Properties of the specific heat and chemical potentials of hadronic matter from CERN/RHIC experiments at relativistic and ultrarelativistic collision energies

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The specific heat, mean hadronic mass excited and its fluctuation are connected to particle production yields and properties of baryon and electric charge chemical potentials (value, slope and curvature). A possible divergence of the specific heat as $1/(T_0 - T)^2$ is discussed. A Hagedorn model with $\rho \sim m^{-\tau} \exp(\beta_h m)$ is studied and restriction on τ are analyzed. Limitations imposed by a Q-g transition are mentioned.

I. INTRODUCTION

A statistical model of very high energy collisions can account for particle production yields from heavy ion collisions [1–5] and also in elementary particle collisions, such as electron-positron collisions [6,7]. The same model contains information regarding the thermodynamic properties of this system of particles. One important thermodynamic property is the specific heat. The importance of a study of the specific heat stems from the fact that sudden changes in the specific heat have been used as signals for phase transitions. A classic example of this statement is the lambda transition in liquid helium. The name lambda transition reflects the lambda shape of the specific heat with a very sharp rise followed by a sudden decrease. In the liquid-gas phase transition of nuclear matter at moderate excitation energy or temperature a very similar sharp peak in the specific heat was found in a theoretical model developed in Refs. [8,9]. For the situation discussed here, a rapid rise in the specific heat is associated with large fluctuations in the mass spectrum of excited particles. Event-by event studies [10] have also been stressed along with temperature fluctuations [11]. Large values of the specific heat are associated with large energy fluctuations. The compressibility is associated with density fluctuations [12]. Fluctuations associated with net electric charge and baryon charge have also been of recent interest [13,14] as well as p_t fluctuations [15]. An overview of fluctuations and correlations can be found in Refs. [16,17].

The organization of this paper is as follows. First, results of the statistical model are given for particle production yields and for thermodynamic quantities. The specific heat (here taken as the heat capacity per particle) is then connected to properties of the particle production yields and conserved charges such as baryon number B and electric charge Z. Limiting cases of the specific heat are discussed which show a connection of it to the mass fluctuation in the spectrum of excited particles. Connections of the specific heat with the behavior of the chemical potential (its value, slope, and curvature) are also developed. The distribution of particles obtained from the detailed analysis of fitting the statistical model to hadronic multiplicities in Pb-Pb collisions at $30A$, $40A$ and $80A$ GeV data [3] is then used to study the behavior of the specific heat. A parametrization of the behavior of the chemical potential with T from this analysis may indicate a sharp increase in the specific heat. A resonance gas model description of the behavior of the chemical potential with temperature is also given which has some similar properties.

II. STATISTICAL MODEL

A. Statistical and thermodynamic properties

The statistical model of heavy ion collisions assumes that hadron multiplicities are the result of an established thermal and chemical equilibrium [18] in some interaction volume V at some temperature T. The interaction volume is the freeze-out volume which is the largest volume over which equilibrium is maintained in the evolution of the fireball. The underlying dynamical processes that establish the equilibrium are the collisions between the particles where the elastic collisions establish the thermal equilibrium and the inelastic collisions develop the chemical equilibrium. For example, pion production via a two-body collision is initially through a process $N + N \rightarrow N + N + \pi$, where N is a nucleon. Since densities and energies are initially very high in the collision, three and higher order collision processes leading to two or more pions can also take place. Moreover, once new particles are produced through the initial collision of the nucleons in the colliding target and projectile nuclei in a nucleus-nucleus collision, these new particles can collide with existing nucleons or other newly created particles to produce other particles. As an example of a secondary process, kaons can be produced via pion induced reactions such as $\pi^+ + \pi^- \rightarrow K^+ + K^-$. A set of coupled reaction equations can be developed as in Ref. [18] to study the approach to equilibrium for each particle. The equilibrium thermodynamic model is based on the assumption that the reactions are initially fast enough compared to expansion time scales in the expanding fireball so that a quasiequilibrium is established. Since reaction rates depend on high powers of the density, i.e., the product of the densities of the colliding particles,

the reaction rate decreases quickly and equilibrium cannot be maintained in an expanding system. A dynamical model of hadronization and freeze out can be found in Ref. [19]. The particle multiplicity distributions in an equilibrium model are given by

$$\langle N_j \rangle = g_j \left(V m_j^2 T / 2\pi^2 \right) \sum_k (\pm)^{k+1} K_2(km_j/T)$$
$$\times \mathrm{Exp}[b_j \mu_B k/T + q_j \mu_Q k/T]/k. \quad (1)$$

The b_j and q_j are the baryon number and charge of particle j which has spin degeneracy g_j and mass m_j. The μ_B and μ_Q are the baryon and charge chemical potentials. The strangeness chemical potential μ_S will be set equal to 0. The energy of particle j is

$$\langle E_j \rangle = g_j \left(V m_j^3 T / 2\pi^2 \right) \sum_k (\pm)^{k+1} \left(\left(\frac{3}{4} \right) K_3 + \left(\frac{1}{4} \right) K_1 \right)$$
$$\times \mathrm{Exp}(k(b_j \mu_B + q_j \mu_Q)/T)/k. \quad (2)$$

