Method for constructing relativistic three-particle models of the pion-nucleon system

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The Bakamjian-Thomas procedure is used to develop a method for constructing relativistic, instant form models of the pion-nucleon system. A limited model space is used to illustrate the method. The model space consists of a single-nucleon subspace, a pion-nucleon subspace, a two-pion-nucleon subspace, and a pion-sigma meson-nucleon subspace. A Poincaré invariant mass operator is constructed that includes vertex interactions that couple the various subspaces, as well as renormalization terms. It is shown that the pion-nucleon elastic scattering and production amplitudes can be obtained from the solution of a single, three-dimensional, integral equation of the Lippmann-Schwinger type. The effective pion-nucleon potential that appears in this equation contains contributions from direct and crossed nucleon exchanges along with sigma exchange. The production amplitudes are of the form that arises in isobar models. The elastic scattering and production amplitudes satisfy unitarity. The method developed makes it possible to extend existing coupled-channel models of the pion-nucleon system to include three-particle channels in such a way that the requirements of special relativity and unitarity are satisfied exactly.

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I. INTRODUCTION

The Faddeev [1] formulation of the nonrelativistic three-body problem, as well as alternative formulations [2–5], rather quickly inspired the development of a relativistic three-particle model of the pion-nucleon system. The reduction technique introduced by Blankenbecler and Sugar [6] to produce three-dimensional, relativistic equations was used by Aaron, Amado, and coworkers [7] to develop such a model. Reduction techniques were also used by Garcilazo and Mathelitsch [8] to derive a relativistic model for the coupled πN–$\pi\pi N$ system. Three-dimensional equations for this system were derived by Afnan and Pearce [9] using diagramatic, time-ordered techniques.

Here we will use the Bakamjian-Thomas [10,11] method to develop a formalism for constructing relativistic three-particle models of the πN system. The original motivation for this approach can be traced back to an important paper by Dirac [12] in which he pointed out that there are various possibilities for incorporating interactions in the Poincaré generators. These possibilities Dirac called the instant form, the point form, and the front form. Each form is associated with a three-dimensional hypersurface that is invariant under a subgroup of the Poincaré transformations, $x' = ax + b$, and intersects every world line just once. For the instant, point, and front forms the hypersurfaces can be taken to be $t =$ const., $c^2t^2 - \mathbf{x}^2 = a^2 > 0$ with $t > 0$ and $ct + z = 0$, respectively. In Dirac's approach the generators associated with these hypersurfaces are taken to be noninteracting, and interactions are put into the remaining generators. In the instant form the three-momentum $\mathbf{P}$ and the angular momentum $\mathbf{J}$ are noninteracting, while the Hamiltonian H and the generator of rotationless boosts $\mathbf{K}$ contain interactions.

In the Bakamjian-Thomas approach for the instant form generators the ten generators are expressed in terms of another set of ten operators, $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$, where M is the mass operator, $\mathbf{S}$ is a spin operator, and $\mathbf{X}$ is the so-called Newton-Wigner position operator [11,13]. This second set of operators satisfies much simpler commutation rules than the generators, which facilitates the construction of models. In the Bakamjian-Thomas scheme $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are taken to be noninteracting, and an interaction is put only into the mass operator, M.

It is relatively straightforward to carry out a Bakamjian-Thomas construction for a system in which there are coupled one- and two-particle channels. In particular, models of the pion-nucleon system have been constructed in which there are single-baryon and meson-baryon channels [14–16]. Such models contain vertex interactions that couple single-baryon states to meson-baryon states, as well as interactions that couple meson-baryon states directly to each other. The vertex interactions lead to renormalization effects.

There has been a fair amount of activity devoted to using the Bakamjian-Thomas method to construct relativistic three-particle models in which the number of particles is fixed. Coester [17] analyzed the S matrix that arises in an instant form, three-particle model. The development of three-particle equations in the front form was carried out by Bakker *et al.* [18]. These front form equations have been used in a study of relativistic proton-deuteron scattering [19]. Relativistic effects in three-body bound states have been investigated within the framework of a simplified model of the three-nucleon system, constucted by using the Bakamjian-Thomas scheme [20].

The Bakamjian-Thomas construction has also been used to formulate relativistic quark models of the baryons. In particular Szczepaniak *et al.* [21] have used the construction to derive a Poincaré invariant formulation of the Isgur-Karl quark model for baryons [22,23]. Coester *et al.* [24] have developed a simply solvable three-quark model for the baryons, also by using the Bakamjian-Thomas construction.

