Signals of spinodal hadronization: Strangeness trapping

Volker Koch, Abhijit Majumder, and Jørgen Randrup
Nuclear Science Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, California 94720, USA
(Received 14 September 2005; published 8 December 2005)

If the deconfinement phase transformation of strongly interacting matter is of first order and the expanding chromodynamic matter created in a high-energy nuclear collision enters the corresponding region of phase coexistence, a spinodal phase separation might occur. The matter would then condense into a number of separate blobs, each having a particular net strangeness that would remain approximately conserved during the further evolution. We investigate the effect that such *strangeness trapping* may have on strangeness-related hadronic observables. The kaon multiplicity fluctuations are significantly enhanced and thus provide a possible tool for probing the nature of the phase transition experimentally.

DOI: 10.1103/PhysRevC.72.064903 PACS number(s): 25.75.Nq, 25.75.Gz

I. INTRODUCTION

One of the major goals of high-energy heavy-ion research is to explore the equation of state of strongly interacting matter, particularly its phase structure [1]. Depending on the beam energy, various regions of temperature and baryon density can be explored. Thus systems with a very small net baryon density are formed at the Relativistic Heavy Ion Collider (RHIC) [2]($\sqrt{s} \simeq 200A$ GeV), whereas it is expected that the creation of the highest possible baryon densities occurs at more moderate beam energies ($\sqrt{s} \simeq 10A$ GeV), such as those becoming available at the planned FAIR [3].

Our understanding of the quantum chromodynamics (QCD) phase diagram is best developed at vanishing chemical potential, $\mu_B = 0$, where lattice QCD calculations are most easily carried out. The most recent results indicate that the transformation from a low-entropy hadron resonance gas to a high-entropy quark-gluon plasma occurs smoothly at the temperature is raised, with no real phase transition being present [4]. However, at zero temperature most models predict the occurrence of a first-order phase transition when the density is raised [5], though no firm results are yet available for the corresponding value of the chemical potential, μ_0. However, if the $T = 0$ transformation is in fact of first order, one would expect the phase boundary to extend into the region of finite temperature and terminate at a certain critical end point, (μ_c, T_c) [5]. Indeed, recent lattice QCD results [6] suggest the presence of such a first-order phase transition line and an associated critical end point, though its precise location is not well determined.

It is therefore important to consider how this key issue could be elucidated on the basis of experimental data. Generally, one might expect that if the expanding matter created in a high-energy nuclear collision crosses a first-order phase-transition line then the associated nonmonotonic behavior of the thermodynamic potential might have observational consequences.

From this perspective, the enhancements of the K/π ratio reported for beam energies of 20–30A GeV at the SPS [7] appears intriguing. Because these data present the *only* nonmonotonic behavior seen so far in high-energy heavy-ion collisions, it appears appropriate and timely to study the consequences of a possible first-order phase transition on the production of kaons or, more generally, strange hadrons.

A universal feature of first-order phase transitions is the occurrence of spinodal decomposition, which results from the convex anomaly in the associated thermodynamic potential [8]. This phenomenon occurs when bulk matter, by a sudden expansion or cooling, is brought into the convex region of phase coexistence. Because such a configuration is thermodynamically unfavorable and mechanically unstable, the uniform system seeks to reorganize itself into spatially separate single-phase domains. Moreover, because this spinodal phase separation develops by means of the most unstable collective modes, the resulting domain pattern tends to have a scale characteristic of those modes. This general phenomenon, which is known in many areas of physics and has found a variety of technological applications, appears to be an important mechanism behind the multifragmentation phenomenon in medium-energy nuclear collisions [9,8], where the relevant first-order phase transition is between the nuclear liquid and a gas of nucleons and light fragments. Thus, if a first-order phase transition is encountered during the expansion stage of a high-energy nuclear collision, one might expect that such a spinodal separation might occur. While the resulting enhancement of baryon fluctuations was studied by Bower and Gavin [10] and the prospects for observing such a process via N-body kinematic correlations was discussed by Randrup [11], the present study explores the consequences for the production of strange hadrons.

For a spinodal decomposition to occur, several conditions must be met. First of all, of course, the equation of state must have a first-order phase transition. Though expected, the existence of a first-order phase transition is not yet well established theoretically and it may ultimately have to be determined experimentally by analyzes of the kind considered here. Second, the dynamical trajectory of the bulk matter formed early on must pass through the spinodal region of phase coexistence. Although some calculations suggest that this may happen at FAIR [12–14], this question needs to be investigated more thoroughly. Third, even if the the above conditions are met, the dynamical conditions of the collision must be carefully tuned to ensure, on the one hand, that the bulk of the system is brought into the spinodal region sufficiently quickly to achieve

0556-2813/2005/72(6)/064903(14)/$23.00

a quench, yet, on the other hand, the overall expansion should be slowed down to a degree that will allow the dominant instabilities to grow sufficiently to cause the bulk to break up. Although the conditions for achieving this delicate balance are hard to ascertain theoretically, they may be found by a systematic variation of the beam energy. These open questions notwithstanding, we shall here assume that the matter created in a heavy-ion collision somehow breaks into a number of subsystems, blobs, which subsequently expand and hadronize independently and we then investigate the consequences for the production of strange hadrons. In particular, we wish to ascertain whether such a breakup could lead to an enhancement of the K/π ratio and its fluctuations.

In such a scenario, if the breakup is sufficiently rapid, then whatever net strangeness happens to reside within the region of the plasma that forms a given blob will effectively become trapped and, consequently, the resulting hadronization of the blob will be subject to a corresponding constraint on the net strangeness. As we shall demonstrate, this type of canonical constraint will enhance the multiplicity of strangeness-carrying hadrons, as compared to the conventional (grand-canonical) scenario where global chemical equilibrium is maintained through the hadronic freeze-out [15]. (This occurrence of an enhancement is qualitatively easy to understand, because the presence of a finite amount of strangeness in the hadronizing blob enforces the production of a corresponding minimum number of strange hadrons.) The fluctuations in the multiplicity of strange hadrons, such as kaons, are enhanced even more, thus offering a possible means for the experimental exploration of the phenomenon.

The remainder of this article is organized as follows: First we introduce a suitable idealized model framework and develop the necessary formal tools for the required canonical calculations, with some of the formal manipulations being relegated to appendices. The key features are then brought out in a schematic scenario containing only charged kaons. Subsequently, we present instructive numerical results and also make a quantitative assessment of the importance of global strangeness conservation. We finally give a concluding discussion. The appendices contain a number of technical details.

II. CALCULATIONAL FRAMEWORK

To establish a framework for investigating the effect of the strangeness trapping mechanism, we adopt the following schematic scenario: We start by considering the expanding system when it is still in the plasma phase. At this stage the system is spatially uniform and the strange quarks and antiquarks can be considered as being randomly distributed throughout the system, irrespective of what the net baryon density happens to be. We imagine that the bulk of the expanding and cooling plasma enters the region of phase coexistence and that the associated spinodal instability will cause it to break up into separate subsystems, blobs, which are assumed to all have the same size, as they would tend to have in a spinodal breakup. Each of these blobs now proceed to expand and hadronize while maintaining its net strangeness. The resulting assembly of hadrons is determined at freeze-out by a sampling of the statistical phase space, subject to the appropriate canonical strangeness constraint.

To assess the effect of the strangeness trapping, it is useful to compare the results against the standard scenario, in which the system is assumed to evolve to freeze-out while remaining macroscopically uniform and maintaining global statistical equilibrium. Focusing on a particular subvolume V_h, we describe the resulting hadron gas in the classical grand-canonical approximation. The abundance of a particular hadron specie k is then as follows:

$$\bar{n}_k = \frac{g_k}{2\pi^2}\frac{V_h T_h^3}{\hbar^3 c^3}\,\tilde{K}_2\left(\frac{m_k c^2}{T_h}\right) e^{(\mu_B B_k+\mu_Q Q_k+\mu_S S_k)/T_h}, \qquad (1)$$

where $\tilde{K}_2(x) \equiv x^2 K_2(x)$ is regular at $x=0$ and a hadron of the specie k has baryon number B_k, electric charge Q_k, and strangeness S_k. The average values of baryon number, charge, and strangeness in the volume V_h then readily follow,

$$\bar{B} = \sum_k B_k \bar{n}_k, \quad \bar{Q} = \sum_k Q_k \bar{n}_k, \quad \bar{S} = \sum_k S_k \bar{n}_k. \qquad (2)$$

The values of the freeze-out temperature T_h and the three chemical potentials μ_B, μ_Q, and μ_S will be determined by fits to the experimental yield ratios (see later). In this treatment, the individual hadron species are statistically independent and the associated multiplicities have Poisson distributions, so the multiplicity variances are equal to the mean values, $\sigma_k^2 = \bar{n}_k$.

We now return to the particular spinodal scenario described above, where we assume that the plasma has broken up into separate blobs. We first consider the distribution of strangeness within the blobs and then treat their subsequent hadronic freeze-out.

If a given plasma blob is only a small part of the total system, its statistical properties may be treated in the grand-canonical approximation. The various quark (and gluon) species are then independent. Furthermore, because there is no bias on the overall strangeness, the s and $\bar{s}$ quarks have identical partition functions, $\mathcal{Z}_s = \mathcal{Z}_{\bar{s}}$, where

$$\ln \mathcal{Z}_s = g_q \int \frac{d^3\mathbf{r}_s d^3\mathbf{p}_s}{h^3}\, \ln[1+e^{-\epsilon_s/T_q}] \qquad (3)$$

$$\approx \frac{3}{\pi^2}\frac{V_q T_q^3}{\hbar^3 c^3}\tilde{K}_2\left(\frac{m_s}{T_q}\right) \equiv \zeta_s. \qquad (4)$$

Here $g_q = 6$ is the quark spin-color degeneracy, T_q is the plasma temperature, and V_q is the volume of the particular blob considered at the time when its strangeness is frozen in. The energy ϵ_s is given by $\epsilon_s^2 = p_s^2 + m_s^2$, where we use the mass $m_s = 150\,\mathrm{MeV}$. Whereas the first expression is the exact fermionic form, the last relation emerges in the classical limit which we shall adopt here for simplicity (see Appendix A). Then the s and $\bar{s}$ multiplicities, ν and $\bar{\nu}$, have Poisson distributions characterized by the mean value $\zeta_s = \ln \mathcal{Z}_s$. The total strangeness content in the blob is then $S_0 = \bar{\nu} - \nu$. Because the different quark flavors are distributed independently, the resulting probability for ending up with a given blob strangeness S_0 is independent of the prevailing baryon and charge contents and can be expressed as a modified

Bessel function,

$$P(S_0) = \sum_{\nu\bar{\nu}} \frac{\zeta_s^{\nu}\zeta_s^{\bar{\nu}}}{\nu!\bar{\nu}!} e^{-2\zeta_s} \delta_{\bar{\nu}-\nu,S_0} = I_{S_0}(2\zeta_s)\, e^{-2\zeta_s}. \quad (5)$$

We note that the corresponding ensemble average value of S_0 vanishes, $\prec S_0 \succ = 0$, while its ensemble variance is $\sigma_{S_0}^2 = \sum_{S_0} S_0^2 P(S_0) = 2\zeta_s$.

