PHYSICAL REVIEW C **72**, 064333 (2005)

Number of states for nucleons in a single-j shell

Y. M. Zhao[1,2,3,*] and A. Arima[4]
[1]*Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China*
[2]*Cyclotron Center, Institute of Physical Chemical Research (RIKEN), Hirosawa 2-1, Wako-shi, Saitama 351-0198, Japan*
[3]*Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou 730000, China*
[4]*Science Museum, Japan Science Foundation, 2-1 Kitanomaru-koen, Chiyodaku, Tokyo 102-0091, Japan*
(Received 28 September 2005; published 30 December 2005)

In this paper we obtain number of states with a given spin I and a given isospin T for systems with three and four nucleons in a single-j orbit, by using sum rules of six-j and nine-j symbols obtained in earlier works.

DOI: 10.1103/PhysRevC.72.064333 PACS number(s): 05.30.Fk, 05.45.−a, 21.60.Cs, 24.60.Lz

Recently, there have been efforts to obtain algebraic formulas for the number of spin I states (denoted by D_I in this paper) for fermions in a single-j shell (with j a half integer) [1–5]. So far most discussions have been restricted to identical particles. In nuclear physics, there are two types of valence nucleons: protons and neutrons. Therefore, it is also interesting to obtain the formulas of number of states with a given spin I and isospin T (denoted by D_{IT} in this paper), which automatically includes D_I for identical particles because D_I for identical particles studied in earlier works equals D_{IT} with $T = T_{\max}$.

In Refs. [6] and [7] sum rules of six-j and nine-j symbols were studied by using the summation (trace) of diagonal matrix elements of individual J pairing interactions for three and four identical particles in a single-j shell. If one takes all two-body matrix elements to be 1 (i.e., the strength G_J of all J pairing interactions equals 1), the summation of traces over J must equal $n(n-1)/2$ multiplied by the number of spin I states. This is nothing but the trace of identity 1.

In this paper we shall go in the reverse direction: We obtain formulas of D_{IT} by using the sum rules of six-j and nine-j symbols obtained in Refs. [6] and [7]. Similarly to Ref. [7], we first define the J-pairing interaction H_{JT} for nucleons in a single-j shell as follows:

$$\begin{aligned} H_{JT} &= G_{JT} \sum_{M=-J}^{J} A^{(JT)\dagger}_{MM_T} A^{(JT)}_{MM_T}, \\ A^{(JT)\dagger}_{MM_T} &= \frac{1}{\sqrt{2}} [a^\dagger_{jt} a^\dagger_{jt}]^{(JT)}_{MM_T}, \\ A^{(JT)}_{M} &= -(-1)^{M+M_T} \frac{1}{\sqrt{2}} [\tilde{a}_{jt}\tilde{a}_{jt}]^{(JT)}_{-M-M_T}, \\ \tilde{A}^{(JT)} &= -\frac{1}{\sqrt{2}} [\tilde{a}_{jt}\tilde{a}_{jt}]^{(JT)}, \end{aligned} \tag{1}$$

where $[]^{(JT)}_{MM_T}$ means an operator in which two nucleons are coupled to spin J and isospin T with spin projection M and isospin projection M_T. We take $G_{JT} = 1$ throughout this paper.

In this paper we shall exemplify our method by using three and four nucleons in a single-j orbit. The same method can be applied to three and four bosons with F spin of the interacting boson model (IBM) II [8], IBM-III, and IBM-IV [9].

First, one can prove that for $n = 4$ the summation of all nonzero eigenvalues of $H = H_{JT}$ is the trace of the H_{JT} matrix with total spin I, and this trace is given by summing the diagonal matrix elements

$$\begin{aligned} &\langle 0 | [A^{(JT_2)} A^{(KT_2')}]^{(IT)}_{MM_T} [A^{(JT_2)\dagger} A^{(KT_2')\dagger}]^{(IT)}_{MM_T} | 0 \rangle \\ &= 1 + (-)^{I+T}\delta_{JK} - 4(2J+1)(2K+1)(2T_2+1)(2T_2'+1) \\ &\quad \times \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T_2 \\ \frac{1}{2} & \frac{1}{2} & T_2' \\ T_2 & T_2' & T \end{Bmatrix} \end{aligned} \tag{2}$$

over K, T_2, and T_2'. Here T_2 (T_2') and T are isospins for two and four nucleons, respectively.