There has been a modest amount of effort devoted to using the Bakamjian-Thomas method to construct

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three-particle models for systems in which the particle number is not conserved. Such a model has been developed for the NN–πNN system by Betz and Coester [25] and applied by Betz and Lee [26]. Pichowsky *et al.* [27] have used the Bakamjian-Thomas construction to develop a model that describes $\pi\pi$ scattering from threshold up to 1400 MeV. Their model properly includes unitarity cuts for one-, two, and three-hadron states.

Most relevant to the present work is Klink's [28] Bakamjian-Thomas construction of point form mass operators from vertex interactions. He considered a simplified model in which a scalar "nucleon" interacts with a "scalar" pion. A truncated Hilbert space consisting of the direct sum of N and πN states leads to an eigenvalue problem for the physical nucleon mass, as well as a Lippmann-Schwinger equation for πN scattering. Another truncation consisting of NN and πNN states leads to an eigenvalue problem for the "deuteron", along with a model for NN scattering with pion production.

Here we will use a Bakamjian-Thomas construction to develop an instant form, three-particle model for a system that has many of the features of the pion-nucleon system. The model space consists of single-nucleon states, πN states, $\pi\pi N$ states, and $\pi\sigma N$ states. The model specifies a Poincaré invariant mass operator that includes vertex interactions along with renormalization terms. The vertex interactions couple N to πN states, πN states to $\pi\pi N$ states, and πN states to $\pi\sigma N$ states. By eliminating the N, $\pi\pi N$, and $\pi\sigma N$ states, we will show that the elastic scattering and production amplitudes can be obtained from the solution of a single, three-dimensional integral equation of the Lippmann-Schwinger type. The driving term for this equation is an effective, energy-dependent, pion-nucleon potential that describes direct and crossed nucleon exchanges along with sigma exchange. It turns out that this effective potential can be identified with expressions derived from effective Lagrangians by using time-ordered perturbation theory. The production amplitudes have the well-known isobar form. It is demonstrated explicitly that the various amplitudes satisfy two- and three-particle unitarity.

The outline of the paper is as follows. In Sec. II a brief description of the Bakamjian-Thomas method is given, and the restrictions on a mass operator necessary to ensure Poincaré invariance are stated. The various states that enter into the construction of the Hilbert space of the model are described and precisely defined in Sec. III. The mass operator is developed in Sec. IV, and its Poincaré invariance is verified in Sec. V. In Sec. VI integral equations are derived that describe the coupling between the various subspaces of the model, and it is shown that the amplitudes for elastic scattering and production can be obtained by solving a single integral equation. Section VII presents specific models for the various vertex functions that are an essential part of the mass operator, and these models are used to derive explicit expressions for the effective pion-nucleon potential. The proof of unitarity is given in Sec. VIII. Section IX presents a discussion of the results along with suggestions for future work on the model. Many of the details on the basis states and the relationships among them are given in Appendix A.

Throughout we work in units in which $\hbar = c = 1$.

II. GENERAL BACKGROUND

In a satisfactory relativistic quantum mechanics there exist unitary operators $U(a, b)$ that correspond to the Poincaré transformations, $x' = ax + b$ and map quantum mechanical state vectors from the x frame to the x' frame. For proper transformations these unitary operators can be expressed in terms of ten generators, four of which are the components of the four-momentum operator, $P = (H, \mathbf{P})$, while the other six are the components of the three-vector operators $\mathbf{J}$ and $\mathbf{K}$. Here H is the Hamiltonian operator, $\mathbf{P}$ is the three-momentum operator, $\mathbf{J}$ is the angular momentum operator, and $\mathbf{K}$ is the generator of rotationless boosts.