To avoid spurious correlations, we take account of the fact that the presence of a certain net strangeness in a given blob introduces a bias on its baryon number and charge, relative to the overall grand-canonical averages $\bar{B}$ and $\bar{Q}$ of the grand-canonical reference scenario, Eq. (2). Indeed, each particular value of S_0 determines a canonical subensemble of blobs that have modified distributions of baryon number and charge. It is elementary to show that these are shifted by amounts proportional to S_0, so the corresponding conditional expectation values become

$$\langle B\rangle_{S_0} = \bar{B} - \tfrac{1}{3}S_0, \quad \langle Q\rangle_{S_0} = \bar{Q} + \tfrac{1}{3}S_0. \quad (6)$$

It also follows that the ensemble correlations of B and Q with S are given by the following:

$$\sigma_{BS} = \prec BS \succ = -\tfrac{1}{3}\sigma_{S_0}^2, \quad (7)$$

$$\sigma_{QS} = \prec QS \succ = +\tfrac{1}{3}\sigma_{S_0}^2. \quad (8)$$

Each isolated blob is assumed to expand further while hadronizing, until the freeze-out volume $V_h = \chi V_q$ has been reached. Its temperature is then $T_h \leqslant T_q$. Rough approximations to the equation of state [16] and the demand of energy conservation suggest that the expansion factor is $\chi \approx 3$, to within a factor of 2 or so. The blob has now transformed itself into a hadron resonance gas that we describe as a canonical ensemble characterized by the strangeness S_0 of the precursor plasma blob. The baryon number and charge are treated grand-canonically and the demand that the expected baryon and charge contents match the above conditional values (6) then determines the associated *biased* chemical potentials μ'_B and μ'_Q, where the primes indicate that these pertain to the biased (canonical) ensemble characterized by the particular value of S_0. Though required for formal consistency and included in the calculations, this refinement is not quantitatively important.

For the description of the hadron gas, we include 124 hadronic species $\{k\}$, from the $\pi^0(135)$ and up though the $\Omega^-(1672)$. Each specie is characterized by its one-particle partition function,

$$\zeta_k = g_k \int \frac{d^3\mathbf{r}_k d^3\mathbf{p}_k}{h^3} e^{-(\epsilon_k - \mu'_B B_k - \mu'_Q Q_k)/T_h} \quad (9)$$

$$= \frac{g_k}{2\pi^2}\frac{V_h T_h^3}{\hbar^3 c^3}\tilde{K}_2\left(\frac{m_k c^2}{T_h}\right) e^{(\mu'_B B_k + \mu'_Q Q_k)/T_h}. \quad (10)$$

The nonstrange hadrons, which are not affected directly by the canonical strangeness constraint, have grand-canonical distributions governed by the biased chemical potentials $\mu'_B(S_0)$ and $\mu'_Q(S_0)$. The total partition function is therefore of the form $\mathcal{Z} = \mathcal{Z}_{\rm gc}^{\{S=0\}}\mathcal{Z}_{S_0}^{\{S\neq 0\}}$. We describe briefly below how we obtain the canonical partition function for the strange hadrons, $\mathcal{Z}_{S_0}^{\{S\neq 0\}}$, and refer to Appendix B for more details.

For this task, we organize the hadron species according to their strangeness (an alternate approach was employed in Refs. [17,18]). For each strangeness class $S = \pm 1, \pm 2, \pm 3$, we introduce the *effective* one-particle partition function $\zeta_S = \sum_\kappa \zeta_\kappa \delta_{S_\kappa,S}$, which accounts for all hadron species having the particular strangeness S. The corresponding *generic* multiplicities are denoted by N_S, the multiplicity of hadrons having the specified strangeness S (e.g., $N_{+1} = n_{K^0} + n_{K^+} + n_{\bar{\Lambda}} + \cdots$). It should be noted that ζ_S is not simply related to ζ_{-S} when the system has a net baryon density. In terms of these quantities we then have the following:

$$\mathcal{Z}_{S_0}^{\{S\neq 0\}} = \prod_{S\neq 0}\left[\sum_{N_S\geqslant 0}\frac{\zeta_S^{N_S}}{N_S!}\right]\delta\left(\sum_S N_S S - S_0\right). \quad (11)$$

It is convenient to obtain this total partition function recursively by adding two conjugate strangeness classes $\{\pm S\}$ at a time,

$$\mathcal{Z}_{S_0}^{\{S=\pm1,\pm2\}} = \sum_{S_2=0,\pm2,\ldots}\mathcal{Z}_{S_0-S_2}^{\{\pm1\}}\mathcal{Z}_{S_2}^{\{\pm2\}}, \quad (12)$$

$$\mathcal{Z}_{S_0}^{\{S=\pm1,\pm2,\pm3\}} = \sum_{S_3=0,\pm3,\ldots}\mathcal{Z}_{S_0-S_3}^{\{\pm1,\pm2\}}\mathcal{Z}_{S_3}^{\{\pm3\}}, \quad (13)$$

where the canonical partition function for a single pair of conjugate strangeness classes is given by an expression analogous to Eq. (5) [18],

$$\mathcal{Z}_{S_0}^{\{\pm S\}} = \sum_{N_+N_-}\frac{\zeta_{+S}^{N_+}}{N_+!}\frac{\zeta_{-S}^{N_-}}{N_-!}\delta_{(N_+-N_-)S,S_0}$$

$$= \left(\frac{\zeta_{+S}}{\zeta_{-S}}\right)^{S_0/2} I_{S_0}(2\sqrt{\zeta_{+S}\zeta_{-S}}). \quad (14)$$

The corresponding correlated conditional distribution of the generic multiplicities $\{N_S\}$ is then determined. Furthermore, recursive expressions can readily be derived for the associated multiplicity moments, such as $\langle N_S\rangle_{S_0}$ and $\langle N_S N_{S'}\rangle_{S_0}$ (see Appendix B). Once we know the number of hadrons with a given strangeness S, N_S, we can obtain the multiplicities $\{n_\kappa\}$ of the individual strange species from the corresponding multinomial, for example, distributions,

$$P_\kappa(n_\kappa) = \frac{N_S!}{\zeta_S^{N_S}}\frac{\zeta_\kappa^{n_\kappa}}{n_\kappa!}\frac{(\zeta_S-\zeta_\kappa)^{N_S-n_\kappa}}{(N_S-n_\kappa)!}. \quad (15)$$

By proceeding as described above, it is possible to treat the entire ensemble of possible blobs and by numerical simulation generate a sample of "events" consisting of the resulting primordial hadrons. (The importance of subsequent decays is discussed later.) These "final states" can then be analyzed as idealized experimental data. This can be done both in the spinodal scenario where individual blobs are treated canonically as well as in the grand-canonical reference scenario. For instructive purposes, we also consider a restricted canonical scenario in which the strangeness of a blob is always required to vanish, $S_0 = 0$.

The inclusion of the bias effect [see Eq. (6)], as expressed through μ'_B and μ'_Q in Eq. (10), ensures that the ensemble correlation of baryon number and charge with strangeness remains the same as it were in the plasma, even though our treatment of the hadron production conserves B and Q only on the average for each S_0,

$$\sigma_{BS} \equiv \prec BS \succ - \prec B \succ \prec S \succ = \sum_{S_0} \langle B \rangle_{S_0} S_0 P(S_0)$$
$$= \sum_{S_0} \left(\bar{B} - \frac{1}{3} S_0 \right) S_0 P(S_0) = -\frac{1}{3} \sigma_{S_0}^2, \quad (16)$$
$$\sigma_{QS} \equiv \prec QS \succ - \prec Q \succ \prec S \succ = \sum_{S_0} \langle Q \rangle_{S_0} S_0 P(S_0)$$
$$= \sum_{S_0} \left(\bar{Q} + \frac{1}{3} S_0 \right) S_0 P(S_0) = +\frac{1}{3} \sigma_{S_0}^2. \quad (17)$$

III. SCHEMATIC SCENARIO

Before discussing the results of such numerical simulations, it is instructive to first illustrate the effect of strangeness trapping in a schematic scenario. For this purpose, assume that plasma blobs of strangeness S_0 hadronize into charged kaons only. We furthermore ignore the chemical potentials (μ_B is ineffective for mesons and μ_Q is anyway rather small). The resulting $K^\pm$ multiplicity distributions are then given in terms of the (common) one-particle partition function,

$$\zeta_K = g_K \int \frac{d^3\mathbf{r} d^3\mathbf{p}}{h^3} e^{-\epsilon_K/T_0} = \frac{g_K}{2\pi^2} \frac{V_h T_h^3}{\hbar^3 c^3} \tilde{K}_2 \left(\frac{m_K}{T_h} \right). \quad (18)$$

In particular, the average kaon multiplicities resulting from blobs of strangeness S_0 can be expressed as an asymptotic expansion in $1/\zeta_K$,

$$\langle n_{K^\pm} \rangle_{S_0} = \zeta_K \pm \frac{1}{2} S_0 - \frac{1}{4} + \frac{4S_0^2 - 1}{32\zeta_K} + \mathcal{O}\left(\frac{1}{\zeta_K^2} \right). \quad (19)$$

Ignoring the relatively small effects of quantum statistics on the quark multiplicity distribution, we found above that the ensemble of values $S_0 = \bar{\nu} - \nu$ is characterized by $\prec S_0 \succ = 0$ and $\sigma_{S_0}^2 = 2\zeta_s$. The ensemble average $K^\pm$ multiplicities are then as follows:

$$\prec n_{K^\pm} \succ = \zeta_K - \frac{1}{4} + \frac{8\zeta_s - 1}{32\zeta_K} + \mathcal{O}\left(\frac{1}{\zeta_K^2} \right). \quad (20)$$

Furthermore, the corresponding expressions for the ensemble multiplicity (co-)variances are as follows:

$$\sigma_{K^+}^2 = \sigma_{K^-}^2 = \frac{1}{2}\zeta_K + \frac{1}{2}\zeta_s + \mathcal{O}\left(\frac{1}{\zeta_K} \right), \quad (21)$$

$$\sigma_{K^+K^-}^2 = \frac{1}{2}\zeta_K - \frac{1}{2}\zeta_s + \mathcal{O}\left(\frac{1}{\zeta_K} \right). \quad (22)$$

It should be noted that these expressions imply that $\sigma^2(n_{K^+} - n_{K^-}) = 2\zeta_s = \sigma_{S_0}^2$ as required by the fact that $S_0 = n_{K^+} - n_{K^-}$ in each blob.

In the special case where ζ_s happens to equal ζ_K we recover the grand-canonical result, $\prec n_{K^\pm} \succ = \sigma_{K^\pm}^2 = \zeta_K$ and $\sigma_{K^+K^-} = 0$, reflecting the fact that the grand-canonical ensemble can be built from an ensemble of canonical subensembles. But generally the above results differ from those of the grand-canonical scenario. In particular, when $\zeta_K < \zeta_s = (1 + \Delta)\zeta_K$, as tends to be the case because of the large degeneracy of the quarks, the kaon production is *enhanced* by the strangeness trapping,

$$\prec n_{K^\pm} \succ \approx \zeta_K + \tfrac{1}{4}\Delta, \quad (23)$$

$$\sigma_{n_{K^\pm}}^2 \approx \zeta_K + \tfrac{1}{2}\Delta\zeta_K, \quad (24)$$

$$\sigma_{n_{K^+}n_{K^-}} \approx -\tfrac{1}{2}\Delta\zeta_K. \quad (25)$$

The negative value of the covariance reflects the fact positive (negative) values of S_0 favors positive (negative) kaons, so an excess of positive kaons is likely to be accompanied by a deficit of negative kaons and vice versa. We note that while the relative increase of the average multiplicities is of the order of Δ/ζ_K, the relative effect on the fluctuations is of the order of Δ, thus being enhanced by a factor of ζ_K. This basic feature suggests that the fluctuations are preferable to the averages for probing the strangeness trapping phenomenon.

IV. RESULTS AND DISCUSSION

We now discuss results obtained by numerical sampling of the canonical partition function described above. In our calculations, which serve to merely illustrate the effect of the strangeness trapping mechanism, we use the freeze-out hadron populations directly in the analysis and make no attempt to include subsequent electroweak decays. Obviously, this complication should be addressed before a quantitative confrontation with data can be made.