The arguments of the Bessel K functions in Eq. (2) are the same as in Eq. (1). The sum over k gives the degeneracy corrections, with the $k = 1$ term the nondegenerate limit. The plus $(+)$ sign is for bosons and the minus $(-)$ sign is for fermions. For nonrelativistic particles in the nondegenerate limit,

$$\langle N_j \rangle = g_j \left(V/\lambda_j^3 \right) \mathrm{Exp}((-m_j + b_j \mu_B + q_j \mu_Q)/T). \quad (3)$$

The $\lambda_j = h/(2\pi m_j T)^{1/2}$. The $\langle E_j \rangle = \langle N_j \rangle (m_j + (3/2)T)$. For zero mass particles: $\langle N_j \rangle = g_j (V T^3/\pi^2) \, \mathrm{Exp}(b_j \mu_B + q_j \mu_Q)/T)$, $\langle E_j \rangle = 3T \langle N_j \rangle$.

B. Features of the chemical potentials μ_B and μ_Q

The two chemical potentials are determined by the constraint conditions on total baryon number B and total charge Q or Z. Namely, $B = \Sigma_j b_j \langle N_j \rangle$ and $Z = \Sigma_j q_j \langle N_j \rangle$. Moreover, the derivative of these chemical potentials with respect to T (all partial derivatives are at constant V in this paper as is the heat capacity C_V) can be obtained from the conditions $\partial B/\partial T = 0$ and $\partial Z/\partial T = 0$. For $\langle N_j \rangle$ given by Eq. (3), then

$$(3B/2) + M_B/T + R_{bb} f_B + R_{bq} f_Q = 0,$$
$$(3Z/2) + M_Q/T + R_{bq} f_B + R_{qq} f_Q = 0. \quad (4)$$

The $M_B = \Sigma_j m_j b_j N_j$, $M_Q = \Sigma_j m_j q_j N_j$, $R_{bb} = \Sigma_j b_j^2 N_j$, $R_{qq} = \Sigma_j q_j^2 N_j$, $R_{bq} = \Sigma_j b_j q_j N_j$, $f_B = \partial \mu_B/\partial T - \mu_B/T$, and $f_Q = \partial \mu_Q/\partial T - \mu_Q/T$. In these equations $N_j = \langle N_j \rangle$. The coupled equations for f_B and f_Q can be solved to give $f_B = (-(3B/2 + M_B/T)R_{qq} + (3Z/2 + M_Q/T)R_{bq})/R$, $f_Q = (-(3Z/2 + M_Q/T)R_{bb} + (3B/2 + M_B/T)R_{bq})/R$. The $R = R_{bb}R_{qq} - R_{bq}^2$. For massless pions the f_B and f_Q become $f_B = (-(3B/2 + M_B/T) R_{qq} + (3Z/2 + 3Z_\pi/2 + M_Q/T) R_{bq})/R$, and $f_Q = (-(3Z/2 + 3Z_\pi/2 + M_Q/T)R_{bb} + (3B/2 +$

$M_B/T)R_{bq})/R$. The $Z_\pi = N_{\pi^+} - N_{\pi^-}$, while the Z is the total conserved charge.

C. Expressions for the heat capacity and specific heat of hadronic matter

The heat capacity C_V in the nondegenerate and nonrelativistic limit for all particles is

$$C_V = \left(\frac{3}{2} \right) \sum_j N_j + \sum_j \left(\frac{3}{2} + \frac{m_j}{T} \right)^2 N_j$$
$$+ \left\{ -\left(\frac{3B}{2} + \frac{M_B}{T} \right)^2 R_{qq} - \left(\frac{3Z}{2} + \frac{M_Q}{T} \right)^2 R_{bb} \right.$$
$$\left. + 2 \left(\frac{3B}{2} + \frac{M_B}{T} \right) \left(\frac{3Z}{2} + \frac{M_Q}{T} \right) R_{bq} \right\} \bigg/ R$$
$$= \left(\frac{3}{2} \right) \sum_j N_j + \sum_j \left(\frac{3}{2} + \frac{m_j}{T} \right)^2 N_j - R_{bb}(f_B)^2$$
$$- R_{qq}(f_Q)^2 - 2R_{bq} f_B f_Q. \quad (5)$$