In the Bakamjian-Thomas scheme [10,11] the ten generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ are expressed in terms of another set of ten Hermitian operators, $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$, by means of the relations

$$H = (\mathbf{P}^2 + M^2)^{1/2}, \tag{2.1a}$$

$$\mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S}, \tag{2.1b}$$

$$\mathbf{K} = -\frac{1}{2}(\mathbf{X}H + H\mathbf{X}) - \frac{\mathbf{P} \times \mathbf{S}}{M + H}. \tag{2.1c}$$

Here M is the mass operator, $\mathbf{S}$ is the spin operator, and $\mathbf{X}$ is the Newton-Wigner position operator [11,13]. The operators $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are chosen to be the same as those for the system of particles without interactions, while the mass operator M contains interactions. The commutation rules for $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are then automatically satisfied, and in order to guarantee Poincaré invariance it is only necessary to ensure that

$$[M, \mathbf{P}] = [M, \mathbf{S}] = [M, \mathbf{X}] = 0. \tag{2.2}$$

With this procedure the generators $\mathbf{P}$ and $\mathbf{J}$ are noninteracting, while H and $\mathbf{K}$ contain interactions. This defines an *instant form* of relativistic quantum mechanics, since the Poincaré transformations related to the noninteracting generators map a Minkowski space, $t =$ const. hypersurface into itself.

III. MODEL SPACE

The model constructed here describes the nucleon, N, two pions, π_1 and π_2, and two sigma mesons, σ_1 and σ_2, with the possible types of state given by $|N\rangle$, $|N\pi_1\rangle$, $|N\pi_2\rangle$, $|N\pi_1\pi_2\rangle$, $|N\pi_1\sigma_2\rangle$, and $|N\pi_2\sigma_1\rangle$. A state of any type is orthogonal to any state of another type, e.g., $\langle N|N\pi_2\rangle = 0$.

The various energies that will be encountered are given by

$$E_a(\mathbf{p}) = \left(\mathbf{p}^2 + m_a^2\right)^{1/2}, \tag{3.1a}$$

$$W_{ab}(\mathbf{q}) = E_a(\mathbf{q}) + E_b(\mathbf{q}), \tag{3.1b}$$

$$E_{ab}(\mathbf{p}, \mathbf{q}) = \left[\mathbf{p}^2 + W_{ab}^2(\mathbf{q})\right]^{1/2}, \tag{3.1c}$$

$$W_{abc}(\mathbf{k}, \mathbf{q}) = E_a(\mathbf{k}) + E_{bc}(\mathbf{k}, \mathbf{q}), \tag{3.1d}$$

$$E_{abc}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \left[\mathbf{p}^2 + W_{abc}^2(\mathbf{k}, \mathbf{q})\right]^{1/2}. \tag{3.1e}$$

In general W will indicate a c.m. energy. For example, $W_{\pi N}(\mathbf{q})$ is the energy of a pion and a nucleon in a c.m. frame in which the pion has three-momentum $\mathbf{q}$ and the nucleon has three-momentum $-\mathbf{q}$, while $E_{\pi N}(\mathbf{p}, \mathbf{q})$ is their energy in a frame in which their total three-momentum is $\mathbf{p}$. The total four-momentum of a set of particles with total energy E and total three-momentum $\mathbf{p}$ is given by

$$p = (p^0, \mathbf{p}) = (E, \mathbf{p}). \tag{3.2}$$

To keep subscripts to a minimum, where convenient we set $\omega = E_\pi$, $\varepsilon = E_N$, and $\sigma = E_\sigma$.

States of total four-momentum p are obtained by boosting a c.m. or rest-frame state, using the unitary operator $U[l_c(p)]$ that corresponds to the so-called canonical boost $l_c(p)$. This particular boost is defined by

$$x = l_c(p) x_{\text{c.m.}}, \tag{3.3a}$$

$$x^0 = \frac{p^0 x^0_{\text{c.m.}} + \mathbf{p} \cdot \mathbf{x}_{\text{c.m.}}}{W}, \tag{3.3b}$$

$$\mathbf{x} = \mathbf{x}_{\text{c.m.}} + \left(x^0_{\text{c.m.}} + \frac{\mathbf{p} \cdot \mathbf{x}_{\text{c.m.}}}{p^0 + W} \right) \frac{\mathbf{p}}{W}, \tag{3.3c}$$

$$W = +(p \cdot p)^{1/2}. \tag{3.3d}$$

The inverse boost is obtained by interchanging x and $x_{\text{c.m.}}$ and letting $\mathbf{p} \to -\mathbf{p}$. The state of a nucleon with three-momentum $\mathbf{p}$, isospin component i, and spin component m is defined by [see Eq. (A17)]

$$|\mathbf{p} i m\rangle = U[l_c(p)] |\mathbf{0} i m\rangle [m_N/\varepsilon(\mathbf{p})]^{1/2}, \tag{3.4}$$