The overall grand-canonical reference environment is determined as follows. Using the baryon chemical potential μ_B as a control parameter, we obtain the freeze-out temperature T_h from the fit to the data obtained in Ref. [15], yielding $T_h(\mu_B) \approx T_0[1 - (\mu_B/m_N)^{5/2}]$ with $T_0 = 170$ MeV. Subsequently, we perform a grand-canonical iteration to determine those values of μ_Q and μ_S that ensure $\bar{Q} = \alpha\bar{B}$ and $\bar{S} = 0$, where $\alpha = 0.4$, which is representative of Z/A for gold. The resulting values are shown in Fig. 1 as functions of μ_B. As μ_B is increased,

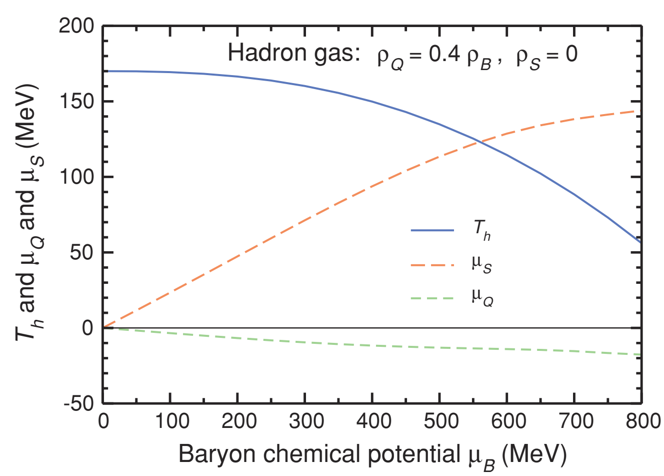

FIG. 1. (Color online) The employed freeze-out values of the temperature T_h and the chemical potentials μ_Q and μ_S as functions of μ_B, as obtained by fitting to the result for $T_h(\mu_B)$ [15] and demanding that $\bar{Q} = 0.4\bar{B}$ and $\bar{S} = 0$.

baryons become favored over antibaryons, so there will a bias of hyperons over antihyperons. To counterbalance the associated net negative strangeness (recall that S_Λ is -1), μ_S must take on a positive value to ensure a compensating excess of kaons over antikaons, leading to an approximate proportionality between μ_S and μ_B. This balancing of the strangeness in turn leads to a small bias toward positive net charge and μ_Q must therefore be correspondingly negative. Although the resulting value is also approximately proportional to μ_B, the absolute value is rather small and one might for most purposes simply take μ_Q to be zero.

In our spinodal scenario the value of S_0 is sampled at the plasma stage and then canonically conserved though the hadronization process. Each plasma blob has the volume V_q (usually taken to be $50\,\mathrm{fm}^3$) and its strangeness content is determined by sampling ν and $\bar{\nu}$ at the plasma temperature T_q, which is either taken to be equal to T_0 $(= 170\,\mathrm{MeV})$ or to equal the hadronic freeze-out temperature $T_h(\mu_B)$, which should provide approximate upper and lower bounds. At $\mu_B = 0$, where $T_q = 170\,\mathrm{MeV}$, the width of the strangeness distribution in such a blob is $\sigma_{S_0} = 5.74$. If we take $T_q = T_h$, the distribution will grow narrower as μ_B is increased and at $\mu_B = 400\,\mathrm{MeV}$, where the temperature has decreased to 150 MeV, we have $\sigma_{S_0} = 4.65$.

The blob then expands until the freeze-out volume $V_h = \chi V_q$ is reached. (Guided by rough approximations to the equation of state, we usually employ an expansion factor of $\chi = 3$, i.e., $V_h = 150\,\mathrm{fm}^3$.) At this point, the blob has transformed itself into an assembly of hadrons whose abundances are assumed to be governed by the canonical distribution Eq. (15) associated with the particular value of S_0, and the modified chemical potentials $\mu'_B(S_0)$ and $\mu'_Q(S_0)$ [see Eq. (10)] that have been adjusted for each particular blob strangeness S_0 to ensure matching of the corresponding biased baryon and charge contents, as explained earlier.

A. Kaon multiplicity distributions

We first discuss the results for the charged kaon multiplicity distributions. Figure 2 shows the resulting average $K^\pm$ multiplicities for three different scenarios. The first scenario is the usual global grand-canonical treatment (see above), whereas the second is our spinodal scenario in which the blob strangeness is determined at the plasma stage and then kept fixed during the hadronization. Both of these scenarios consider all possible values of S_0 but while the distribution of this quantity is determined at the hadronic freeze-out in the former, it is determined already in the plasma in the latter. In the third "restricted canonical" scenario the blob strangeness is always required to vanish, $S_0 = 0$.

The various scenarios lead to rather similar results. The K^- multiplicity decreases steadily as a result of the strangeness balancing explained above, whereas the K^+ yield initially increases, for the same reason, but then, as the freeze-out temperature $T_h(\mu_B)$ begins to decrease noticeably, the overall hadron production decreases, thus yielding a decreasing behavior of the K^+ curve. Generally, the average multiplicities are affected very little by the blob formation, relative to the grand-canonical treatment, and the spinodal results are therefore omitted for K^-, while only the results for $T_q = T_0$ are shown for K^+. These exhibit a progressively increasing enhancement that amount to 2% at $\mu_B = 400\,\mathrm{MeV}$. By contrast, for the restricted scenario, where only blobs having $S_0 = 0$ are admitted, there is a reduction in the averages by a few percentages for all μ_B.

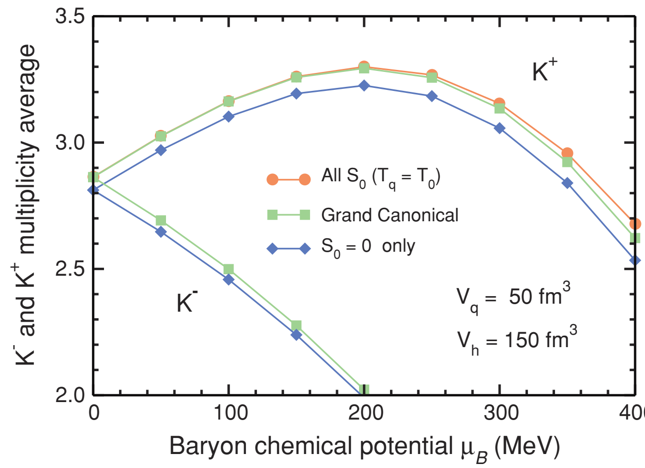

FIG. 2. (Color online) The average $K^\pm$ multiplicities as functions of the baryon chemical potential μ_B in three scenarios: (a) the standard grand-canonical treatment where the hadronic freeze-out occurs in the volume V_h at the temperature $T_h(\mu_B)$, (b) our spinodal treatment in which the blob strangeness S_0 is sampled in the plasma volume V_q at the temperature $T_q = T_0$ and then conserved through the hadronic freeze-out, and (c) the restricted canonical treatment admitting only $S_0 = 0$.

The picture changes when the multiplicity fluctuations are considered, as illustrated in Fig. 3, where the corresponding multiplicity variances are shown for the positive kaons. While the overall behavior is qualitatively similar to the behavior of the averages, there are several important differences. First,

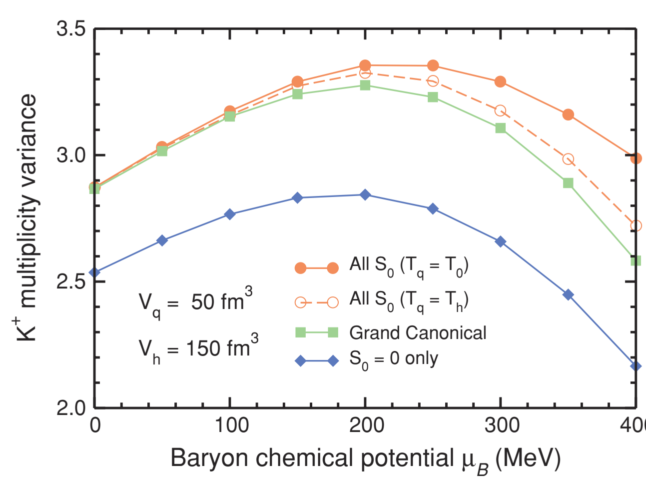

FIG. 3. (Color online) The variance in the K^+ multiplicity as functions of the baryon chemical potential μ_B for the various scenarios considered in Fig. 2, using either T_0 or T_h for the plasma temperature T_q in the spinodal scenario.

in the restricted scenario (where only $S_0 = 0$ is included) the suppression of the variance is significantly larger than was the case for the average, amounting here to 8–10%. Furthermore, at the larger values of μ_B, where the net baryon density becomes significant, the grand-canonical variance suffers more from the decreasing temperature than the spinodal variance. This important divergence is a result of the fact that the larger average baryon number implies a correspondingly larger baryon-number variance as well and therefore also a larger variance in the strangeness (because strangeness in the plasma is carried exclusively by quarks and antiquarks that also carry baryon number). Most importantly, there is a significant dependence on the employed plasma temperature T_q. To illustrate this effect, we show results for two extreme values, namely $T_q = T_0 = 170$ MeV and $T_q = T_h(\mu_B)$, and for these the difference amounts to 10% at $\mu_B = 400$ MeV.

B. Multiplicity ratios

From the experimental perspective, it is more convenient to consider multiplicity *ratios*, and we therefore show in Fig. 4 the average K^+/π^+ and K^-/π^- ratios, for the same three scenarios. When μ_B is positive there is a preference for K^+ over K^- and hence K^+/π^+ will increase while K^-/π^- decreases. This behavior is practically linear because the suppression from the decreasing temperature affects all hadrons species. Because there is a (small) tendency for the π and K multiplicities to vary in concert, the difference between the various scenarios is reduced. In particular, there is hardly any difference to be seen for the K^-/π^- ratio.

We now turn to the corresponding variances. Because the variance of the K/π ratio decreases in inverse proportion to the size of the system, it is convenient to multiply by the mean pion multiplicity and thus obtain a result that approaches a constant for large volumes, $\langle\pi^\pm\rangle\sigma^2(K^\pm/\pi^\pm)$. The resulting results for the positively charged hadrons are shown in Fig. 5. They are qualitatively similar to those for $\langle K^\pm/\pi^\pm\rangle$ (Fig. 4). But although variances in the ratios are less sensitive to the

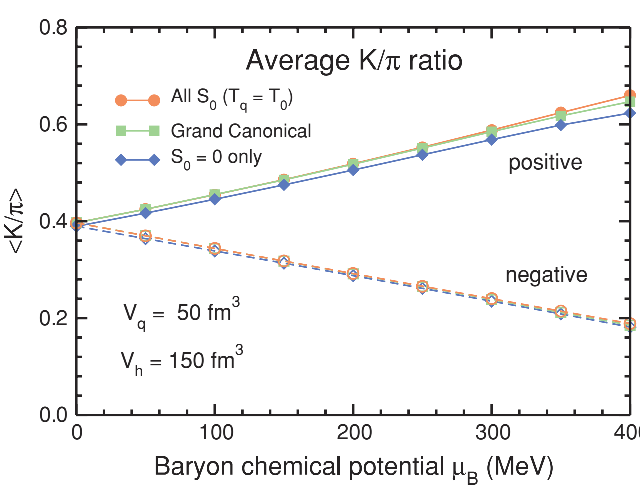

FIG. 4. (Color online) The average value of the K/π ratio for either positive (increasing) or negative charges (decreasing) for the three scenarios considered in Figs. 2 and 3.