The procedure to obtain D_{IT} is straightforward. From the sum rule of two-body coefficients of fractional parentage, one obtains $n(n-1)/2$ multiplied by D_{IT}, if one sums Eq. (2) over all allowed J, K, T_2, and T_2'; namely,

$$\begin{aligned} &\sum_J \sum_\alpha \langle j^4 \alpha I T | H_J | j^4 \alpha I T \rangle \\ &\quad = \sum_{JKT_2T_2'} \langle 0 | [A^{(JT_2)} A^{(KT_2')}]^{(IT)}_{MM_T} [A^{(JT_2)\dagger} A^{(KT_2')\dagger}]^{(IT)}_{MM_T} | 0 \rangle \\ &\quad = 6 D_{IT}. \end{aligned} \tag{3}$$

For $n = 4$, the maximum isospin T ($T_{\max}$) should be equal to 2. For this case, T_2 and T_2' in Eq. (3) equal 1, and J and K must take even values. Then we have

$$6D_{I(T=2)} = \sum_{\text{even } J \text{ even } K} \left(1 + (-)^I \delta_{JK} - 36(2J+1)(2K+1) \times \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \end{Bmatrix} \right)$$

*Electronic address: ymzhao@sjtu.edu.cn

$$= \sum_{\text{even } J \text{ even } K} \left(1 + (-)^I \delta_{JK} - 4(2J+1)(2K+1) \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \right). \tag{4}$$

The right-hand side of Eq. (4) is just $6D_I$. This can be easily understood by confirming that the right-hand side of Eq. (4) is equal to Eq. (8) of Ref. [7]. Although the formulas of $D_{I(T=2)}$ were available in Eqs. (3)–(5) of Ref. [2], we present $D_{I(T=2)}$ in new forms that are simpler than other forms in practice. For $I \geqslant 2j-3$, the formula of $D_{I(T=2)}$ in Ref. [2] is very simple and can be easily applied. When $I \leqslant 2j-3$, let us define $I = 6k+\kappa$, $L = [(j - \frac{6k+3)}{2})/3]$, and $m = [(j-(6k+3)/2) \bmod 3] \equiv (j-3/2) \bmod 3$. We have

$$D_{I,T=2} = (3k+1)L + km + 1 + 3\left[\frac{k}{2}\right]\left(\left[\frac{k}{2}\right]+1\right) + (k \bmod 2)\left(3\left[\frac{k}{2}\right]+2\right) \tag{5}$$

when $\kappa = 0$;

$$D_{I,T=2} = (3k+2)L + (k+1)m\left[\frac{k}{2}\right]\left(3\left[\frac{k}{2}\right]+2\right) + (k \bmod 2)\left(3\left[\frac{k}{2}\right]+2\right) \tag{6}$$

when $\kappa = 2$; and

$$D_{I,T=2} = (3k+3)L + (k+1)m + 3 + \left[\frac{k}{2}\right]\left(3\left[\frac{k}{2}\right]+7\right) + (k \bmod 2)\left(3\left[\frac{k}{2}\right]+4\right) \tag{7}$$

when $\kappa = 4$.

When $\kappa = 4$, $D_{I(T=2)}$ can be further simplified. If $I = 12k+4$,

$$D_{I,T=2} = (2k+1)j - \tfrac{1}{2}[34k + 18k(k-1) + 3]$$

and if $I = 12k+10$,

$$D_{I,T=2} = (2k+2)j - \tfrac{1}{2}[26k + 9k(k-1) + 8].$$

As in Ref. [7], we denote

$$S_I(j^4, \text{ condition } X \text{ on } J \text{ and } K) = \sum_{X} 4(2J+1)(2K+1)\begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \tag{8}$$

for sake of simplicity.