where the rest-frame state $|\mathbf{0} i m\rangle$ is an SU(2) basis state under three-rotations [see Eq. (A2)]. This state, as well as all of the following states, have delta-function-Kronecker-delta normalization as given by (A11). The pion-nucleon states are defined by

$$\begin{aligned} |\mathbf{p}(\mathbf{k}u) i m\rangle_a &= U[l_c(p)] |\mathbf{k}u\rangle_a \otimes |-\mathbf{k}, i m\rangle \\ &\quad \times [W_{\pi N}(\mathbf{k})/E_{\pi N}(\mathbf{p}, \mathbf{k})]^{1/2}, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} |\mathbf{p}(\mathbf{k} i m) u\rangle_a &= U[l_c(p)] |\mathbf{k} i m\rangle \otimes |-\mathbf{k}, u\rangle_a \\ &\quad \times [W_{N\pi}(\mathbf{k})/E_{N\pi}(\mathbf{p}, \mathbf{k})]^{1/2}, \end{aligned} \tag{3.5b}$$
$$a = 1, 2,$$

where in Eqs. (3.5a) and (3.5b) $\mathbf{k}$ is the rest-frame three-momentum of the pion or the nucleon, respectively. Here $|\mathbf{k}u\rangle_a$ is a state of pion π_a with three-momentum $\mathbf{k}$ and isospin component u and is obtained from its rest-frame state in a manner similar to Eq. (3.4). The parentheses around $\mathbf{k}u$ and $\mathbf{k}im$, and in Eqs. (3.6)–(3.10), will play a role when the mass operator interactions are defined. For now they indicate only the state on the left in the direct products. The $\pi_a \pi_b N$ states are defined by

$$\begin{aligned} |\mathbf{p}(\mathbf{k}u) \boldsymbol{\rho} t i m\rangle_a &= U[l_c(p)] |\mathbf{k}u\rangle_a \otimes |-\mathbf{k}(\boldsymbol{\rho} t) i m\rangle_b \\ &\quad \times [W_{\pi\pi N}(\mathbf{k}, \boldsymbol{\rho})/E_{\pi\pi N}(\mathbf{p}, \mathbf{k}, \boldsymbol{\rho})]^{1/2}, \end{aligned} \tag{3.6}$$
$$a = 1, 2; \quad b = 1, 2; \quad a \neq b.$$

Here $|-\mathbf{k}(\boldsymbol{\rho} t) i m\rangle_b$ is obtained from Eq. (3.5a) by letting $a \to b$, $\mathbf{p} \to -\mathbf{k}$, $\mathbf{k} \to \boldsymbol{\rho}$, and $u \to t$. It is important to note that in Eq. (3.6) $\mathbf{k}$ is the three-momentum of π_a in the $\pi_a \pi_b N$ c.m. frame, while $\boldsymbol{\rho}$ is the three-momentum of π_b in a $\pi_b N$ c.m. frame obtained by an inverse canonical boost from the $\pi_a \pi_b N$ c.m. frame. An $N\sigma_b$ state is given by

$$\begin{aligned} |\mathbf{p}(\mathbf{k} i m)\rangle_b &= U[l_c(p)] |\mathbf{k} i m\rangle \otimes |-\mathbf{k}\rangle_b \\ &\quad \times [W_{N\sigma}(\mathbf{k})/E_{N\sigma}(\mathbf{p}, \mathbf{k})]^{1/2}, \quad b = 1, 2, \end{aligned} \tag{3.7}$$

where $|-\mathbf{k}\rangle_b$ is a state of σ_b with three-momentum $-\mathbf{k}$. A $\pi_a N \sigma_b$ state is defined by

$$\begin{aligned} |\mathbf{p}(\mathbf{k}u) \boldsymbol{\xi} i m\rangle_a &= U[l_c(p)] |\mathbf{k}u\rangle_a \otimes |-\mathbf{k}(\boldsymbol{\xi} i m)\rangle_b \\ &\quad \times [W_{\pi N\sigma}(\mathbf{k}, \boldsymbol{\xi})/E_{\pi N\sigma}(\mathbf{p}, \mathbf{k}, \boldsymbol{\xi})]^{1/2}, \end{aligned} \tag{3.8}$$
$$a = 1, 2; \quad b = 1, 2; \quad a \neq b,$$