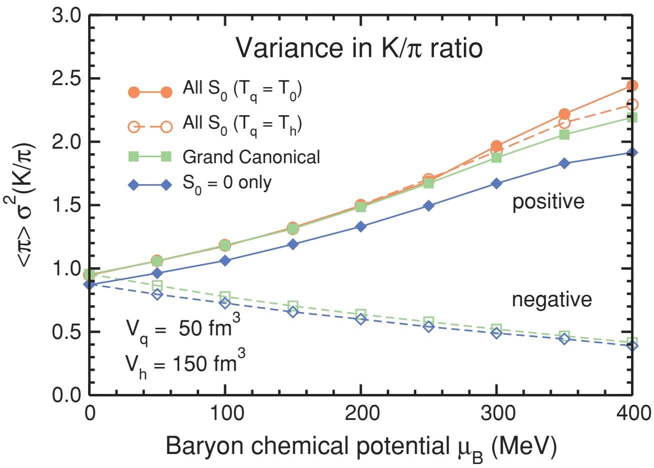

FIG. 5. (Color online) The normalized variance in the K/π ratio for either positive (increasing) or negative charges (decreasing) for the three scenarios considered in Figs. 2 and 3. The variances have been multiplied by the corresponding average pion multiplicity to make the result scale invariant.

specific scenario than the kaon variances themselves (Fig. 3), the differences are still clearly brought out. Furthermore, as expected from Fig. 3, an increase of the plasma temperature T_q increases the resulting values. The fluctuations in the K^+/π^+ ratio may thus offer a suitable observable that is sensitive to a clumping-induced trapping of strangeness in the expanding matter prior to the hadronization.

C. Dependence on the expansion factor

The above results have been obtained for a given expansion factor $\chi \equiv V_h/V_q = 3$. This value should be taken only as a rough approximation to what might actually happen and because the results are sensitive to this parameter it is of interest to consider also other degrees of expansion. This aspect is illustrated in Fig. 6, where the average of the K^+/π^+ ratio and its variance (normalized by $\langle\pi^+\rangle$) are shown as a function of χ for $\mu_B = 300$ MeV. Since we keep the hadronic freeze-out volume equal to $V_h = 150\,\mathrm{fm}^3$ to facilitate the comparisons, a larger expansion factor χ implies a smaller plasma volume V_q. Because both averages and variances are proportional to volume, a smaller V_q shrinks the distribution of the blob strangeness S_0 (so it has smaller mean and variance). Consequently, the resulting values $\langle K^+/\pi^+\rangle$ and $\langle\pi^+\rangle\sigma^2(K^+/\pi^\pm)$ are decreasing functions of χ. However, this dependence is not dramatic: A doubling of χ from 2 to 4 reduces the average ratio by less than 2% and the normalized variance by less than 10%.

It is important to recognize that there are two opposing effects: One is the basic fact there are more degrees of freedom in the deconfined plasma phase than in the confined hadron gas, which enhances the fluctuations. (At $T = T_0$ and $\mu_B = 0$ we have $\zeta_s \approx 0.33\,V_q$, whereas $\zeta_{S=\pm1} \approx 0.081\,V_h$, so the effective degeneracy of the plasma is approximately 4 times larger than that of the hadron gas, which would then be compensated with an expansion factor of $\chi \approx 4$, as indeed borne out by the calculation.) However, as χ is increased from unity, this

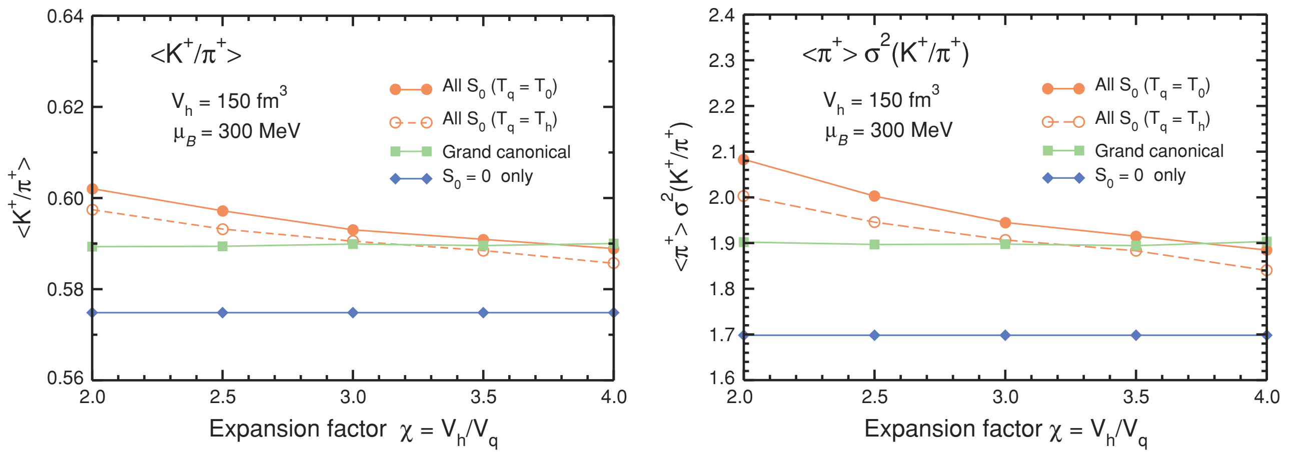

FIG. 6. (Color online) The average of the K^+/π^+ ratio (left) and its variance multiplied by the mean π^+ multiplicity (right) as obtained at $\mu_B = 300$ MeV for the scenarios considered in Fig. 5, shown as a function of the volume expansion factor $\chi = V_h/V_q$.

advantage is being steadily offset by the ever larger volume of the freeze-out configuration, $V_h = \chi V_q$. The crossover happens to occur at $\chi \approx 3.3$, a value rather near our adopted estimate, $\chi = 3$. Should it turn out that there is less change in volume from the formation of the plasma blob to the hadronic freeze-out, then the relative effect of the strangeness trapping will be considerably larger. For example, if χ were only 10% smaller, the effect would be about twice as large.

D. Effect of source mixing

The above studies have considered only the hadrons resulting from a single blob which, presumably, populate a certain limited kinematical region centered around the velocity of the original blob [11]. However, even if a spinodal decomposition into kinematically separated blobs were to occur, it would not be experimentally feasible to restrict the measurement to include only those hadrons resulting from a single blob. Rather, one should generally expect that a given detection acceptance will admit hadrons originating from more than one single blob. Thus one needs to address the fact that a mixing of hadrons from different sources will degrade the strangeness-trapping signal.

To elucidate this practically important feature, we calculate the production from several different blobs and combine the resulting hadrons into a single "event" before performing the analysis. The resulting variance in the K/π ratio (multiplied by the mean π^+ multiplicity) is shown in Fig. 7 as a function of the number of blobs whose products have been combined. Although the value drops by about a factor of 2 when going from a single blobs to two combined blobs, it subsequently stabilizes and quickly approaches a constant as ever more blobs are combined. Furthermore, the relative increase when going from the standard grand-canonical scenario one of our spinodal scenarios remains nearly unaffected by the numbers of blobs combined in the analysis. This result indicates that the suggested signal of the strangeness trapping is robust against the inevitable source mixing and, consequently, it may in fact be practically observable.

E. Global strangeness conservation

We finally analyze in some detail the role played by the overall conservation of strangeness in each event.

In the above studies, we have treated the strangeness in a given plasma blob as a grand-canonical variable, which is well justified when the blob volume forms only a small part of the total system. To investigate the quality of this approximation, we consider the effect of constraining the combined strangeness of N individual blobs to zero. The corresponding canonical partition function for the combined system is then

$$\mathcal{Z}_0^{(1\cdots N)} = \prod_{i=1}^{N}\left[\sum_{S_i}\mathcal{Z}_{S_i}^{(i)}\right]\delta_{S_1+\ldots+S_N,0}, \tag{26}$$

where $\mathcal{Z}_{S_i}^{(i)}$ is the canonical partition function for blob i having the strangeness S_i. For simplicity, we shall assume

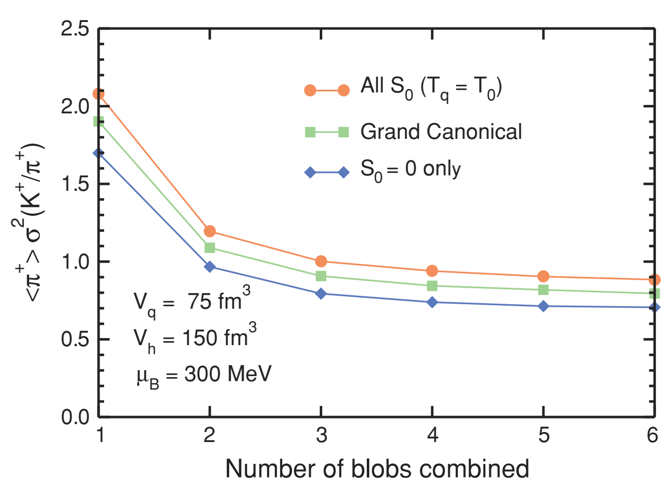

FIG. 7. (Color online) The effect of combining the hadrons from several separate plasma blobs before extracting the variance of the K^+/π^+ ratio, for the three different treatments. (For these results we used an expansion ratio of $\chi = 2$.)

that all N blobs are similar (as we have done above). First, to obtain the correlated distribution of the blob strangenesses $(S_1, \ldots, S_N)$, we consider the problem at the plasma level where $\mathcal{Z}_0 = I_0(2N\zeta_s)$. We then find the following:

$$P_0(S_1, \ldots, S_N) = \frac{\delta_{S_1+\cdots+S_N,0}}{I_0(2N\zeta_s)} \prod_{i=1}^{N} [I_{S_i}(2\zeta_s)]. \tag{27}$$

as described in more detail in Appendix C.

To bring out the effect of the global strangeness constraint, we consider the multiplicity of a particular strange hadron κ resulting from the combined observation of some of the blobs, say those labeled $1, \cdots, N'$, i.e., $n_\kappa = n_\kappa^{(1)} + \cdots + n_\kappa^{(N')}$, where $n_\kappa^{(i)}$ is the multiplicity contributed by the blob i. If all the blobs are similar, the average multiplicity is of the form

$$\langle n_\kappa \rangle = N' n_\kappa^{\text{ave}}(N), \tag{28}$$

where $n_\kappa^{\text{ave}}(N)$ is the average multiplicity arising from any single blob. Furthermore, the variance in the total multiplicity n_κ has the following form,

$$\sigma_\kappa^2 = N' \left[\sigma_\kappa^{\text{var}}(N) + \frac{N'-1}{N-1} \sigma_\kappa^{\text{cov}}(N) \right], \tag{29}$$

where $\sigma_\kappa^{\text{var}}(N)$ is the variance in multiplicity from any single source and $\sigma_\kappa^{\text{cov}}(N)$ is the covariance between the multiplicity from any one source and the combined multiplicity from the all the other $N-1$ sources. The overall restriction of the total strangeness to zero reduces the individual multiplicities $\{n_\kappa^{(i)}\}$, leading to smaller values of both n_κ^{ave} and $\sigma_\kappa^{\text{var}}$. Furthermore, a higher-than-average strangeness in one blob introduces a bias toward lower-than-average values in the others, in that particular "event," thus producing an anticorrelation among the individual strangeness values $\{S_n\}$. This in turn leads to negative covariances, $\sigma_\kappa^{\text{cov}} < 0$.

To illustrate these effects, we consider the production of positive kaons and show in Fig. 8 the dependence of the coefficients $n^{\text{ave}}, \sigma^{\text{var}}$, and σ^{cov} on N, the total number of blobs that are subject to the global constraint. (We may here employ either analytical recursion relations or statistical simulation, as discussed in Appendix C.) For $N=1$ the strangeness of each blob must vanish and we obtain the restricted canonical scenario considered earlier. In the opposite extreme, $N \to \infty$, the global constraint becomes ineffective and the standard grand-canonical scenario is recovered. To a good approximation, the deviations of the coefficients from their grand-canonical values are inversely proportional to N, for example,

$$\sigma_\kappa^{\text{var}}(N) \approx \sigma_\kappa^{\text{var}}(\infty) + \frac{1}{N}\left[\sigma_\kappa^{\text{var}}(1) - \sigma_\kappa^{\text{var}}(\infty)\right]. \tag{30}$$

One may then readily judge the effect for a given N on the basis of the change in going from $N=\infty$ to $N=1$. For the average number of K^+ emitted by each blob, n^{ave}, this drop is only about 4.5% (2.5%), whereas it amounts to 25% (16%) for the corresponding variance, σ^{var} (the numbers in paranthesis refer to $\chi = 3$). This behavior may be contrasted with the approximate constancy of $\sigma^{\text{var}} + \sigma^{\text{cov}}$, the variance of the total K^+ multiplicity divided by N, which decreases by only 4% (2%) when going from $N \gg 1$ to $N=1$. These results illustrate the fact that the variance in the total multiplicity is always smaller than the corresponding mean when the system is subject to a canonical constraint.