This is obtained from $C_V = (\partial E/\partial T)_V$ using Eqs. (2) and (3). The resulting equation will involve terms like $\partial \mu_B/\partial T$ which are evaluated using the constraint equations that lead to Eq. (4). Derivations of the fundamental equations are given in Appendix A of this paper. The first term $(3/2)$ on the right-hand side of Eq. (5) is just the ideal gas specific heat at constant V of each nonrelativistic particle, with both mesons and baryons contributing. The second term involves the mass spectrum of all particles produced. The curly bracket or third term in the first equality has three contributions and involves the three coefficients R_{bb}, R_{qq}, and R_{bq}. The second and the third term arise from the possibility that particle distributions change with T or $\partial N_j/\partial T \neq 0$. The second term will be canceled by the third term for a system which has $\partial N_j/\partial T = 0$, for all j. Specifically, consider a system of N_p protons and N_n neutrons, so that $N_p = Z$ and $N_p + N_n = B$. Then $R_{bb} = B$, $R_{qq} = Z$, and $R_{bq} = Z$. When these results are substituted into Eq. (5), along with the results $M_B = m_N B$ and $M_Q = m_N Z$, the second and the third term exactly cancel and $C_V = (3/2)B$, the ideal gas result. The last equality in Eq. (5) replaces the curly bracket term with the functions $-R_{bb}(f_B)^2 - R_{qq}(f_Q)^2 - 2R_{bq} f_B f_Q$ and connects C_V to μ_B and μ_Z.

When pions are taken in the zero mass and nondegenerate limit, the C_V is modified to

$$C_V = \left(\frac{3}{2} \right) \sum_j N_j + \sum_j \left(\frac{3}{2} + \frac{m_j}{T} \right)^2 N_j + 12(N_{\pi^+} + N_{\pi^-} + N_{\pi^0})$$
$$+ \left\{ -\left(\frac{3B}{2} + \frac{M_B}{T} \right)^2 R_{qq} - \left(\frac{3Z}{2} + \frac{3Z_\pi}{2} + \frac{M_Q}{T} \right)^2 R_{bb} \right.$$
$$\left. + 2 \left(\frac{3B}{2} + \frac{M_B}{T} \right) \left(\frac{3Z}{2} + \frac{3Z_\pi}{2} + \frac{M_Q}{T} \right) R_{bq} \right\} \bigg/ R. \quad (6)$$

The first two sums over j on the right-hand side of this equation exclude the pion contribution in their evaluation. The pion

contribution is now contained in the following terms in that equation. The third term is the direct contribution of the pion as if it were independent of the charge conservation law and the curly bracket term arises from the chemical potentials and associated constraints. The independent pion contribution can also be calculated using the results of Eqs. (1) and (2). The exact expression for $C_{V,m}/N_m$ of an unconstrained meson or boson including statistical corrections reads

$$
\begin{aligned}
&C_{V,m}/N_m \\
&= (m/T)^2 \left\{ \sum_k K_2(km/T) + 3(T/m)K_3(km/T)/k \right\} \Big/ \\
&\qquad \sum_k K_2(km/T)/k.
\end{aligned} \tag{7}
$$

The $m \to 0$ limit of this equation is $C_{V,m}/N_m = 12\varsigma(4)/\varsigma(3)$. If statistical corrections are neglected this limit would be 12. The nondegenerate and large m/T limit of Eq. (7) is $C_{V,m}/N_m = (3/2 + m/T)^2 + 3/2$, the characteristic dependence of the first two terms in Eq. (6).

D. Simplified model 1; only conserved baryon charge

To gain some further insight into properties of C_V, a simplified situation of conserved baryon charge will be considered. Then all charge and neutral states of the same baryon will have equal yields. Mesons and baryons will also completely decouple. The specific heat C_V will be a sum of independent contributions from mesons given by Eq. (7) and constrained baryons. The contribution of baryons to $C_{V,B}$:

$$
C_{V,B} = \frac{3}{2} \sum_j N_{j,B} + \sum_j \left(\frac{3}{2} + \frac{m_j}{T} \right)^2 N_{j,B} - \left(\frac{1}{R_{bb}} \right) \left(\frac{3B}{2} + \frac{M_B}{T} \right)^2. \tag{8}
$$

The sums over j are over both baryons and antibaryons. Antibaryons are usually a small fraction of the baryon number. If we allow only baryons with $b_j = 1$, then $R_{bb} = B$, and

$$
C_{V,B} = \frac{3}{2} B + B \frac{1}{T^2} \left(\langle m_B^2 \rangle - \langle m_B \rangle^2 \right) = \frac{3}{2} B + B \frac{1}{T^2} (\delta m_B^2), \tag{9}
$$

where $\langle m_B \rangle = \Sigma_j m_j N_{j,B}/B$, and $\langle m_B^2 \rangle = \Sigma_j m_j^2 N_{j,B}/B$. The enhancement to $C_{V,B} = (3/2)B$ involves the mean square fluctuation in the resonance mass excitation. The mean mass $\langle m_B \rangle = -3T/2 + (R_{bb}/B)(\mu_B/T - \partial\mu_B/\partial T)$. The condition $\partial^2 B/\partial T^2 = 0$ can be used to obtain an expression for the mean square fluctuation and $C_{V,B}/B$ as

$$
\begin{aligned}
(\delta m_B)^2 &= \langle m_B^2 \rangle - \langle m_B \rangle^2 = -3T^2/2 - T^3 \partial^2\mu_B/\partial T^2, \\
C_{V,B}/B &= 3/2 + (\delta m_B)^2/T^2 = -T\partial^2\mu_B/\partial T^2.
\end{aligned} \tag{10}
$$

E. Role of antibaryons

The presence of antibaryons will modify the results given in Sec. II D. For collision energies $\leqslant 100A$ GeV antibaryons make up a few percent of B. From Ref. [3], the antiproton, proton ratio is $\sim 2\%$ for the $80A$ GeV Pb + Pb collision.