where $\mathbf{k}$ is the three-momentum of π_a in the $\pi_a N \sigma_b$ c.m. frame, while $\boldsymbol{\xi}$ is the three-momentum of N in a $N\sigma_b$ c.m. frame obtained by an inverse canonical boost from the $\pi_a N \sigma_b$ c.m. frame. A $\pi_a \sigma_b$ state is given by

$$\begin{aligned} |\mathbf{p}(\mathbf{k}u)\rangle_a &= U[l_c(p)] |\mathbf{k}u\rangle_a \otimes |-\mathbf{k}\rangle_b \\ &\quad \times [W_{\pi\sigma}(\mathbf{k})/E_{\pi\sigma}(\mathbf{p}, \mathbf{k})]^{1/2}, \end{aligned} \tag{3.9}$$
$$a = 1, 2; \quad b = 1, 2; \quad a \neq b.$$

An $N\pi_a\sigma_b$ state is defined by

$$\begin{aligned} |\mathbf{p}(\mathbf{k} i m) \boldsymbol{\zeta} u\rangle_a &= U[l_c(p)] |\mathbf{k} i m\rangle \otimes |-\mathbf{k}(\boldsymbol{\zeta} u)\rangle_a \\ &\quad \times [W_{N\pi\sigma}(\mathbf{k}, \boldsymbol{\zeta})/E_{N\pi\sigma}(\mathbf{p}, \mathbf{k}, \boldsymbol{\zeta})]^{1/2}, \end{aligned} \tag{3.10}$$
$$a = 1, 2,$$

where $\mathbf{k}$ is the three-momentum of N in the $N\pi_a\sigma_b$ c.m. frame, while $\boldsymbol{\zeta}$ is the three-momentum of π_a in a $\pi_a\sigma_b$ c.m. frame obtained by an inverse canonical boost from the $N\pi_a\sigma_b$ c.m. frame. It should be noted that the states in Eqs. (3.5) are simply related by $|\mathbf{p}(\mathbf{k}u) i m\rangle_a = |\mathbf{p}(-\mathbf{k}, i m) u\rangle_a$. Also, in Eq. (3.6) the $a = 1$ states are related to the $a = 2$ states, since both are $\pi_1\pi_2 N$ states, and the states defined by Eqs. (3.8) and (3.10) are related to each other, since both are $\pi_a \sigma_b N$ states. The connections between these pairs of states are provided by Eq. (A27).

IV. MASS OPERATOR

The mass operator that acts in the space spanned by the states of Eqs. (3.4)–(3.6), (3.8), and (3.10) is of the form

$$M = M_0 + V, \tag{4.1a}$$

$$V = V_N + \sum_{a=1}^{2} \left(V_a^\pi + V_a^N \right), \tag{4.1b}$$

where M_0 is the noninteracting mass operator and V contains the interactions. The noninteracting mass operator is defined by its action on our basis states, i.e.,

$$M_0 |\mathbf{p} i m\rangle = m_N |\mathbf{p} i m\rangle, \tag{4.2a}$$

$$M_0 |\mathbf{p}(\mathbf{k}u) i m\rangle_a = W_{\pi N}(\mathbf{k}) |\mathbf{p}(\mathbf{k}u) i m\rangle_a, \tag{4.2b}$$

$$M_0 |\mathbf{p}(\mathbf{k}u) \boldsymbol{\rho} t i m\rangle_a = W_{\pi\pi N}(\mathbf{k}, \boldsymbol{\rho}) |\mathbf{p}(\mathbf{k}u) \boldsymbol{\rho} t i m\rangle_a, \tag{4.2c}$$

$$M_0 |\mathbf{p}(\mathbf{k}u) \boldsymbol{\xi} i m\rangle_a = W_{\pi N\sigma}(\mathbf{k}, \boldsymbol{\xi}) |\mathbf{p}(\mathbf{k}u) \boldsymbol{\xi} i m\rangle_a, \tag{4.2d}$$

$$M_0 |\mathbf{p}(\mathbf{k} i m) \boldsymbol{\zeta} u\rangle_a = W_{N\pi\sigma}(\mathbf{k}, \boldsymbol{\zeta}) |\mathbf{p}(\mathbf{k} i m) \boldsymbol{\zeta} u\rangle_a. \tag{4.2e}$$

