2}+V_{\rm int}\right]\chi = M\chi, \quad (64)$$

where $V_{\rm int}$ is some interaction operator acting only through internal variables (spins and relative momentum), and $\hat{q}$ is a momentum operator of one of the particles in the center of mass frame (relative momentum). Rearrangement of (64) gives

$$[\hat{q}^2 + 2mV]\chi = q^2\chi, \quad (65)$$

where

$$q^2 = \frac{M^2}{4} - \frac{m_1^2+m_2^2}{2} + \frac{\left(m_1^2-m_2^2\right)^2}{4M^2}, \quad (66)$$

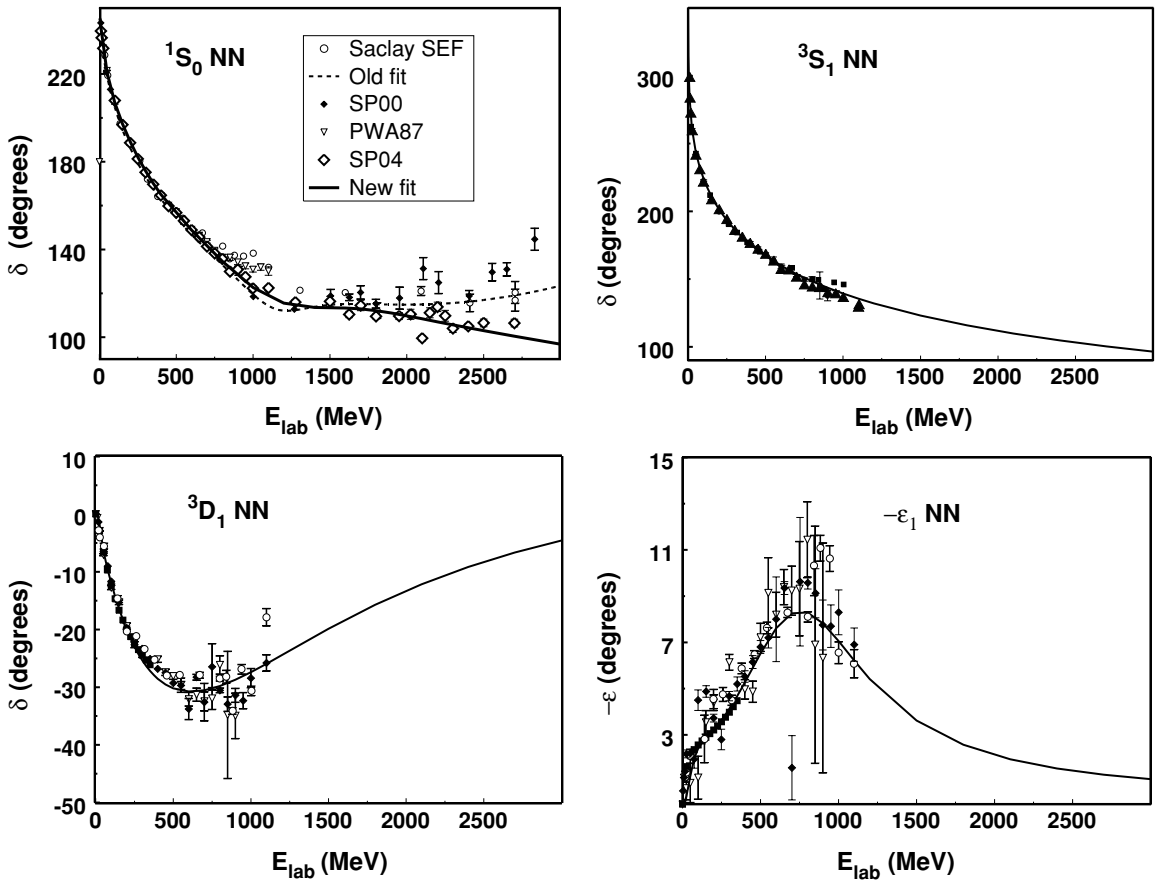

FIG. 2. Phase shifts and mixing parameter. Results for optical potentials are indistinguishable from results for real potentials. PWA data are from Refs. [35,36]. For S waves, the original data set from Refs. [35,36] is raised 180°. To leave the S matrix unchanged, we change the sign of the mixing parameter ϵ_1 for the MP. The SP00 and SP04 data are from Ref. [36].