Figure 9 illustrates the dependence of the canonical effect on the ratio N'/N by displaying the ratio of the variance to the mean for the K^+ multiplicity resulting from a combined system of N blobs of which N' are being observed. For any given value of the expansion ratio $\chi = V_h/V_q$, which governs the effective statistical weight of the plasma relative to that of the hadron gas, the change from the limit where only a small fraction of the combined system is being observed to the situation when all the kaons are collected is seen to follow a universal curve that is linear in N'/N, to a very good aproximation.

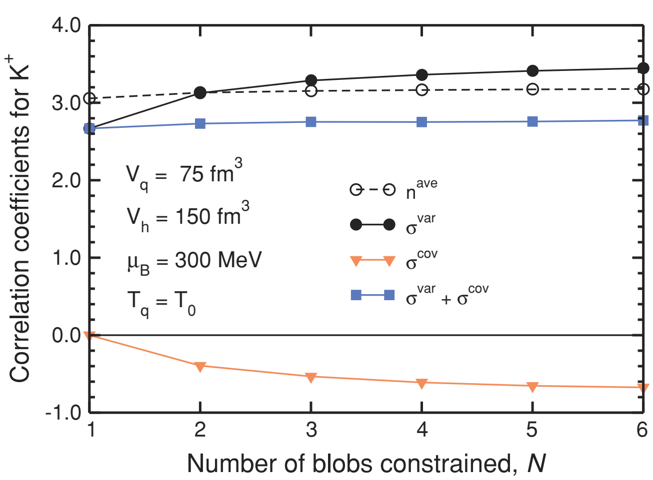

FIG. 8. (Color online) The dependence of the coefficients $n^{\text{ave}}, \sigma^{\text{var}}, \sigma^{\text{cov}}$ on the total number of blobs that are subject to the global strangeness constraint, $S_{\text{tot}}=0$, for K^+ in a scenario having $\mu_B = 300$ MeV, $T_q = T_0$, and $\chi = 2$, as in Fig. 7.

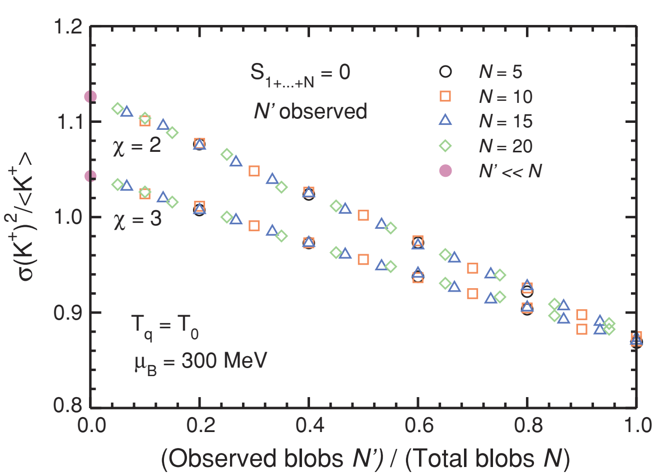

FIG. 9. (Color online) The ratio of the variance to the mean for the K^+ multiplicity resulting from a combined system of N blobs of which N' are being observed, plotted as a function of N'/N. These results were obtained for $\mu_B = 300$ MeV and $T_q = T_0$ with either $\chi = 2$ (upper curve) or $\chi = 3$ (lower curve).

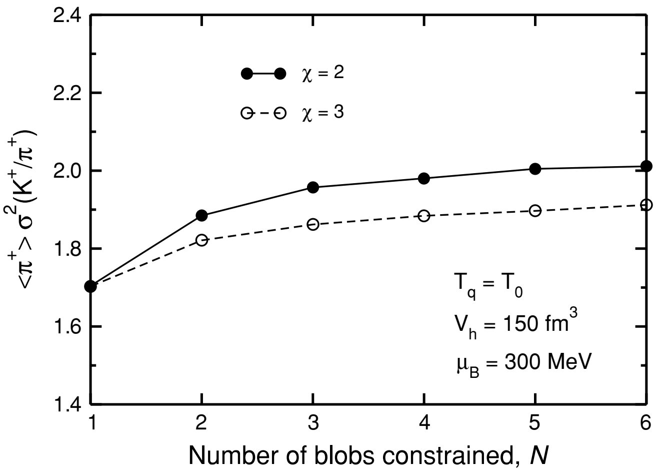

FIG. 10. The variance of the K^+/π^+ ratio for a single blob, multiplied by the mean π^+ multiplicity, when the combined net strangeness of N blobs is required to vanish. These results were obtained for $\mu_B = 300$ MeV and $T_q = T_0$ with either $\chi = 2$ (upper curve) or $\chi = 3$ (lower curve), as in Fig. 9.

The corresponding effect on the fluctuations of the K/π ratio is illustrated in Fig. 10. It shows the value of $\langle\pi^+\rangle\sigma^2(K^+/\pi^+)$ obtained for the hadrons emitted from a single blob when N such blobs satisfy a combined canoncial constraint on their total strangeness contents, $S_{\mathrm{tot}} = 0$. The canonical constraint is obviously most effective in the extreme case of a single blob, $N = 1$, where it is identical to the restricted scenario in which only $S_0 = 0$ is allowed for each individual blob (bottom curve in Fig. 7). As more blobs share the burden of the constraint, its effect rapidly diminishes and the results approach those of our standard spinodal scenario, where the strangeness of each individual plasma blob is determined grand-canonically (top curve in Fig. 7); this approach occurs approximately as $\sim 1/N$. Thus already at $N \approx 5$ the effect is hardly significant. Consequently, if a given blob represents less than 20% of the total system, the vanishing of the overall strangeness should have no noticeable effect on the observables considered here.

V. CONCLUDING DISCUSSION

Most observables considered so far in high-energy collision experiments behave rather smoothly as functions of the control parameters, such as bombarding energy and centrality. However, preliminary data analysis of fixed-target experiments at the SPS by the NA49 collaboration [7] have revealed two intriguing exceptions. First, there appears to be a significant enhancement of the K/π ratio at beam energies of 20–30A GeV. Second, in the same energy range, the fluctuation of this ratio (expressed relative to mixed events) is strongly enhanced [19]. Both enhancements are localized within about 10A GeV in beam energy. Though otherwise rather successful in describing the hadron yields, statistical models cannot reproduce the observed energy dependence of these enhancements [19,20]. Nor does a hadron gas and its resonance-induced correlations account for the strong enhancement in the ratio fluctuations [21,22]. One might then wonder whether these strong fluctuations might result from the enhanced fluctuations associated with a second-order phase transition [5]. However, if this were the case, one would expect particularly strong fluctuations in the pions. However, while the fluctuations of the K/π ratio are observed to be enhanced, those of the p/π ratio follow the expectations from transport models [23].

If the statistical models [15] indeed provide reasonable estimates for the relation between beam energy and the thermodynamic parameters characterizing the chemical freeze-out, i.e., the temperature and the chemical potentials, then the anomalous behavior reported by NA49 would be consistent with a first-order phase transition having occurred prior to the chemical freeze-out, at a somewhat higher temperature and about the same chemical potential. To evaluate the plausibility of such a speculation, we have in this article studied the effects on yields and fluctuations if the bulk of the system indeed undergoes a spinodal-like breakup as it hadronizes. The key physical effect is then that the amount of strangeness residing within the plasma that forms a given blob remains effectively trapped, thus imposing a canoncial strangeness constraint on its conversion into hadrons.

Considering idealized scenarios in which an equilibrated plasma forms a number of separate blobs that subsequently expand and hadronize independently, we have developed the relevant statistical tools. These results may be of general applicability. With a view toward the specific observations at the SPS, we have studied the production of pions and kaons in such a scenario relative to the standard picture in which statistical equilibrium is maintained through freeze-out. While the average hadron yields are essentially unaffected by the breakup, the fluctuations in the strange-particle multiplicities are significantly enhanced, especially at larger values of the chemical potential. From the experimental perspective, the yield *ratios* are particularly interesting and we have especially studied the K/π ratio. Depending on the degree of expansion during the stage of strangeness trapping, the fluctuations in the K^+/π^+ ratio can be enhanced significantly (of the order of 10%) relative to the standard scenario where global equilibrium is maintained.

Thus, the present (rather idealized) studies suggest that a spinodal decomposition might indeed lead to enhancements of the magnitude observed by NA49. However, before such a conclusion could be made with confidence, further studies would be required. In particular, both strong resonance decays and weak decays must be taken into account. Moreover, the enhancement of the K/π fluctuations should be correlated with other expected effects, such as N-body kinematic correlations [11]. Of particular interest are more precise calculations of the equation of state in the relevant baryon-rich environments, to better ascertain the location of the expected phase coexistence region, as well as refined calculations of the collision dynamics to determine whether the spinodal region is in fact likely to be encountered and, if so, to what degree the phase decomposition may actually develop. We hope that the present findings will provide added incentive for these challenging undertakings.

APPENDIX A

Canonical fermions

We discuss here the canonical treatment of fermions, in which the total number of quanta present is kept fixed to a given value N_0 (the number of particles). The appropriate partition function is then

$$\mathcal{Z}_{N_0}(\beta) = \prod_i \left[\sum_{n_i=0}^{1} e^{-\beta n_i \varepsilon_i}\right] \delta\left(\sum_i n_i - N_0\right) = \sum_{\{n_i\}_{N_0}} e^{-\beta E\{n_i\}}, \quad E\{n_i\} = \sum_i n_i \varepsilon_i. \tag{A1}$$

Here n_i is the number of quanta present in the particular "single-particle" state i of energy ε_i. Furthermore, $\{n_i\}_{N_0}$ denotes the subset of all configurations $\{n_i\}$ whose total particle number is constrained to be equal to the specified value N_0, $N\{n_i\} \equiv \sum_i n_i \doteq N_0$. The fermionic restriction on n to be at most one complicates the evaluation of the partition function. However, it is possible to derive the following recursion relation,

$$\mathcal{Z}_N^F(\beta) = \frac{1}{N}\sum_{n=1}^{N}(-)^{n-1}\mathcal{Z}_{N-n}^F(\beta)\mathcal{Z}_1^F(n\beta), \tag{A2}$$

with $\mathcal{Z}_0^F(\beta) = 1$. N can be readily sampled numerically on the basis of $P(N) = \mathcal{Z}_N/\mathcal{Z}$, where $\mathcal{Z}^F = \sum_{N\geqslant 0}\mathcal{Z}_N^F$ is the grand-canonical partition function.

One way to derive the above recursion relation reorganizes the basic expression for Z_N,

$$Z_N = \sum_{i_1<i_2<\cdots<i_N} e^{a_{i_1}+a_{i_2}+\cdots+a_{i_N}} \tag{A3}$$

$$= \frac{1}{N!}\sum_{i_1\neq i_2\neq\cdots\neq i_N} e^{a_{i_1}+a_{i_2}+\cdots+a_{i_N}}, \tag{A4}$$

where $a_i \equiv -\beta\epsilon_i$ and no indices may be equal in the second sum. With $\zeta_n \equiv \mathcal{Z}_1(n\beta)$, the first few terms are as follows:

$$\mathcal{Z}_2 = \frac{1}{2!}\sum_{i\neq j} e^{a_i+a_j} = \frac{1}{2}\left[\sum_{ij} e^{a_i+a_j} - \sum_i e^{2a_i}\right] = \frac{1}{2}\left[\mathcal{Z}_1(\beta)^2 - \mathcal{Z}_1(2\beta)\right] = \frac{1}{2}\left[\mathcal{Z}_1\zeta_1 - \mathcal{Z}_0\zeta_2\right], \tag{A5}$$

$$\mathcal{Z}_3 = \frac{1}{3!}\sum_{i\neq j\neq k} e^{a_i+a_j+a_k} = \frac{1}{3!}\left[\sum_{ijk} e^{a_i+a_j+a_k} - \cdots\right] = \frac{1}{3}[\mathcal{Z}_2\zeta_1 - \mathcal{Z}_1\zeta_2 + \mathcal{Z}_0\zeta_3], \tag{A6}$$

The first term is the classical limit, $\mathcal{Z}_N = \mathcal{Z}_1^N/N! + \cdots$.