This ratio rises to $\sim 5\%$ for the $158A$ GeV collision. For a $\mu_B \sim 300$ MeV and $T \sim 150$, $\exp(-2\mu_B/T) = \exp(-4.) \sim 2\%$, which determines the antiparticle/particle ratio in the absence of μ_Z. The presence of both antibaryons with fraction $y = N_{\bar{B}}/(N_B + N_{\bar{B}})$ and baryons with fraction $x = N_B/(N_B + N_{\bar{B}})$ leads to a modified form for the baryonic (plus antibaryonic) $C_{V,B}$:

$$
\begin{aligned}
C_{V,B}/(N_B+N_{\bar{B}}) &= 3/2 + (x(\delta m_B)^2 + y(\delta m_{\bar{B}})^2)/T^2 \\
&\quad + xy(3 + (\langle m_B \rangle + \langle m_{\bar{B}} \rangle)/T)^2 \\
&= 3/2 + (\delta m_B)^2/T^2 + 4xy(3/2 + \langle m_B \rangle/T)^2.
\end{aligned} \tag{11}
$$

The last equality arises from $\langle m_B \rangle = \langle m_{\bar{B}} \rangle$ and $\delta m_B = \delta m_{\bar{B}}$. The specific heat is also

$$
\begin{aligned}
C_{V,B}/(N_B + N_{\bar{B}}) &= ((N_B + N_{\bar{B}})/B)(T\partial^2\mu_B/\partial T^2) \\
&\quad + 4(N_B N_{\bar{B}}/B^2) f_B^2 \\
&= -\coth(\mu_B/T)T\partial^2\mu_B/(\partial T)^2 \\
&\quad + \mathrm{csch}^2(\mu_B/T)(\partial\mu_B/\partial T - \mu_B/T)^2 \\
&= -\partial(\coth(\mu_B/T)T^2(\partial(\mu_B/T)/\partial T))/\partial T.
\end{aligned} \tag{12}
$$

When $N_{\bar{B}} \to 0$, or μ_B/T very large, the rhs of Eqs. (11) and (12) reduces to Eq. (10). The specific heat now involves both the curvature and slope of the chemical potential and the value of the chemical potential itself. In the limit $x = y = 1/2$ the $C_{V,B}$ of Eq. (12) becomes the unconstrained limit: $C_{V,B}/(N_B + N_{\bar{B}}) = 3/2 + (9/4 + 3\langle m_B \rangle/T + \langle m_B^2 \rangle/T^2)$. The ideal gas limit of $C_{V,B}/B = 3/2$ is realized in the limit $x = 1$, $y = 0$ and $\delta m_B = 0$. The $\langle m_B \rangle$ and $\langle m_B^2 \rangle - \langle m_B \rangle^2 = (\delta m_B)^2$ are given by

$$
\begin{aligned}
\langle m_B \rangle/T &= -3/2 - \coth(\mu_B/T)(\partial\mu_B/\partial T - \mu_B/T), \\
(\delta m_B)^2/T^2 &= -3/2 - \coth(\mu_B/T)T\partial^2\mu_B/(\partial T)^2 \\
&\quad + \mathrm{csch}^2(\mu_B/T)(\partial\mu_B/\partial T - \mu_B/T)^2.
\end{aligned} \tag{13}
$$

Thus, the values of μ_B, its derivative and curvature also contain the information necessary to evaluate various quantities of interest regarding the mass excitation.

F. Some features of the statistical model analysis of CERN/RHIC data

The result of the previous sections can be used to evaluate some features of recent CERN/RHIC data. In Ref. [3], the T dependence of μ_B is given as $T = .167 - .153\mu_B^2$. The T and μ_B are in GeV. It should be stressed that the values of μ_B of Ref. [3] are on the freeze-out surface which has a volume dependence not included in the above expressions. Here, a more general form for the T dependence of μ_B is used, namely $T = T_0 - a(\mu_B)^{1/\beta}$. In the next section a discussion of how such a form may arise is developed in a resonance gas model with V constant. Here it is used as a simple form to study. Then substituting this result into Eqs. (13), (14) gives for $T \to T_0$

$$
\begin{aligned}
\langle m_B \rangle/T &\to \beta \cdot T_0/(T_0 - T), \quad C_{V,B}/(N_B + N_{\bar{B}}) \propto (\delta m_B)^2/T^2 \\
&\to \beta \cdot T_0^2/(T_0 - T)^2.
\end{aligned} \tag{14}
$$

TABLE I. Values of $C_{V,B}$, $\langle m_B \rangle$, and δm_B in MeV excited in various collisions. Main analysis A is the first set of numbers in each row, main analysis B is the second set.