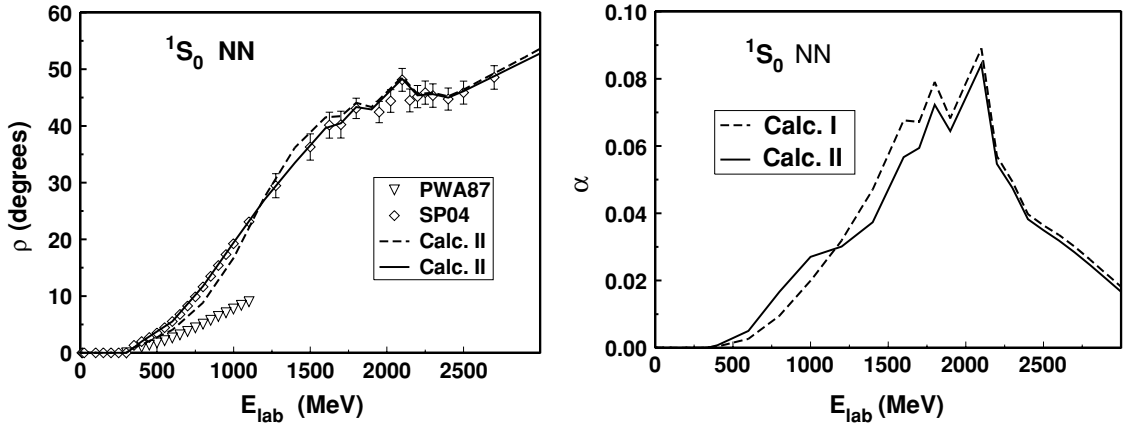

FIG. 3. Left: Inelasticity parameters ρ for $^1S_0$ *NN* wave. PWA data are from [35]. Right: Parameter α for the same wave.

m is taken as a nonrelativistic reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}, \tag{67}$$

and V is an operator acting like $V_{\rm int}$ only through internal variables. In the case of two particles with equal masses $m_1 = m_2 \equiv 2m$,

$$q^2 = \frac{M^2}{4} - 2m^2. \tag{68}$$

Equation (65) is identical in form to the Schrôdinger equation. The quasicoordinate representation corresponds to the realization $\mathbf{q} = -i\frac{\partial}{\partial \mathbf{r}}$, $V = V(\mathbf{r})$. This formal coincidence allows us to apply our inversion algorithm.

IV. RESULTS AND DISCUSSION

We applied the described algorithm of inversion to reconstruction of the nucleon-nucleon potential. We chose for our analysis the singlet $^1S_0$ wave that must be described by simplest DF potentials in the Lax form and the coupled $^3SD_1$ waves where only D-wave potential may contain more complex terms with different radial dependence of the real and imaginary parts. As input data for this reconstruction, we used modern phase shift analysis data (single-energy solutions) up to 1100 MeV for the $^3SD_1$ state and up to 3 GeV for the $^1S_0$ state of the *NN* system [35,36]. The deuteron properties were taken from Ref. [37]. These data allow us to construct Moscow-type *NN* partial potentials sustaining forbidden bound states. In this way, we constructed the *NN* optical potentials for $^1S_0$ and $^3SD_1$ partial waves. These potentials describe part of the deuteron properties and the phase shift analysis data by the construction. The $^1S_0$ phase shifts of MP begin from π. Parameters of the forbidden bound state for the $^1S_0$ partial MP were fitted to make this potential close to the $^3S_1$ potential. The $^3S_0$ phase shifts of the MP begin from 2π. The mixing parameter ϵ_1 of the MP differs from that of the traditional repulsive core potential by sign. The real parts of the constructed partial potentials are presented in Fig. 1. The calculated values of the deuteron properties are compared with the experimental data [37] in Table I. Three parameters are fixed as input data of inversion problem for the $^3SD_1$ state. These parameters are energy, A_S, and $\eta_{d/s}$. The last two define elements of the M_1 matrix. This matrix