Another approach starts from the expansion of $\ln\mathcal{Z}$,

$$\ln\mathcal{Z} = \sum_i \ln[1+e^{-\beta\epsilon_i}] = \sum_i\sum_{n\geqslant 0}(-)^{n-1}\frac{1}{n}e^{-n\beta\epsilon_i} = \sum_{n\geqslant 0}(-)^{n-1}\frac{1}{n}\zeta_n = \zeta_1 - \frac{1}{2}\zeta_2 + \frac{1}{3}\zeta_3 + \cdots, \tag{A7}$$

then exponentiates $\ln\mathcal{Z} = [\zeta_1 - \frac{1}{2}\zeta_2 + \frac{1}{3}\zeta_3 + \cdots]$,

$$\mathcal{Z} = e^{[\,\cdot\,]} = \sum_{N\geqslant 0}\frac{1}{N!}[\,\cdot\,]^N = 1 + [\,\cdot\,]\big\{1 + \tfrac{1}{2}[\,\cdot\,]\big[1 + \tfrac{1}{3}[\,\cdot\,]\big(1 + \tfrac{1}{4}[\,\cdot\,](1+\cdots)\big)\big]\big\}, \tag{A8}$$

and finally reorganizes the series in powers of $e^{\beta\epsilon}$,

$$\mathcal{Z} = \sum_{N\geqslant 0}\mathcal{Z}_N = 1 + \zeta_1 + \underbrace{\tfrac{1}{2}\big[\zeta_1^2 - \zeta_2\big]}_{\mathcal{Z}_2} + \tfrac{1}{3}\Big[\underbrace{\tfrac{1}{2}(\zeta_1^2 - \zeta_2)}_{\mathcal{Z}_2}\zeta_1 - \zeta_1\zeta_2 + \zeta_3\Big] + \cdots. \tag{A9}$$

1. Degeneracy and volume

The above derivations have been made for nongenerate systems, $g = 1$. In the general case, when $g > 1$, the situation is more complicated because a given energy level may accommodate up to g quanta. There is then no simple relation between $\mathcal{Z}[g=1]$ and $\mathcal{Z}' \equiv \mathcal{Z}[g>1]$. However, using the above expansion [Eq. (A7)] of $\ln\mathcal{Z}$ in terms of ζ_n we find

$$\ln\mathcal{Z}' = g\sum_i \ln[1+e^{-\beta\epsilon_i}] = \cdots = \zeta_1' - \frac{1}{2}\zeta_2' + \frac{1}{3}\zeta_3' + \cdots, \tag{A10}$$

where $\zeta_n' \equiv g\zeta_n = g\mathcal{Z}_1(n\beta)$. The partition function may thus be obtained by replacing ζ_n by $\zeta_n' = g\zeta_n$ in the above procedure,

$$\mathcal{Z}' = \sum_N \mathcal{Z}_N', \quad \mathcal{Z}_N' = \mathcal{Z}_N\{\zeta_n \to \zeta_n' = g\zeta_n\}. \tag{A11}$$

Consequently, the recursion relation takes the form,

$$\mathcal{Z}_N' = \frac{1}{N}\sum_{n=1}^{N}(-)^{n-1}\mathcal{Z}_{N-n}'\zeta_n', \quad \mathcal{Z}_0 = 1, \tag{A12}$$

with $\zeta_n' \equiv g\zeta_n = g\sum_i e^{-n\beta\epsilon_i}$, and $\mathcal{Z}_N'$ becomes an N^{th} order polynomial in the degeneracy g,

$$\mathcal{Z}_0' = 1,\ \mathcal{Z}_1' = g\zeta_1,\ \mathcal{Z}_2' = \tfrac{1}{2}g^2\zeta_1^2 - \tfrac{1}{2}g\zeta_2, \tag{A13}$$

$$\mathcal{Z}_3' = \tfrac{1}{6}g^3\zeta_1^3 - \tfrac{1}{2}g^2\zeta_1\zeta_2 + \tfrac{1}{3}g\zeta_3, \ldots. \tag{A14}$$

We note that the dependence on the *volume* V is similar to the dependence on the degeneracy g, since both enter as overall factors in the elementary partition functions, $\zeta_n \sim gV$.

2. Multiplicity distribution

The mean multiplicity is enhanced/reduced for bosons/fermions,

$$\langle N\rangle = \frac{1}{\mathcal{Z}}\sum_N N\mathcal{Z}_N = \frac{g}{2\pi^2}\frac{VT^3}{\hbar^3c^3}\sum_{n>0}(\pm)^{n-1}\frac{1}{n^3}\tilde{K}_2\left(\frac{nm}{T}\right) = \sum_{n>0}(\pm)^{n-1}\zeta_n = \zeta_1 \pm \zeta_2 + \cdots, \tag{A15}$$

and so is the corresponding multiplicity variance,

$$\sigma_N^2 = \frac{g}{2\pi^2}\frac{VT^3}{\hbar^3c^3}\sum_{n>0}(\pm)^{n-1}\frac{1}{n^2}\tilde{K}_2\left(\frac{nm}{T}\right) = \sum_{n>0}(\pm)^{n-1}n\zeta_n = \zeta_1 \pm 2\zeta_2 + 3\zeta_3 \pm \cdots, \quad \text{(A16)}$$

and both are strictly proportional to both the volume V and the degeneracy g, because $\zeta_n \sim gV$, as just noted above.

3. Chemical potentials

The above results apply in the absence of a chemical potential. The presence of a chemical potential effectively replaces the energy ϵ_i by a shifted value $\epsilon_i - \mu$ and consequently all the manipulations can be carried through as above. So, with

$$\zeta_n(\alpha,\beta) \equiv \zeta_n^\circ(\beta)\,e^{-\alpha n},\ \zeta_n^\circ(\beta) \equiv \mathcal{Z}_1(n\beta) = \sum_i e^{-n\beta\epsilon_i}, \quad \text{(A17)}$$

we see that all the terms in $\mathcal{Z}_N$ contain the same power of the fugacity $e^{-\alpha}$ and we find

$$\mathcal{Z}_N(\alpha,\beta) = \mathring{\mathcal{Z}}_N\{\mathring{\zeta}(n\beta)\}\,e^{-\alpha N}. \quad \text{(A18)}$$

However, the multiplicity moments have a complicated α dependence,

$$\langle N\rangle = \zeta_1^\circ\,e^{-\alpha} \pm \zeta_2^\circ\,e^{-2\alpha} + \zeta_3^\circ\,e^{-3\alpha} \pm \cdots, \quad \text{(A19)}$$

$$\sigma_N^2 = \zeta_1^\circ\,e^{-\alpha} \pm 2\zeta_2^\circ\,e^{-2\alpha} + 3\zeta_3^\circ\,e^{-3\alpha} \pm \cdots. \quad \text{(A20)}$$

Furthermore, the corresponding expressions for the associated antiparticle are

$$\langle \bar{N}\rangle = \zeta_1^\circ\,e^{+\alpha} \pm \zeta_2^\circ\,e^{+2\alpha} + \zeta_3^\circ\,e^{+3\alpha} \pm \cdots, \quad \text{(A21)}$$

$$\sigma_{\bar{N}}^2 = \zeta_1^\circ\,e^{+\alpha} \pm 2\zeta_2^\circ\,e^{+2\alpha} + 3\zeta_3^\circ\,e^{+3\alpha} \pm \cdots. \quad \text{(A22)}$$

In the case of a plasma blob, which has no strangeness bias, the distributions of N and $\bar{N}$ must be identical. Consequently, in that situation, we must have $\alpha = 0$. Thus, if the s and $\bar{s}$ quarks are embedded in an environment where μ_B (and/or μ_Q) have finite values, a strangeness symmetric distribution can be established by adjusting μ_S appropriately: $\mu_S = \frac{1}{3}(\mu_B - \mu_Q)$ (since $B_s = \frac{1}{3}$, $Q_s = -\frac{1}{3}$, and $S_s = -\frac{1}{3}$).

It may seem odd to introduce a chemical potential within the context of a canonical treatment, but a given scenario may well require a canonical treatment with respect to one attribute (e.g., strangeness), while a grand-canonical treatment suffices with respect to another (e.g., baryon number). Of course, as the above analysis brings out, the consideration of several attributes is relevant only when the system contains several species that combine the attributes differently. (The inherent correlation between baryon number and strangeness in the quark-gluon plasma was recently proposed as a useful diagnostic for strongly interacting matter [24].)

APPENDIX B

General statistical treatment of strange hadrons

We derive here the expressions needed for the general (classical) statistical treatment of a gas of hadrons characterized by a temperature T and chemical potentials for baryons and electric charge, μ_B and μ_Q, with its total strangeness S_0 being kept fixed. The strategy will be to group the strange hadrons species $\{\kappa\}$ together according to their strangeness S_κ and then build up the complete partition function by pairwise inclusion of classes having opposite value of their strangeness.

1. Generic multiplicities

The one-particle partition function for a given strange hadronic specie κ is given by

$$\zeta_\kappa(T,\mu_B,\mu_Q) = \zeta_\kappa^\circ(T)e^{(\mu_B B_\kappa + \mu_Q Q_\kappa)/T}, \quad \text{(B1)}$$

where its value for vanishing chemical potentials is

$$\zeta_\kappa^\circ(T) = \frac{g_\kappa}{2\pi^2}\frac{VT^3}{\hbar^3c^3}\left(\frac{m_\kappa c^2}{T}\right)^2 K_2\left(\frac{m_\kappa c^2}{T}\right). \quad \text{(B2)}$$

The effective one-particle partition function for a class of hadrons having a common strangeness S is then

$$\zeta_S(T,\mu_B,\mu_Q) = \sum_\kappa \delta_{S_\kappa,S}\zeta_\kappa(T,\mu_B,\mu_Q) = \sum_\kappa \delta_{S_\kappa,S}\zeta_\kappa^\circ(T)\,e^{(\mu_B B_\kappa + \mu_Q Q_\kappa)/T}, \quad \text{(B3)}$$

which generally differs from ζ_{-S}. The number of hadrons having the strangeness S in a given system is denoted by N_S and the probability that such a hadron belongs to the particular specie κ is given by ζ_κ/ζ_S. The associated first and second multiplicity moments are then $\langle n_{\kappa'}\rangle = \langle N_{S'}\rangle\zeta_{\kappa'}/\zeta_{S'}$ and $\langle n_{\kappa'}n_{\kappa''}\rangle = \langle N_{S'}N_{S''}\rangle\zeta_{\kappa'}\zeta_{\kappa''}/\zeta_{S'}\zeta_{S''}$. We may therefore concentrate on finding the generic multiplicities $\{N_S\}$.