Energy	T	μ_B	$\langle m_B \rangle$	δm_B	$C_{V,B}/B$	$\langle m_B \rangle$	δm_B	$C_{V,B}/(N_B + N_{\bar{B}})$
11.6	118.1	555	1061	277	7.0	1071	277	7.0
	119.1	578	1094	286	7.3	1077	267	7.3
30	139.5	428.6	1295	593	19.6	1297	611	20.7
	140.3	428.	1316	612	20.6	1313	633	21.9
40	147.6	380.3	1513	853	34.9	1517	914	39.9
	145.5	375.4	1426	770	29.5	1444	812	32.8
80	153.7	297.7	1771	1216	64.0	1856	1380	82.1
	151.7	288.9	1629	1070	51.2	1715	1187	62.7
158	157.8	247.3	2113	1676	114.3	2331	2038	168.4
	154.8	244.5	1804	1313	73.4	1952	1513	97.0

Thus, β does not appear as an exponent in the divergence of the specific heat which is quadratic or $1/(T_0 - T)^2$ for all β. The $\langle m_B \rangle \to \infty$ as $T \to T_0$. Using values for μ_B and T given in Ref. [3] for Pb + Pb collisions at 30, 40, 80, and 158A GeV and for Au + Au 11.6A GeV collisions, and the above parametrization of the behavior of μ_B with T with $\beta = 1/2$, the values of $\langle m_B \rangle$, $(\langle m_B^2 \rangle - \langle m_B \rangle^2)^{1/2} \equiv \delta m_B$ in MeV and $C_{V,B}/B$ are shown in Table I. The last three columns are obtained from results in Sec. II E that include antibaryons. The previous three columns are without antibaryons obtained from expressions in Sec. II D. The curvature and slope functions are obtained from this parametrization. The antibaryon case also used this parametrization to evaluate the chemical potential, while the case with just baryons used the chemical potential of Ref. [3]. The error bars in T and μ_B are not given and generate large error bars in the results for the mean mass, mass fluctuation, and specific heat, especially at the higher temperatures. These errors are typically ±20%. Two sets of numbers for each energy appear in the table since Ref. [3] has two main analysis of the data, called A and B. The results show that $C_{V,B}/(N_B + N_{\bar{B}})$ are very different from the ideal gas value 1.5 and increase very sharply.

The contribution of decoupled mesons to C_V can be obtained from Eq. (7). In Ref. [3]: $\pi's = 1356$, $K's = 160$, $K^{*'}s = 66$, $\rho's = 149$, $\eta = 49$, $\omega = 40$, are the multiplicity yields for the 80 A GeV Pb + Pb collision for the low-lying well-known mesons. Using Eq. (7), each of these mesons contributes to C_V as follows: $C_{V,\pi}/N_\pi = 13.2$, $C_{V,K}/N_K = 28.8$, $C_{V,\eta}/N_\eta = 32$, $C_{V,\omega}/N_\omega = 49.5$, $C_{V,\rho}/N_\rho = 48.5$, $C_{V,K^*}/N_{K^*} = 59$. Thus mesons make a very large contribution to the total C_V.

G. Resonance gas model

The behavior of the chemical potential used in the previous section suggests a further investigation in terms of a Hagedorn resonance gas model. The Hagedorn model involves an exponentially increasing density of states that reads $\rho = D_\tau m^{-\tau} \exp(\beta_h m)$, where D_τ is a constant and β_h is the reciprocal of the limiting temperature T_0. For a system with net baryon number ($B = N_B - N_{\bar{B}}$), the constraint imposed by baryon conservation leads to

$$(N_B - N_{\bar{B}})/V)(2\pi/T)^{3/2}$$
$$= 2D_\tau (y/m_0^2)^{\tau-5/2} \left(\int_y^\infty (dx e^{-x}/x^{(\tau-3/2)}) \right) \sinh(\mu_B/T)$$