Now we discuss the case of $T = 1$. Here (T_2, T_2') can take the following values: (1,0), (0,1), and (1,1). Because of the Pauli principle, there are requirements on J and K values. The corresponding requirements for (J, K) are (J = even, K = odd), (J = odd, K = even), and (J = even, K = even), respectively. We obtain

$$\begin{aligned} 6D_{I(T=1)} = & \sum_{\text{even } J \text{ even } K} \left(1 - (-)^I \delta_{JK} - 36(2J+1)(2K+1) \times \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{Bmatrix} \right) \\ & + \sum_{\text{odd } J \text{ even } K} \left(1 - (-)^I \delta_{JK} - 12(2J+1)(2K+1) \times \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 1 \end{Bmatrix} \right) \\ & + \sum_{\text{even } J \text{ odd } K} \left(1 - (-)^I \delta_{JK} - 12(2J+1)(2K+1) \times \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{Bmatrix} \right) \\ = & \sum_{\text{even } J \text{ even } K} [1 - (-)^I \delta_{JK}] + 2\sum_{\text{even } J \text{ odd } K} [1 - (-)^I \delta_{JK}] \\ & - S(j^4, \text{even } J \text{ odd } K). \end{aligned} \tag{9}$$

In the derivation of Eq. (9) we used following relations:

$$\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 1 \end{Bmatrix} = 1/6.$$

To simplify Eq. (9), one should consider the number of combinations of J and K, as exemplified in Ref. [7]. We note that J, K, and I must satisfy the triangle relation for vector couplings.

When $n = 4$ and $T = 1$, $I_{\max}$ equals $4j-3$ (odd value). When $I \geqslant 2j$, let us define $I = I_{\max} - 2I_0$ for odd I and $I_{\max} - 2I_0 - 1$ for even I. Using Eqs. (9) and (22) of Ref. [7], one can obtain

$$D_{IT=1} = \left(\left[\frac{I_0}{2}\right]+1\right)\left(\left[\frac{I_0}{2}\right]+1+(I_0 \bmod 2)\right). \tag{10}$$

When $I \leqslant 2j$, we use Eqs. (9) and (21) of Ref. [7] and obtain

$$D_{I(T=1)} = (I_0+1)j - \left(1 + 4\left[\frac{I_0}{2}\right] + 6\left[\frac{I_0}{2}\right]^2 + (I_0 \bmod 2)\left(6\left[\frac{I_0}{2}\right]+3\right)\right) \Big/ 2, \tag{11}$$

where $I_0 = (I-1)/2$ and $I \leqslant 2j$.

Next we discuss the case of $T=0$. Here (T_2, T_2') can take the values (1,1) and (0,0), and the corresponding requirements for (J, K) are ($J=$ even, $K=$ even) and ($J=$ odd, $K=$ odd), respectively. Similarly, we obtain the following results:

$$\begin{aligned}6D_{I(T=0)} = &\sum_{\text{even } J \text{ even } K}\left(1+(-)^I\delta_{JK} - 36(2J+1)(2K+1)\right.\\ &\left.\times\begin{Bmatrix} j & j & J\\ j & j & K\\ J & K & I\end{Bmatrix}\begin{Bmatrix}\frac12 & \frac12 & 1\\ \frac12 & \frac12 & 1\\ 1 & 1 & 0\end{Bmatrix}\right)\\ &+\sum_{\text{odd } J \text{ odd } K}\left(1+(-)^I\delta_{JK} - 4(2J+1)(2K+1)\right.\\ &\left.\times\begin{Bmatrix} j & j & J\\ j & j & K\\ J & K & I\end{Bmatrix}\begin{Bmatrix}\frac12 & \frac12 & 0\\ \frac12 & \frac12 & 0\\ 0 & 0 & 0\end{Bmatrix}\right)\\ = &\sum_{\text{even } J \text{ even } K}[1+(-)^I\delta_{JK}]\\ &+\sum_{\text{odd } J \text{ odd } K}(1+(-)^I\delta_{JK}) - \frac12 S(j^4, \text{odd } J \text{ odd } K)\\ &+\frac12 S(j^4, \text{even } J \text{ even } K).\end{aligned} \quad (12)$$

In the derivation of Eq. (12) we used the following relations:

$$\begin{Bmatrix}\frac12 & \frac12 & 1\\ \frac12 & \frac12 & 1\\ 1 & 1 & 0\end{Bmatrix} = -\frac{1}{18}, \qquad \begin{Bmatrix}\frac12 & \frac12 & 0\\ \frac12 & \frac12 & 0\\ 0 & 0 & 0\end{Bmatrix} = 1/2.$$

To simplify Eq. (12), again one should consider the number of combinations of J and K, which are very complicated. In the following we show our final results.