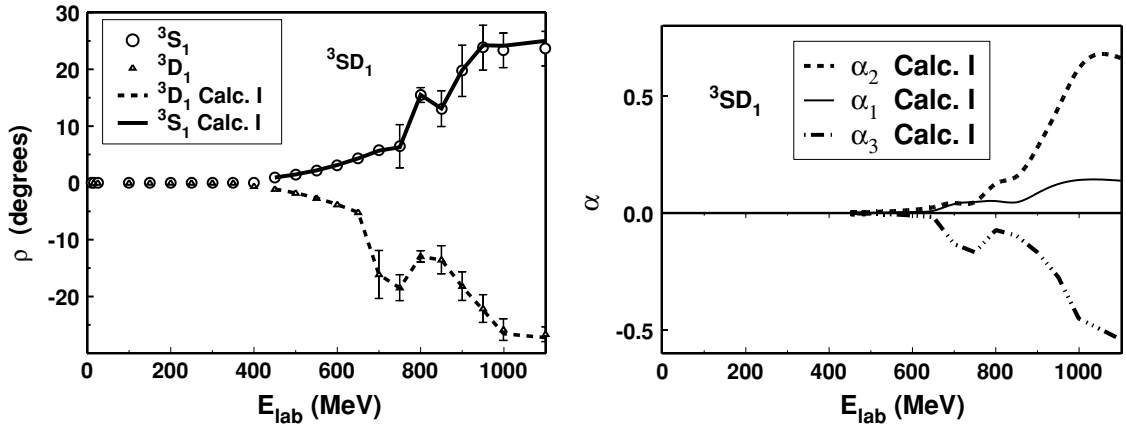

FIG. 4. Same as Fig. 3, but for $^3SD_1$ waves; PWA data are from Ref. [36].

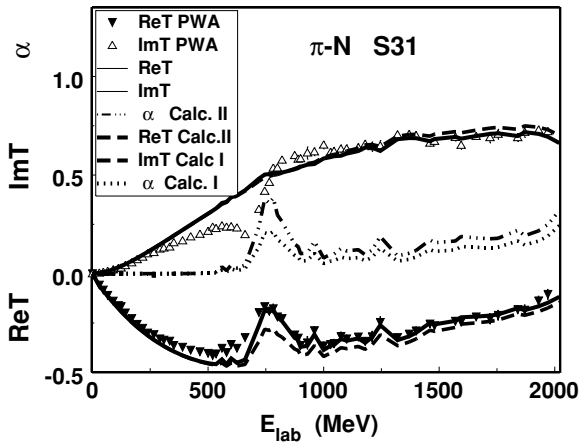

FIG. 5. T matrix and α for $S31$ $\pi^- N$; data are from [36]. Phase shift begins from 180° (potential with forbidden state).

corresponds to the deuteron state. We have two independent elements of the M_2 matrix. This matrix corresponds to the forbidden state, and its elements as well as energy of the forbidden state are free. One of the elements and energy of the forbidden state were fitted to make the $^3S_1$ potential close to the $^1S_0$ potential. The second independent element of M_2 was fitted to make the potential matrix (18) regular at $r = 0$. Our numerical experiments show that such regularization is always possible. This possibility is a direct consequence of the Marchenko transformation method [2] which converts an SS matrix potential into an SD matrix potential (there is a family of phase equivalent SD matrix potentials with different behavior at $r = 0$). Figures 1 and 2 demonstrate how changes of δ influence the partial potential for the $^1S_0$ wave.