$$\text{(15)}$$

with $y = (T_0 - T)m_0/T_0 T$ and m_0 the lowest mass. For $\tau < 5/2$, μ_B will go to zero as $(T_0 - T)^{(5/2-\tau)}$, when $T \to T_0$. Thus, in the resonance gas model, the Hagedorn limiting temperature is identified with the limiting value of the chemical freeze-out temperature. The $I_{y,\tau} \equiv \int_y^\infty (dx e^{-x}/x^{(\tau-3/2)}) \to \Gamma(5/2 - \tau)$ for $y \to 0$. For $\tau > 5/2$, μ_B will go to a constant that depends on τ. For $\tau = 5/2 + \eta$, with $\eta > 0$, $(y)^{\tau-5/2} \int_y^\infty dx e^{-x}/x^{(\tau-3/2)} = 1/\eta$ as $y \to 0$. When $5/2 < \tau < 7/2$, the constant $1/\eta$ is approached with ∞ slope. As an example $y^{1/2} I_{y,3} = 2(1 - (\pi y)^{1/2})$. For $7/2 < \tau$, the slope is also finite: $y^{3/2} I_{y,4} = (2/3)(1 - 2y)$, $y^2 I_{y,9/2} = (1/2)(1 - y)$. When $\mu_B \to (T_0 - T)^{(5/2-\tau)}$ is substituted into Eq. (14), a divergence in $C_{V,B}/(N_B + N_{\bar{B}})$ as $1/(T_0 - T)^2$ results. The $(N_B + N_{\bar{B}})/V$ also diverges as $1/(T_0 - T)^{5/2-\tau}$, and $C_{V,B} \sim 1/(T_0 - T)^{9/2-\tau}$. The $N_{\bar{B}}$, N_B asymmetry $(N_B - N_{\bar{B}})/(N_B + N_{\bar{B}}) = \tanh(\mu_B/T)$, with $B = N_B - N_{\bar{B}}$ fixed. This behavior of $C_{V,B}$ is the $B = 0$, $\mu_B = 0$ result of Ref. [20]. Earlier results at $\mu_B \neq 0$, $N_{\bar{B}} = 0$ are in the second paper of Ref. [18]. A more recent discussion can be found in Ref. [21]. An ∞ baryon density presents a problem if baryons are not pointlike. The model is thus limited to densities and T where composite baryons and mesons do not overlap. At some density ~ 1 hadron/fm^3 and $T = T_{Qg} < T_0$, a transition to a quark-gluon phase occurs which truncates behaviors based on this particular model. A discussion of this feature can be found in Ref. [21] where the quark-gluon phase is treated in a statistical model of quark-gluon bags along with the hadron phase.

How well does Eq. (15) describe the μ_B of Table I? When $\tau = 2$, the $\mu_B \to (T_0 - T)^{(1/2)}$ as $T \to T_0$ which is the form of Ref. [3] which will be used. Setting $\mu_B = 375.5$ at $T = 145.5$ and using $m_0/T_0 = 6 \sim m_p/T_0$ while also holding V constant, gives $\mu_B = 625$, 434, 375, 295, and 250 at $T = 119.1$, 140.3, 145.5, 151.7, and 154.8, respectively. These values for μ_B

can be compared to those in Table I which read for 578, 428, 375, 289, 245 for the same T's, respectively, which are on the freeze-out surface. The V dependence of μ_B is contained in Eq. (15), but details of the freeze-out volume must be given. If X is constant then $(\partial\mu_B/\partial T)_X - (\partial\mu_B/\partial T)_V = (\partial\mu_B/\partial V)_T(\partial V/\partial T)_X$. In spite of this difference, Eq. (15) gives a good qualitative representation of μ_B. The Hagedorn model has higher values of $C_{V,B}/(N_B + N_{\bar{B}})$: 11.4, 30.8, 45.4, 84.4, and 129, for these same T's.

III. SUMMARY AND CONCLUSIONS

Properties of the specific heat of hadronic matter produced in very high energy nucleus-nucleus collisions such as those at RHIC and CERN are studied in this paper. A grand canonical statistical model is used to develop expressions for the particle multiplicity distribution and energy caloric equation of state which is then used to obtain the specific heat. The constraints associated with baryon number and electric charge conservation are included to obtain an expression for the specific heat which contains the particle yields, the mass spectrum of produced particles, and three coefficients R_{bb}, R_{qq}, and R_{bq} associated with these constraints. The specific heat is not simply a sum of independent contributions arising from each type of particle. Rather C_V has additional terms which significantly alter its value from this independent particle result. Resonance excitations allow for the possibility that individual particle yields change with T and redistribute the conserved charge and baryon number on other particles. The behavior of C_V is studied in some limiting cases to see how various quantities such as baryonic charge conservation and the production of antibaryons manifest themselves. Using properties of the constraint equations, the specific heat and mass spectrum of excited hadrons are related to properties of μ_B and μ_Q such as their curvature, slope, and value. A recent parametrization of μ_B with T as $\mu_B = a(T_0 - T)^{1/2}$ may lead to a very rapid increase in the baryonic component of the specific heat. In general, if an expression such as $\mu_B = a(T_0 - T)^\beta$ correctly describes the behavior of μ_B near a limiting temperature T_0, then $C_{V,B}/(N_B + N_{\bar{B}})$ would diverge as $1/(T_0 - T)^2$. Moreover, the exponent 2 is independent of the functional form of β used. A resonance gas description of the behavior of the chemical potential and associated specific heat is also given. The mass prefactor exponent τ in a Hagedorn spectrum $\rho = D_B m^{-\tau} \exp(\beta_h m)$ is important in the behavior of the chemical potential. Specifically, if $\mu_B \to 0$, as $T \to T_0$, then $\tau \leqslant 5/2$, with $\mu_B \to (T - T_0)^{5/2-\tau}$. Thus, the Hagedorn limiting temperature of a resonance gas model can be identified with the limiting value of the chemical freeze-out temperature. Moreover, the behavior of μ_B may possibly be used to study τ. The limitations of the Hagedorn result are discussed with respect to a Q-g phase transition.