We first combine two conjugate classes $\{+S\}$ and $\{-S\}$ to form the class $\{\pm S\}$. The partition function for the resulting combined system of hadrons having $S_\kappa = \pm S$ is then

$$\mathcal{Z}_{S_0}^{\{\pm S\}} = \sum_{N_{+S},N_{-S}} \frac{\zeta_{+S}^{N_{+S}}}{N_{+S}!}\frac{\zeta_{-S}^{N_{-S}}}{N_{-S}!}\delta_{(N_{+S}-N_{-S})S,S_0} = \left(\frac{\zeta_{+S}}{\zeta_{-S}}\right)^{\frac{1}{2}S_0} I_{S_0}(2\zeta_0), \quad \text{(B4)}$$

where $N_{\pm S} \geqslant 0$ denotes the number of hadrons having the strangeness $S_\kappa = \pm S$ and $\zeta_0^2 = \zeta_{+S}\zeta_{-S}$. We note that $\mathcal{Z}_{S_0}^{\{\pm S\}}$ and the corresponding multiplicity moments vanish unless S_0 is a multiple of S, i.e., $S_0 = 0, \pm S, \pm 2S, \ldots$. The factorial multiplicity moments can also readily be obtained,

$$\langle N_{+S}(N_{+S}-1)\ldots(N_{+S}-m+1)\rangle_{S_0}^{\{\pm S\}} = \zeta_{+S}^m\left(\frac{\zeta_{-S}}{\zeta_{+S}}\right)^{\frac{1}{2}mS}\frac{I_{S_0-mS}(2\zeta_0)}{I_{S_0}(2\zeta_0)}. \quad \text{(B5)}$$

Furthermore, the mixed multiplicity moments can be obtained by use of the constraint $(N_{+S} - N_{-S})S = S_0$.

These relations provide a complete treatment of one pair of conjugate classes. Imagine now that we have thus obtained the partition functions $\mathcal{Z}_{S_0}^{\{\pm S\}}$ and the corresponding multiplicity moments $\langle N_{\pm S}^m\rangle_{S_0}^{\{\pm S\}}$ for $|S| = 1, 2, 3$. The classes $\{\pm S\}$ may then be combined recursively. Thus, combining first $\{\pm 1\}$ with $\{\pm 2\}$, we find the partition function for the combined ensemble $\{\pm1, \pm2\}$,

$$\mathcal{Z}_{S_0}^{\{\pm1,\pm2\}} = \sum_{S_2=0,\pm2,\ldots} \mathcal{Z}_{S_0-S_2}^{\{\pm1\}} \mathcal{Z}_{S_2}^{\{\pm2\}}, \tag{B6}$$

where S_0 is the specified total strangeness. It follows that if we have a self-conjugate system (i.e., for each hadron specie κ included, the corresponding antispecie $\bar{\kappa}$ is also included) with strangeness $S_\kappa = \pm1, \pm2$ and whose total strangeness is S_0, then the probability that those with $S_\kappa = \pm1$ have a combined strangeness of S' is given by

$$P_{S_0}^{\{\pm1,\pm2\}}(S^{\{\pm1\}} = S') = \mathcal{Z}_{S'}^{\{\pm1\}} \mathcal{Z}_{S_0-S'}^{\{\pm2\}} \big/ \mathcal{Z}_{S_0}^{\{\pm1,\pm2\}}, \tag{B7}$$

which vanishes unless $S_0 - S'$ is even. Consequently, after the classes $\{\pm1\}$ and $\{\pm2\}$ have been combined, the multiplicity moments for hadrons with $S_\kappa = \pm1$ are

$$\big\langle N_{S=\pm1}^m \big\rangle_{S_0}^{\{\pm1,\pm2\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2\}}\big)^{-1} \times \sum_{S'=S_0,S_0\pm2,\ldots} \mathcal{Z}_{S'}^{\{\pm1\}} \mathcal{Z}_{S_0-S'}^{\{\pm2\}} \big\langle N_{S=\pm1}^m \big\rangle_{S'}^{\{\pm1\}}, \tag{B8}$$

whereas those for hadrons with $S_\kappa = \pm2$ are

$$\big\langle N_{S=\pm2}^m \big\rangle_{S_0}^{\{\pm1,\pm2\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2\}}\big)^{-1} \times \sum_{S_2=0,\pm2,\pm4,\ldots} \mathcal{Z}_{S_0-S_2}^{\{\pm1\}} \mathcal{Z}_{S_2}^{\{\pm2\}} \big\langle N_{S=\pm2}^m \big\rangle_{S_2}^{\{\pm2\}}. \tag{B9}$$

Similar expressions hold for the mixed moments, e.g.,

$$\langle N_{-1} N_{+1} \rangle_{S_0}^{\{\pm1,\pm2\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2\}}\big)^{-1} \times \sum_{S'=S_0,S_0\pm2,\ldots} \mathcal{Z}_{S'}^{\{\pm1\}} \mathcal{Z}_{S_0-S'}^{\{\pm2\}} \langle N_{-1} N_{+1} \rangle_{S'}^{\{\pm1\}}. \tag{B10}$$

It is straightforward to verify the following sum rule expressing strangeness conservation,

$$\sum_{S=\pm1,\pm2} \langle N_S \rangle_{S_0}^{\{\pm1,\pm2\}} S = S_0. \tag{B11}$$

We note that the above recursion scheme holds even if there are no hadrons with $S = \pm2$. In that case, the corresponding effective partition functions vanish, $\zeta_{\pm2} = 0$, and, as noted earlier, the combined partition function is unity for $S_0 = 0$ and vanishes otherwise, $\mathcal{Z}_{S_0}^{\{\pm2\}} = \delta_{S_0,0}$. As a consequence, the combined partition function remains unchanged by the incorporation of $S = \pm2$, $\mathcal{Z}_{S_0}^{\{\pm1,\pm2\}} = \mathcal{Z}_{S_0}^{\{\pm1\}}$, and the multiplicity moments remain unchanged as well.

Further conjugate strangeness classes may be incorporated analogously. Thus, the inclusion of $S = \pm3$ yields the following partition function for $\{\pm1, \pm2, \pm3\}$,

$$\mathcal{Z}_{S_0}^{\{\pm1,\pm2,\pm3\}} = \sum_{S_3=0,\pm3,} \mathcal{Z}_{S_0-S_3}^{\{\pm1,\pm2\}} \mathcal{Z}_{S_3}^{\{\pm3\}}, \tag{B12}$$

and the generic multiplicity moments are given by

$$\big\langle N_{S=\pm1}^m \big\rangle_{S_0}^{\{\pm1,\pm2,\pm3\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2,\pm3\}}\big)^{-1} \times \sum_{S'=S_0,S_0\pm3,\ldots} \mathcal{Z}_{S'}^{\{\pm1,\pm2\}} \mathcal{Z}_{S_0-S'}^{\{\pm3\}} \big\langle N_{S=\pm1}^m \big\rangle_{S'}^{\{\pm1,\pm2\}}, \tag{B13}$$

$$\big\langle N_{S=\pm2}^m \big\rangle_{S_0}^{\{\pm1,\pm2,\pm3\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2,\pm3\}}\big)^{-1} \times \sum_{S'=S_0,S_0\pm3,\ldots} \mathcal{Z}_{S'}^{\{\pm1,\pm2\}} \mathcal{Z}_{S_0-S'}^{\{\pm3\}} \big\langle N_{S=\pm2}^m \big\rangle_{S'}^{\{\pm1,\pm2\}}, \tag{B14}$$

$$\big\langle N_{S=\pm3}^m \big\rangle_{S_0}^{\{\pm1,\pm2,\pm3\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2,\pm3\}}\big)^{-1} \times \sum_{S_3=0,\pm3,\pm6,\ldots} \mathcal{Z}_{S_0-S_3}^{\{\pm1,\pm2\}} \mathcal{Z}_{S_3}^{\{\pm3\}} \big\langle N_{S=\pm3}^m \big\rangle_{S_3}^{\{\pm3\}}. \tag{B15}$$

and the sum rule becomes

$$\sum_{S=\pm1,\pm2,\pm3} \langle N_S \rangle_{S_0}^{\{\pm1,\pm2,\pm3\}} S = S_0. \tag{B16}$$

Recursion expressions for the correlations between the generic multiplicities can be obtained in a similar manner. For example, the correlations between N_{-1} and N_{+1} follow from the corresponding mixed moment,

$$\langle N_{-1} N_{+1} \rangle_{S_0}^{\{\pm1,\pm2,\pm3\}} = \big(\mathcal{Z}_{S_0}^{\{\pm1,\pm2,\pm3\}}\big)^{-1} \times \sum_{S'=S_0,S_0\pm3,\ldots} \mathcal{Z}_{S'}^{\{\pm1,\pm2\}} \mathcal{Z}_{S_0-S'}^{\{\pm3\}} \langle N_{-1} N_{+1} \rangle_{S'}^{\{\pm1,\pm2\}}. \tag{B17}$$

This procedure readily generalizes to the combination of any number of self-conjugate classes.

2. Individual hadron species

The above treatment allows us to determine the canonical moments of the generic hadron multiplicities $\{N_S\}$ in any blob with a specified strangeness S_0. We now consider the multiplicities of the individual hadron species.

For a given value of S_0, the mean number of a particular species κ' is given by

$$\langle n_{\kappa'} \rangle_{S_0} = \frac{1}{\mathcal{Z}_{S_0}} \prod_\kappa \left[\sum_{n_\kappa \geqslant 0} \frac{\zeta_\kappa^{n_\kappa}}{n_\kappa!} \right] n_{\kappa'} \delta\left(\sum_\kappa S_\kappa n_\kappa - S_0 \right), \tag{B18}$$

and the correlation between the multiplicities of any two species κ' and κ'' is given by a similar expression with $n_{\kappa'}$ replaced by $n_{\kappa'} n_{\kappa''}$. To use these expressions with the above expressions for the generic multiplicities $\{N_S\}$, which no longer contain the individual species multiplicities $\{n_\kappa\}$, we note that the probability that a generic hadron of strangeness $S = S_\kappa$ is of a particular specie κ is given by $p_\kappa \equiv \zeta_\kappa / \zeta_{S_\kappa}$. Consequently,

$$\langle n_{\kappa'} \rangle_{S_0} = p_{\kappa'} \big\langle N_{S_{\kappa'}} \big\rangle_{S_0}. \tag{B19}$$

The second multiplicity moments are more complicated to obtain, because they receive contributions both from the correlated internal multiplicity fluctuations within the separate S classes characterized by each particular generic multiplicity set $\{N_S\}$ and from the correlated fluctuations of these generic multiplicities. It is convenient to introduce the correlation coefficient $\sigma_{\kappa'\kappa''} \equiv \delta_{S'S''}\, p_{\kappa'} (\delta_{\kappa'\kappa''} - p_{\kappa''})$ that expresses the degree of correlation between two species in the same strangeness class. Thus it vanishes if $S' \neq S''$. When $S' = S''$ we have $\sigma_{\kappa'\kappa''} = -p_{\kappa'} p_{\kappa''}$ when the two species differ, $\kappa' \neq \kappa''$, whereas $\sigma_{\kappa'\kappa''} = p_{\kappa'} q_{\kappa'}$ when $\kappa' = \kappa''$, where $p_{\kappa'} \equiv \zeta_{\kappa'} / \zeta_{S'}$ is the probability that a hadron of the class S' belongs to the specie κ' and $q_{\kappa'} \equiv 1 - p_{\kappa'}$ is its complement. We then find

$$\langle n_{\kappa'} n_{\kappa''}\rangle_{S_0} = p_{\kappa'} p_{\kappa''} \langle N_{S'} N_{S''}\rangle_{S_0} + \delta_{S'S''} \langle N_{S'}\rangle_{S_0} \sigma_{\kappa'\kappa''}. \quad \text{(B20)}$$

The multiplicity covariances then involve both the intraclass correlations $\sigma_{N'N''}$ and the interclass correlations $\sigma_{n'n''} = \delta_{S'S''}\langle N'\rangle_{S_0}\sigma_{\kappa'\kappa''}$,

$$\begin{aligned}\sigma^{S_0}_{n_{\kappa'} n_{\kappa''}} &\equiv \langle n_{\kappa'} n_{\kappa''}\rangle_{S_0} - \langle n_{\kappa'}\rangle_{S_0}\langle n_{\kappa''}\rangle_{S_0} \\ &= p_{\kappa'} p_{\kappa''} \sigma_{N'N''} + \delta_{S'S''}\langle N'\rangle_{S_0}\sigma_{\kappa'\kappa''}. \end{aligned} \quad \text{(B21)}$$

The above expressions allow us to evaluate the average multiplicities of individual hadron species as well as the associated (co-)variances, for any given value of the fixed strangeness of the blob, S_0.