When $I \geqslant 2j$ and $T=0$, we use Eq. (12), along with Eqs. (11), (19) and (20) of Ref. [7], and obtain

$$D_{I(T=0)} = \left(\left[\frac{I_0}{3}\right]+1\right)\left(\frac32\left[\frac{I_0}{3}\right]+1+(I_0 \bmod 3)\right), \quad (13)$$

where $I_0 = (I_{\max} - I)/2$ and I is even. When I is odd and $I \geqslant 2j$, one has $D_{I(T=0)} = D_{(I+3)(T=0)}$.

When $I \leqslant 2j$ and $T=0$, we use Eq. (12), along with Eqs. (14), (15), (17) and (18) of Ref. [7]. Let us define $I = 6k+\kappa$, $L = [(j - \frac{6k+3}{2})/3]$, and $m = \{[j-(6k+3)/2] \bmod 3\} \equiv (j-3/2) \bmod 3$. For $\kappa = 0$, we obtain

$$D_{I=6k(T=0)} = (2+6k)L + (2k+1)m + \tfrac32 k(k+3) + 1, \quad (14)$$

and for $\kappa = 3$, we have

$$D_{I=6k+3(T=0)} = (2+6k)L + (2k+1)m + \tfrac32 k(k+1). \quad (15)$$

We can see the following relation:

$$D_{I=6k(T=0)} - D_{I=6k+3(T=0)} = 3k+1. \quad (16)$$

For $\kappa = 1$, we have

$$D_{I=6k+1(T=0)} = 2kj - \tfrac12 k(9k+1). \quad (17)$$

Note that $D_{I=1,T=0} = 0$. For $\kappa = 4$ we obtain

$$D_{I=6k+4(T=0)} = 2(k+1)j - \tfrac12(k+1)(9k+4). \quad (18)$$

We have the following relation:

$$D_{I=6k+4(T=0)} - D_{I=6k+7(T=0)} = 3(k+1). \quad (19)$$

For $\kappa = 2$, we have

$$D_{I=6k+2(T=0)} = (4+6k)L + (2k+1)m + \tfrac12(k+1)(3k+4), \quad (20)$$

and for $\kappa = 5$, we have

$$D_{I=6k+5(T=0)} = (4+6k)L + (2k+1)m + \tfrac12 k(3k+1). \quad (21)$$

One can notice that

$$D_{I=6k+2(T=0)} - D_{I=6k+5(T=0)} = 3k+2. \quad (22)$$

We similarly obtain $D_{I(T=1/2)}$ for three nucleons:

$$\begin{aligned}D_{I(T=1/2)} = &\sum_{\text{even } J}\left(1-6(2J+1)\begin{Bmatrix} j & j & J\\ j & I & J\end{Bmatrix}\begin{Bmatrix}1/2 & 1/2 & 1\\ 1/2 & 1/2 & 1\end{Bmatrix}\right)\\ &+\sum_{\text{odd } J}\left(1-2(2J+1)\begin{Bmatrix} j & j & J\\ j & I & J\end{Bmatrix}\begin{Bmatrix}1/2 & 1/2 & 0\\ 1/2 & 1/2 & 0\end{Bmatrix}\right)\\ = &\sum_{\text{even } J}\left(1-(2J+1)\begin{Bmatrix} j & j & J\\ j & I & J\end{Bmatrix}\right)\\ &+\sum_{\text{odd } J}\left(1+(2J+1)\begin{Bmatrix} j & j & J\\ j & I & J\end{Bmatrix}\right),\end{aligned} \quad (23)$$

where the following six-j symbols are used:

$$\begin{Bmatrix}1/2 & 1/2 & 1\\ 1/2 & 1/2 & 1\end{Bmatrix} = \frac16, \qquad \begin{Bmatrix}1/2 & 1/2 & 0\\ 1/2 & 1/2 & 0\end{Bmatrix} = -\frac12. \quad (24)$$

By using Eq. (23) and the sum rules of six-j symbols (Eqs. (A3) and (A8) obtained in Ref. [6]), one can easily obtain $D_{I(T=1/2)}$ for three nucleons. For $I \leqslant j$,

$$D_{I(T=1/2)} = 1 + 2\left[\frac{(I-1/2)}{3}\right] + \delta_{[(I-1/2) \bmod 3],2}, \quad (25)$$

and for $I \geqslant j$, we have

$$D_{I(T=1/2)} = 1 + \left[\frac{(I_{\max} - I)}{3}\right], \quad (26)$$

where $I_{\max} = 3j-1$.