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APPENDIX A: DERIVATION OF MAIN EQUATIONS

This appendix contains details of the derivation of some of the equations used in this paper. To keep the notation simple, $\langle\ \rangle$ will be omitted and all partial derivatives with respect to T are at constant V.

The heat capacity given by Eq. (4) is obtained from

$$C_V = (\partial E/\partial T)_V = \frac{\partial}{\partial T}\sum_j N_j\left(m_j + \frac{3T}{2}\right)$$

$$= \frac{3}{2}\sum_j N_j + \sum_j\left(m_j + \frac{3T}{2}\right)\frac{\partial N_j}{\partial T}. \qquad (A1)$$

Using Eq. (3), the result for $\partial N_j/\partial T$ is

$$\frac{\partial N_j}{\partial T} = \frac{N_j}{T}\left(\frac{3}{2} + \frac{m_j}{T} + b_j f_B + q_j f_Q\right), \qquad (A2)$$

where $f_B = \partial\mu_B/\partial T - \mu_B/T$ and $f_Q = \partial\mu_Q/\partial T - \mu_Q/T$ arise from the partial derivatives of $\exp((b_j\mu_B + q_j\mu_Q)/T)$ with respect to T. Substituting Eq. (A2) into Eq. (A1) leads to

$$\begin{aligned}
C_V &= \frac{3}{2}\sum_j N_j + \sum_j\left(m_j + \frac{3T}{2}\right) \\
&\quad \times \frac{N_j}{T}\left(\frac{3}{2} + \frac{m_j}{T} + b_j f_B + q_j f_Q\right) \\
&= \frac{3}{2}\sum_j N_j + \sum_j\left(\frac{m_j}{T} + \frac{3}{2}\right)^2 N_j \\
&\quad + \sum_j\left(m_j + \frac{3T}{2}\right)\frac{N_j}{T}(b_j f_B + q_j f_Q) \\
&= \frac{3}{2}\sum_j N_j + \sum_j\left(\frac{m_j}{T} + \frac{3}{2}\right)^2 N_j \\
&\quad + \left(\frac{M_B}{T} + \frac{3}{2}B\right)f_B + \left(\frac{M_Z}{T} + \frac{3}{2}Z\right)f_Q. \quad (A3)
\end{aligned}$$

When the results of Eq. (4) are substituted into Eq. (A3) the two forms of Eq. (5) in Sec. II B arise. Equation (8) of Sec. II D, which contains only the baryon constraint, is obtained in a similar manner. The baryonic component to the heat capacity is

$$\begin{aligned}
C_{V,B} &= \frac{3}{2}\sum_j N_{j,B} + \sum_j\left(\frac{m_j}{T} + \frac{3}{2}\right)^2 N_{j,B} + \left(\frac{M_B}{T} + \frac{3}{2}B\right)f_B \\
&= \frac{3}{2}\sum_j N_{j,B} + \sum_j\left(\frac{m_j}{T} + \frac{3}{2}\right)^2 N_{j,B} + \left(\frac{M_B}{T} + \frac{3}{2}B\right) \\
&\quad \times \left(-\frac{1}{R_{bb}}\right)\left(\frac{M_B}{T} + \frac{3}{2}B\right). \quad (A4)
\end{aligned}$$

In the sums over j in the last equation, the $N_{j,B}$ can either be a baryon or antibaryon. Both will contribute a plus sign to the baryonic specific heat. However, the B and M_B both involve the baryonic charge b_j which has an opposite sign for baryons and antibaryons. In obtaining the last result of Eq. (A4), use

was made of Eq. (4) for a situation with a constraint of baryon conservation only. Then Eq. (4) reads

$$\frac{3}{2}B + \frac{M_B}{T} + R_{bb}f_B = 0. \quad (A5)$$

The f_B of Eq. (A5) can be substituted into Eq. (A4) to eliminate its presence in that equation. The results of Eq. (A4) leads to Eq. (9) of Sec. II D (without antibaryons) using

$$R_{bb} = B = N_B,$$
$$M_B = \langle m_B \rangle B, \quad (A6)$$
$$\sum_j \left(\frac{m_j}{T} + \frac{3}{2}\right)^2 N_{j,B} = \left(\frac{\langle m_B^2 \rangle}{T^2} + 3\frac{\langle m_B \rangle}{T} + \frac{9}{4}\right)B.$$