As another example of application, we analyzed the modern $S31$ $\pi^- N$ data up to 2 GeV [36]. This S wave must be described by the Lax potential. Because resonances occur in this wave, we extracted the potential phase shifts and constructed the corresponding OP. Its real part is presented in Fig. 1. This potential sustains a forbidden state. Energy and asymptotic constant of the state were fitted to make the potential close to a Gaussian (quark model motivated).

From (35) and (47)–(54), we calculate α and $\alpha_i (i = 1, 2, 3)$ which define the imaginary parts of potentials. α's predicted by (35) and (47)–(54) may be improved by a simple numerical method. Predicted and improved values of α are shown in Figs. 3–5. "Calc. I" means calculations from predicted values of (35); "Calc. II" means calculations from refined values.

We calculated the energy dependence of α from the PWA data [36] for some NN and $\pi^- N$ partial waves. Parameter $\alpha = 1/\xi = \mathrm{Im}\,[f]\,/\mathrm{Re}\,[f]$ is an averaged characteristic of the $q_T q_P$ scattering. Therefore, it must not be equal for various hadron-hadron systems at equal relative velocities. But in the OLGA, this parameter must be independent of the orbital momentum.

Figure 6 shows values of α calculated for single S and P NN partial waves from the PWA data of Ref. [36]. We calculated these parameters for two different models. First model is the repulsive core NN partial potentials (RCP), and second model is the MP (phase shifts of Ref. [36] are raised by 180°). Results of Fig. 6 are not conclusive. α's for $^1S_0$ and $^3P_0$ waves coincide within the accuracy of the PWA beginning from $E_{\mathrm{lab}} \approx$ 1.5 GeV for RCP as well as for MP. Why do α values for the $^3P_1$ wave presented in Fig. 6 differ from the results for the other two waves? There are two conceivable reasons: (i) the dependance of α on angular momentum J and (ii) wrong single energy values for the $^3P_1$ wave. The last reason we consider as the most likely one. In Refs. [38,39], multiple solutions were found at most energy-angle points. There are substantial differences between PWA results of Refs. [36,39] and of Ref. [38] for isovector partial waves above 1 GeV [39]. But the results for two isovector partial waves, $^1S_0$ and $^3P_0$, are very close.

We examine the $\pi^- N$ PWA data of Ref. [36] in Fig. 7. We consider two possible cases which are compatible with the PWA data. In the first case, the phase shifts for $J = 1/2$ partial waves begin from 0°. In the second case, the phase shifts begin from 180°. This assumes forbidden states in these partial waves. The results favor the last concept. In this case,