The interaction V_N acts in the $\{|N\rangle, |N\pi_1\rangle, |N\pi_2\rangle\}$ subspace and is given by

$$V_N = \sum_{im} \int |\mathbf{p}im\rangle d^3p \big[m_N^{(0)} - m_N\big]\langle \mathbf{p}im| + \frac{1}{\sqrt{2}} \sum_{a=1}^{2} \Bigg\{ \sum_{uim} \sum_{i'm'} \int |\mathbf{p}(\mathbf{k}u)im\rangle_a \times U_{\pi NN}(\mathbf{k}uim, i'm') d^3p\, d^3k \langle \mathbf{p}i'm'| + (\dagger) \Bigg\}. \quad (4.3)$$

The first term on the right-hand side is a mass renormalization term with $m_N^{(0)}$ and m_N the bare nucleon mass and physical nucleon mass, respectively. The second term describes the vertex interactions $N \Leftrightarrow \pi_1 + N$ and $N \Leftrightarrow \pi_2 + N$, where $U_{\pi NN}(\mathbf{k}uim, i'm') = U^*_{\pi NN}(i'm', \mathbf{k}uim)$ is a vertex function, a model for which will be presented in Sec. VII.

The interaction V_a^π describes the vertex interactions $N \Leftrightarrow \pi_b + N$ and $N \Leftrightarrow \sigma_b + N$, with π_a playing the role of a spectator, and also includes a renormalization term. This interaction is defined by

$$V_a^\pi = \sum_{uim} \int |\mathbf{p}(\mathbf{k}u)im\rangle_a\, d^3p\, d^3k\, V_N^\pi(k)\, {}_a\langle \mathbf{p}(\mathbf{k}u)im| + \Bigg\{ \sum_{utim} \sum_{i'm'} \int |\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho} tim\rangle_a\, d^3p\, d^3k\, d^3\rho \times V_{\pi NN}(\boldsymbol{\rho} tim, i'm'; \mathbf{k})\, {}_a\langle \mathbf{p}(\mathbf{k}u)i'm'| + (\dagger) \Bigg\} + \Bigg\{ \sum_{uim} \sum_{m'} \int |\mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im\rangle_a\, d^3p\, d^3k\, d^3\xi \times V_{\sigma NN}(\boldsymbol{\xi} m, m'; \mathbf{k})\, {}_a\langle \mathbf{p}(\mathbf{k}u)im'| + (\dagger) \Bigg\}. \quad (4.4)$$

Note that the parentheses around $\mathbf{k}$ and u draw attention to the variables of the spectator pion π_a. Clearly this interaction is diagonal in these variables, which is consistent with the fact that they describe a spectator particle.

The interaction V_a^N describes the vertex interactions $\pi_a \Leftrightarrow \sigma_b + \pi_a$, with N the spectator particle, and also includes a renormalization term. This interaction is defined by

$$V_a^N = \sum_{imu} \int |\mathbf{p}(\mathbf{k}im)u\rangle_a\, d^3p\, d^3k\, V_\pi^N(k)\, {}_a\langle \mathbf{p}(\mathbf{k}im)u| + \Bigg\{ \sum_{imu} \int |\mathbf{p}(\mathbf{k}im)\boldsymbol{\zeta} u\rangle_a d^3p\, d^3k\, d^3\zeta\, V_{\sigma\pi\pi}(\boldsymbol{\zeta}; \mathbf{k}) \times {}_a\langle \mathbf{p}(\mathbf{k}im)u| + (\dagger) \Bigg\}. \quad (4.5)$$

The first terms on the right-hand sides of Eqs. (4.4) and (4.5) are renormalization terms. The function $V_N^\pi(k)$ describes the renormalization of the nucleon in the presence of the spectator pion, while $V_\pi^N(k)$ describes the renormalization of the pion in the presence of the spectator nucleon. We will see in Sec. VI how these functions are determined. Models for the vertex functions $V_{\pi NN}$, $V_{\sigma NN}$, and $V_{\sigma\pi\pi}$ will be presented in Sec. VII.