The results of Eq. (11) of Sec. II E (with antibaryons and $b_j = \pm 1$ only) also follow from Eq. (A4) using

$$R_{bb} = \sum_j b_j^2 N_{j,B} = N_B + N_{\bar{B}},$$
$$B = \sum_j b_j N_{j,B} = N_B - N_{\bar{B}},$$
$$\sum_j N_{j,B} = N_B + N_{\bar{B}}, \quad (A7)$$
$$\langle m_B \rangle = \langle m_{\bar{B}} \rangle, \quad \langle m_B^2 \rangle = \langle m_{\bar{B}}^2 \rangle,$$
$$M_B \equiv \sum_j m_j b_j N_j = \langle m_B \rangle (N_B - N_{\bar{B}}) = \langle m_B \rangle B.$$

Specifically

$$C_{V,B} = \frac{3}{2} + \left(\frac{\langle m_B^2 \rangle}{T^2} + 3\frac{\langle m_B \rangle}{T} + \frac{9}{4}\right) - \frac{B^2}{(N_B + N_{\bar{B}})^2}$$
$$\times \left(\frac{\langle m_B \rangle^2}{T^2} + 3\frac{\langle m_B \rangle}{T} + \frac{9}{4}\right)$$
$$= \frac{3}{2} + \frac{(\partial m_B)^2}{T^2} + 4xy\left(\frac{3}{2} + \frac{\langle m_B \rangle}{T}\right)^2, \quad (A8)$$

where

$$\frac{B^2}{(N_B + N_{\bar{B}})^2} = 1 - 4\frac{N_B}{N_B + N_{\bar{B}}}\frac{N_{\bar{B}}}{N_B + N_{\bar{B}}} = 1 - 4xy. \quad (A9)$$

The relationship that connects the mean mass excited to the chemical potentials follows from Eq. (4). For the case of baryon conservation only the first result in Eq. (4) with $f_Q \to 0$ can be written as

$$\left(\frac{3}{2} + \frac{\langle m_B \rangle}{T}\right)B + (N_B + N_{\bar{B}})f_B = 0. \quad (A10)$$

Thus

$$\frac{\langle m_B \rangle}{T} = -\frac{3}{2} - \frac{(N_B + N_{\bar{B}})f_B}{(N_B - N_{\bar{B}})}$$
$$= -\frac{3}{2} - \coth\left(\frac{\mu_B}{T}\right)\left(\frac{\partial \mu_B}{\partial T} - \frac{\mu_B}{T}\right). \quad (A11)$$

The mean square fluctuation $(\partial m_B)^2$ is obtained from the constraint condition using $\partial^2 B/\partial T^2 = 0 = T\{(\partial/\partial T)(T\partial B/\partial T)\} \equiv D_2 B$:

$$D_2 B = T\frac{\partial}{\partial T}\sum_j b_j N_j \left(\frac{3}{2} + \frac{m_j}{T} + b_j f_B\right)$$
$$= \sum_j b_j N_j \left(-\frac{m_j}{T} + b_j T\frac{\partial f_B}{\partial T}\right)$$
$$+ \sum_j b_j \left(\frac{m_j}{T} + b_j f_B\right)T\frac{\partial N_j}{\partial T}. \quad (A12)$$

Using

$$T\frac{\partial f_B}{\partial T} = T\frac{\partial^2 \mu_B}{(\partial T)^2} - f_B,$$
$$\frac{3}{2}B + \frac{\langle m_B \rangle}{T}B + R_{bb}f_B = 0, \quad (A13)$$
$$\frac{\langle m_B \rangle}{T}R_{bb}f_B = \frac{\langle m_B \rangle}{T}\left(-\frac{3}{2}B - \frac{\langle m_B \rangle}{T}B\right),$$

the $D_2 B$ can be rewritten as

$$D_2 B = \frac{3}{2}B + R_{bb}T\frac{\partial^2 \mu_B}{(\partial T)^2} + \frac{(\partial m_B)^2}{T^2}B$$
$$- \left(\frac{(R_{bb})^2}{B^2} - 1\right)B(f_B)^2 = 0. \quad (A14)$$

Solving for the mass fluctuation $(\partial m_B)^2/T^2$ gives

$$\frac{(\partial m_B)^2}{T^2} = -\frac{3}{2} - \coth\left(\frac{\mu_B}{T}\right)T\frac{\partial^2 \mu_B}{(\partial T)^2}$$
$$+ \csc h^2\left(\frac{\mu_B}{T}\right)\left(\frac{\partial \mu_B}{\partial T} - \frac{\mu_B}{T}\right)^2. \quad (A15)$$

The result of Eq. (A15) can be simplified to the following expression:

$$-\frac{3}{2} - \frac{\partial}{\partial T}\left\{\coth\left(\mu_B/T\right)(T^2\frac{\partial(\mu_B/T)}{\partial T})\right\}. \quad (A16)$$

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