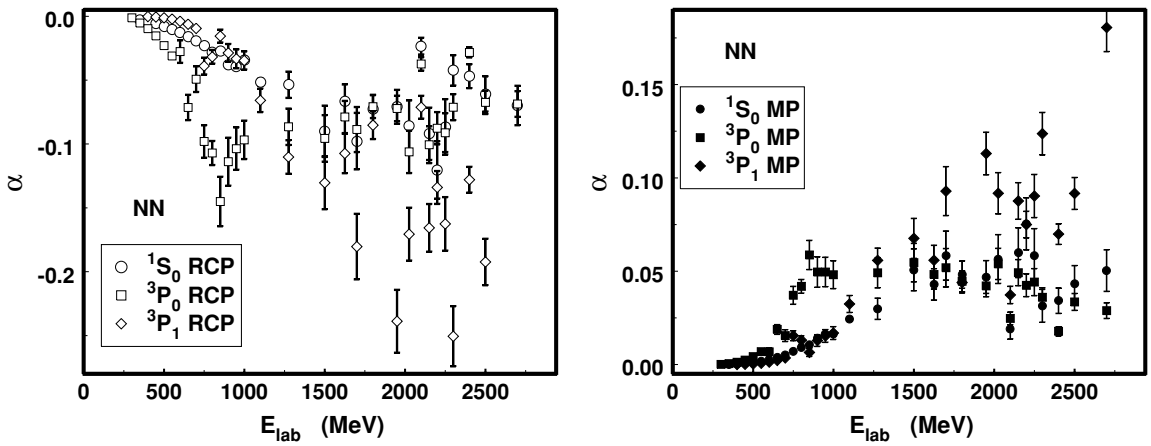

FIG. 6. Parameter α for NN waves predicted by Eq. (35). Left: from data of Ref. [36] (RCP in S and P waves). Right: from data of Ref. [36], but with phase shifts raised up by 180° (MP in S and P waves).

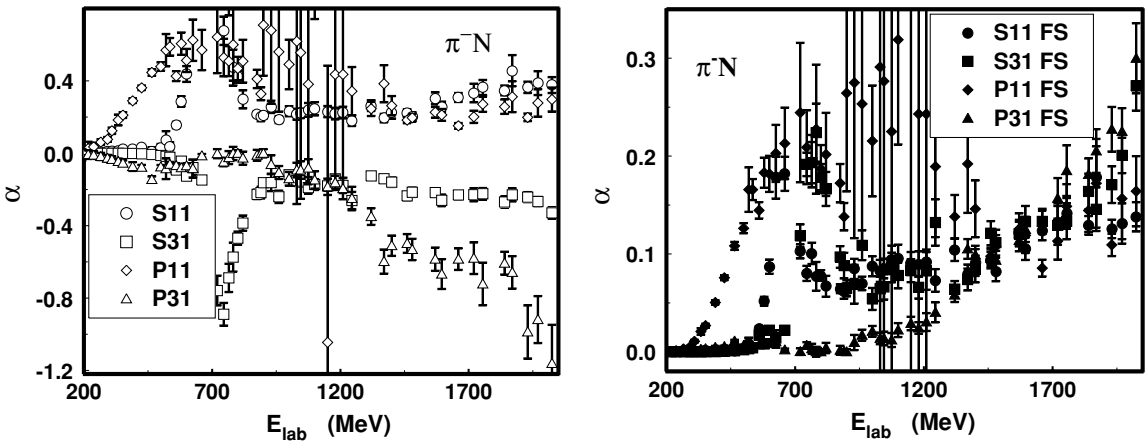

FIG. 7. Parameter α for $\pi^- N$ waves predicted by Eq. (35). Left: from data of [36]. Right: from data of [36] but with phase shifts raised up by 180° (forbidden state, FS, in S and P waves).

the calculated α for all waves is independent on S, L, J from $E_{\text{lab}} \approx 1.3$ GeV.

Our analysis shows that at energies above 1.5 GeV (NN) and above 1.2 GeV ($\pi^- N$), the Lax OP may be valid. Therefore, the general DF potential (61) may be valid at lower energies.

Equation (59) implies that the real and imaginary parts of the Lax interaction do not change sign. Thus, the imaginary part is absorptive everywhere. In our calculations, the imaginary parts of the OP are absorptive from 0 to 1.6 fm at least. The NN D wave is suppressed in this region. Small virtual creation at distances $r > 1.6$ fm is a result of small oscillations of the potentials. These oscillations are characteristic of hadron-hadron inversion potentials [8] (see also NN, $\pi^- N$, and $K^+ N$ inversion potentials of this group in numeric form [40]). Funk, von Geramb, and Amos [8] point out that "the quantum inversion of the SM94 solution (NN) does not give in low partial waves OPEP except on average that might be interpreted as signaling the importance of nonlocality." We may only confirm this statement. On the other hand, the oscillations are inessential for energies above the corresponding thresholds due to uncertainties in PWA. Therefore, the oscillating tails may be cut for these energies without any deterioration of the data description.

Extension of our model to the general case of the DF (61) is model dependent. In this case, we have free parameters that may be calculated in a microscopical quark model of hadrons only. All potentials and inelasticity multipliers (α's) can be accessed via a link to the website [41].

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