According to Eqs. (1) and (2) of Ref. [2], when $I \leqslant j$,

$$D_{I(T=3/2)} = \left[\frac{2I+3}{6}\right]; \quad (27)$$

and when $I \geqslant j$,

$$D_{I(T=3/2)} = \left[\frac{3j-3-I}{6}\right] + \delta_I, \quad (28)$$

where

$$\delta_I = \begin{cases} 0 & \text{if } [(3j-3)-I] \bmod 6 = 1,\\ 1 & \text{otherwise.}\end{cases}$$

Note that $j \leqslant I \leqslant 3j-3$ in Eq. (28). Comparing Eqs. (27) and (28) with Eqs. (25) and (26), one easily sees that $D_{I(T=1/2)} - 2D_{I(T=3/2)}$ has a a modular behavior. We obtain that when $I \leqslant j$

$$D_{I(T=1/2)} - 2D_{I(T=3/2)} = \begin{cases} -1 & \text{if } (I-1/2) \bmod 3 = 1, \\ 1 & \text{otherwise}, \end{cases} \\ = \begin{cases} -1 & \text{if } 2I \bmod 3 = 0, \\ 1 & \text{otherwise}, \end{cases} \\ = 1 - 2\delta_{2I \bmod 3, 0}, \tag{29}$$

and when $I \geqslant j$

$$D_{I(T=1/2)} - D_{I(T=3/2)} \\ = \begin{cases} 1 & \text{if } (3j-1-I) \bmod 6 = 0, \\ 1 & \text{if } (3j-1-I) \bmod 6 = 1, \\ -1 & \text{if } (3j-1-I) \bmod 6 = 2, \\ 2 & \text{if } (3j-1-I) \bmod 6 = 3, \\ 0 & \text{if } (3j-1-I) \bmod 6 = 4, \\ 0 & \text{if } (3j-1-I) \bmod 6 = 5 \end{cases} \\ = \left\{ \left(\left[\frac{3j-1-I}{2} \right] + 1 \right) \bmod 3 \right\} \\ + \begin{cases} -3 & \text{if } (3j-1-I) \bmod 6 = 2, \\ 0 & \text{otherwise}. \end{cases} \tag{30}$$

For $n = 4$, a modular behavior for $D_{I(T=0)} - D_{I(T=2)}$ was recently found in Ref. [10] by Zamick and A. Escuderos, who proved that

$$D_{I(T=0)} - 2D_{I(T=2)} = 2 \sum_{\text{even } J \text{ even } K} (2J+1)(2K+1) \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix}. \tag{31}$$

We can prove that our results are consistent with this relation. Let us take $I = 12k$ (but $I \leqslant 2j-3$) as an example. We define $L = [(j - \frac{12k+3}{2}/3)]$ and $m = (j - 3/2) \bmod 3$. By using Eqs. (5) and (14), we obtain

$$D_{I=0(T=0)} - 2D_{IT=2} \\ = (12k+2)L + (4k+1)m + 3k(2k+3) + 1 \\ - 2[(6k+1)L + 2km + 1 + 3k(k+1)] \\ = 3k + m - 1.$$

This is the right-hand side of Eq. (31) for $I = 12k$, according to Eq. (14) of Ref. [7]. The cases of other I values can be proved similarly.

To summarize, in this paper we obtained, for the first time, algebraic formulas for the number of states with spin I and isospin T for three and four nucleons in a single-j shell by using the sum rules of six-j and nine-j symbols obtained in Ref. [7] for identical particles and the well-known sum rule for two-body coefficients of fractional parentage. We also showed that $D_{I(T=1/2)} - 2D_{I(T=3/2)}$ has a simple modular behavior for $n = 3$. We note without details that the same procedures can be applied to obtain the number of states with given spin I and F spin for three and four bosons in the interacting boson models.

ACKNOWLEDGMENT

One of the authors (YMZ) would like to thank the National Natural Science Foundation of China for supporting this work under Grant Nos. 10545001 and 10575070.

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