APPENDIX C

Global-canonical treatment

We consider here the more complicated situation where N individual blobs are subject to a global-canonical constraint on their combined strangeness, $S_{\rm tot} \equiv \sum_n S_n$. We specialize to the relevant case of $S_{\rm tot}=0$, but the treatment can readily be adapted to any value.

Assuming, as we have throughout, that all the blobs are similar, the joint probability for finding the combined system with ν_n s quarks and $\bar\nu_n$ $\bar s$ antiquarks in the blob n is then given by

$$\begin{aligned}&P(\nu_1,\bar\nu_1;\ldots;\nu_N\bar\nu_N) \\ &= \frac{1}{I_0(2N\zeta_s)} \frac{\zeta_s^{\nu_1}}{\nu_1!}\frac{\zeta_s^{\bar\nu_1}}{\bar\nu_1!}\cdots\frac{\zeta_s^{\nu_N}}{\nu_N!}\frac{\zeta_s^{\bar\nu_N}}{\bar\nu_N!}\delta_{\nu_1+\cdots+\nu_N,\bar\nu_1\cdots+\bar\nu_N}. \end{aligned} \quad \text{(C1)}$$

This expression can be used as a basis for a direct Metropolis sampling of the individual multiplicities $\{\nu_n,\bar\nu_n\}$, starting for example from $\{0,0\}$. However, this may not be optimally efficient, because in fact we only need to know the distribution of the differences $S_n = \bar\nu_n - \nu_n$, which is given by

$$\begin{aligned} P(S_1,\ldots,S_N) &= \prod_{n=1}^{N}\left[\sum_{\nu_n\bar\nu_n}\delta_{\bar\nu_n-\nu_n,S_n}\right] P(\nu_1,\bar\nu_1;\ldots;\nu_N\bar\nu_N) \\ &= \frac{\delta_{S_1+\cdots+S_N,0}}{I_0(2N\zeta_s)}\prod_{n=1}^{N}\left[\sum_{\nu_n\bar\nu_n}\frac{\zeta_s^{\nu_n+\bar\nu_n}}{\nu_n!\bar\nu_n!}\delta_{\bar\nu_n-\nu_n,S_n}\right] \\ &= \frac{\delta_{S_1+\cdots+S_N,0}}{I_0(2N\zeta_s)}\prod_{n=1}^{N}\left[I_{S_n}(2\zeta_s)\right]. \end{aligned} \quad \text{(C2)}$$

The corresponding Metropolis sampling could then start from $\{S_n\}=\{0\}$, for example, and repeatedly exchange one unit of strangeness between two selected subsystems, $S_i \to S'_i = S_i \pm 1$ and $S_j \to S'_j = S_j \mp 1$, with the corresponding ratio of weights being given in terms of ratios of Bessel functions of neighboring order, $W'/W = [I_{S_i\pm1}(2\zeta_s)/I_{S_i}(2\zeta_s)][I_{S_j\mp1}(2\zeta_s)/I_{S_j}(2\zeta_s)]$.

A much simpler procedure is based on the fact that the entire set of configurations $\{\nu_n,\bar\nu_n\}$ can be organized according to the total number of s quarks present, M (which equals the total number of $\bar s$ when $S_{\rm tot}=0$). The class having $M=0$ contains only the empty configuration $\{0,0\}$, whereas the class having $M=1$ has N^2 members, and so on. Thus, class M has $(N^M/M!)^2$ members, each of which has the relative weight ζ_s^{2M}, and it is readily checked that the sum of weights is $I_0(2N\zeta_s)$, the total partition function. The expected number of s quarks is $N\zeta_s$ and the most likely value of M is $[N\zeta_s]$. It is easy to sample M by a Metropolis procedure based on the weight ratios for adjacent values of M, $W_{M+1}/W_M = (N\zeta_s/M+1)^2$ and $W_{M-1}/W_M = (M/N\zeta_s)^2$.

Alternatively, M could be sampled directly from its analytical distribution $P(M) = \zeta_s^{2M}(N^M/M!)^2/I_0(2N\zeta_s)$ in a standard manner. Once M has been selected, it is straightforward to distribute the M quarks and M antiquarks randomly among the N subsystems and thus obtain their strangenesses as $S_n = \bar\nu_n - \nu_n$.

The above discussion brings out the fact that the total plasma partition function can be calculated by either distributing ν_n quarks and $\bar\nu_n$ antiquarks in each subsystem n or distributing M quarks and $\bar M$ antiquarks anywhere within the combined system,

$$\begin{aligned}&\prod_{n=1}^{N}\left[\sum_{\nu_n\bar\nu_n}\frac{\zeta_s^{\nu_n}}{\nu_n!}\frac{\zeta_s^{\bar\nu_n}}{\bar\nu_n!}\right]\delta_{\nu_1+\ldots+\nu_N,\bar\nu_1+\ldots+\bar\nu_N} \\ &\quad = \sum_{M\bar M}\frac{(N\zeta_s)^M}{M!}\frac{(N\zeta_s)^{\bar M}}{\bar M!}\delta_{M,\bar M} = I_0(2N\zeta_s). \end{aligned} \quad \text{(C3)}$$

In any case, once the individual blob strangenesses $\{S_n\}$ have been selected, the various hadron multiplicities can be sampled as in the standard case discussed in Appendix B. It is thus possible, in this manner, to make a complete statistical sampling of the combined system and then perform any analysis of interest.

However, such a full similation is not always necessary. In particular, as is often the case, when the quantities of interest can be expressed in terms of first and second moments of the multiplicity distributions of specified hadron species, it is possible to employ recursion relations in analogy with those derived in Appendix B for $\{\langle n_\kappa^{(i)}\rangle_{S_i}\}$, the average multiplicities of the hadron species κ emitted by the particular blob i, and the corresponding second moments, $\{\langle n_\kappa^{(i)} n_{\kappa'}^{(i)}\rangle_{S_i}\}$. (This approach can of course be extended to higher moments.)

We first note that the canonical partition function for N blobs can be expressed recursively in terms of those for fewer blobs. For example, with $N = N' + (N-N')$,

$$I_S(2N\zeta_s) = \sum_{S'} I_{S'}(2N'\zeta_s) I_{S-S'}[2(N-N')\zeta_s]. \quad \text{(C4)}$$

The inclusive probability for one particular blob to have a given strangeness can then be expressed as follows,

$$\begin{aligned} P_1^{(1\cdots N)}(S_1) &= \sum_{S_2\cdots S_N} P(S_1,\ldots,S_N) \\ &= \frac{I_{S_1}(2\zeta_s) I_{-S_1}[2(N-1)\zeta_s]}{I_0(2N\zeta_s)}, \end{aligned} \quad \text{(C5)}$$

while the inclusive joint probability for two given blobs to have specified strangenesses is of the following form,

$$P_{12}^{(1\cdots N)}(S_1, S_2) = \sum_{S_3\cdots S_N} P(S_1, S_2, \ldots, S_N) = \frac{I_{S_1}(2\zeta_s)I_{S_2}(2\zeta_s)I_{-S_1-S_2}[2(N-2)\zeta_s]}{I_0(2N\zeta_s)}, \tag{C6}$$

It is then straightforward to express the average multiplicity of the hadron species κ arising from a given blob,

$$\langle n_\kappa^{(1)}\rangle_0^{(1\cdots N)} = \sum_{S_1} P_1^{(1\cdots N)}(S_1)\langle n_\kappa^{(1)}\rangle_{S_1}. \tag{C7}$$

A similar expression holds for the average of any power of that multiplicity, $(n_\kappa^{(1)})^m$. It also readily follows that the mixed multiplicity moment for two hadron species emitted from the same blob is

$$\langle n_\kappa^{(1)} n_{\kappa'}^{(1)}\rangle_0^{(1\cdots N)} = \sum_{S_1} P_1^{(1\cdots N)}(S_1)\langle n_\kappa^{(1)} n_{\kappa'}^{(1)}\rangle_{S_1}, \tag{C8}$$

while the corresponding expression for emission from two different blobs is

$$\langle n_\kappa^{(1)} n_{\kappa'}^{(2)}\rangle_0^{(1\cdots N)} = \sum_{S_1 S_2} P_{12}^{(1\cdots N)}(S_1, S_2)\langle n_\kappa^{(1)}\rangle_{S_1}\langle n_{\kappa'}^{(2)}\rangle_{S_2}. \tag{C9}$$

These expressions allow us to use the canonical partition functions for individual blobs obtained in Appendix B to calculate the average multiplicities of specific hadron species and any desired (co-)variances in terms of the corresponding expressions for canonical emission from a single blob.

ACKNOWLEDGMENTS

A.M. thanks F. Beccatini for helpful discussions. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of Nuclear Physics, the Office of Basic Energy Science, Division of Nuclear Science, of the U.S. Department of Energy under contract no. DE-AC03-76SF00098.

[1] R. Stock, J. Phys. G **30**, 633 (2004).
[2] C. Adler *et al.* (STAR Collaboration), Phys. Rev. Lett. **86**, 4778 (2001); **90**, 119903 (E) (2003).
[3] P. Senger, J. Phys. G **30**, 1087 (2004).
[4] F. Karsch, J. Phys. G **30**, S887 (2004).
[5] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998).
[6] Z. Fodor and S. D. Katz, J. High Energy Phys. 04 (2004) 050.
[7] M. Gazdzicki *et al.* (NA49 Collaboration), J. Phys. G **30**, S701 (2004).
[8] Ph. Chomaz, M. Colonna, and J. Randrup, Phys. Rep. **389**, 263 (2004).
[9] B. Borderie *et al.* (INDRA Collaboration), Phys. Rev. Lett. **86**, 3252 (2001).
[10] D. Bower and S. Gavin, J. Heavy Ion Phys. **15**, 269 (2002).
[11] J. Randrup, J. Heavy Ion Phys. **22**, 69 (2005).
[12] B. Friman, W. Nörenberg, and V. D. Toneev, Eur. Phys. J. A **3**, 165 (1998).
[13] H. Weber, C. Ernst, M. Bleicher, L. Bravina, H. Stöcker, W. Greiner, C. Spieles, and S. A. Bass, Phys. Lett. **B442**, 443 (1998).
[14] W. Cassing, E. Bratkovskaya, and S. Juchem, Nucl. Phys. **A674**, 249 (2000).
[15] P. Braun-Munzinger, K. Redlich, and J. Stachel, in *Quark Gluon Plasma 3*, edited by R. C. Hwa and X. N. Wang (World Scientific Publishing, Singapore, 2003), pp. 491–599.
[16] J. Randrup, Phys. Rev. Lett. **92**, 122301 (2004).
[17] A. Majumder and V. Koch, Phys. Rev. C **68**, 044903 (2003).
[18] J. Cleymans, K. Redlich, and L. Turko, Phys. Rev. C **71**, 047902 (2005).
[19] F. Becattini, M. Gazdzicki, A. Keranen, J. Manninen, and R. Stock, Phys. Rev. C **69**, 024905 (2004).
[20] J. Cleymans, H. Oeschler, K. Redlich, and S. Wheaton, Phys. Lett. **B615**, 50 (2005).
[21] S. Jeon and V. Koch, Phys. Rev. Lett. **83**, 5435 (1999).
[22] S. Jeon and V. Koch, in *Quark Gluon Plasma 3*, edited by R. C. Hwa and X. N. Wang (World Scientific Publishing, Singapore, 2003), pp. 430–490.
[23] C. Roland *et al.* (NA49 Collaboration), J. Phys. G **30**, S1381 (2004).
[24] V. Koch, A. Majumder, and J. Randrup, Phys. Rev. Lett. **95**, 182301 (2005).