V. POINCARÉ INVARIANCE

As we saw in Sec. II, in order to ensure that our model is Poincaré invariant, we must verify that our mass operator commutes with $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$. We define the three-momentum operator $\mathbf{P}$ and the Newton-Wigner position operator $\mathbf{X}$ by their representatives in our basis, e.g.,

$${}_a\langle \mathbf{p}(\mathbf{k}u)im|\mathbf{P} = \mathbf{p}\, {}_a\langle \mathbf{p}(\mathbf{k}u)im|, \quad (5.1)$$

$${}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\mathbf{X} = i\nabla_{\mathbf{p}}\, {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|. \quad (5.2)$$

According to Eq. (2.1b) the spin operator $\mathbf{S}$ is determined by $\mathbf{X}$, $\mathbf{P}$, and the total angular momentum operator $\mathbf{J}$. The representatives of $\mathbf{J}$ can be obtained from the fact that states such as those that appear in Eqs. (5.1) and (5.2) rotate according to

$$U(r)|\mathbf{p}(\mathbf{k}u)im\rangle_a = \sum_{m'} |r\mathbf{p}(r\mathbf{k},u)im'\rangle_a D^{(1/2)}_{m'm}(r), \quad (5.3a)$$

$$U(r)|\mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im\rangle_a = \sum_{m'} |r\mathbf{p}(r\mathbf{k},u)r\boldsymbol{\xi},im'\rangle_a D^{(1/2)}_{m'm}(r), \quad (5.3b)$$

where $U(r) = \exp(i\boldsymbol{\theta}\cdot\mathbf{J})$, $r = \exp(i\boldsymbol{\theta}\cdot\mathbf{j})$, and $D^{1/2}(r)$ is a standard SU(2) matrix. These relations follow from Eqs. (A18), (A5), (A12b), (3.5a), and (3.8). Expanding to first order in θ leads to representatives of $\mathbf{J}$, which when combined with Eqs. (2.1b), (5.1), and (5.2) leads to representatives of $\mathbf{S}$. We don't need these representatives, but it is worth noting in passing that for the states in Eqs. (5.1) and (5.2) they are given by

$${}_a\langle \mathbf{p}(\mathbf{k}u)im|\mathbf{S} = \sum_{m'} \left[\mathbf{L}(\mathbf{k})\delta_{mm'} + \frac{1}{2}\boldsymbol{\sigma}_{mm'}\right] {}_a\langle \mathbf{p}(\mathbf{k}u)im'|, \quad (5.4a)$$

$${}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\mathbf{S} = \sum_{m'} \left[\mathbf{L}(\mathbf{k})\delta_{mm'} + \mathbf{L}(\boldsymbol{\xi})\delta_{mm'} + \frac{1}{2}\boldsymbol{\sigma}_{mm'}\right] \times {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im'|, \quad (5.4b)$$

$$\mathbf{L}(\mathbf{x}) = i\nabla \times \mathbf{x}. \quad (5.4c)$$

Note that the representatives of $\mathbf{S}$ don't depend on the total three-momentum $\mathbf{p}$.

Since $\mathbf{P}$, $\mathbf{X}$, and $\mathbf{S}$ are the same as those of the noninteracting system, it follows that

$$[M_0, \mathbf{P}] = [M_0, \mathbf{X}] = [M_0, \mathbf{S}] = \mathbf{0}, \quad (5.5)$$

which in turn implies that it is only necessary to verify that

$$[V, \mathbf{P}] = [V, \mathbf{X}] = [V, \mathbf{S}] = \mathbf{0}. \quad (5.6)$$

This is most easily done by working with matrix elements of the interaction such as

$${}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|V_a^\pi|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_a = \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'} V_{\sigma NN}(\boldsymbol{\xi} m, m'; \mathbf{k}), \quad (5.7)$$

which follows from Eqs. (4.4) and (A11). From here and from

Eqs. (5.1) and (5.2) we find that

$$ {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\left[V_a^\pi, \mathbf{P}\right]|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_a = 0, \tag{5.8} $$

$$ {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\left[V_a^\pi, \mathbf{X}\right]|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_a = 0, \tag{5.9} $$

which when extended to the complete set of basis states gives

$$ \left[V_a^\pi, \mathbf{P}\right] = \left[V_a^\pi, \mathbf{X}\right] = \mathbf{0}. \tag{5.10} $$

To verify that $\mathbf{S}$ commutes with V_a^π let us assume that the vertex function $V_{\sigma NN}$ is rotationally invariant, i.e.,

$$ \sum_{nn'} D_{mn}^{(1/2)}(r^{-1}) V_{\sigma NN}(r\boldsymbol{\xi}, n, n'; r\mathbf{k}) D_{n'm'}^{(1/2)}(r) = V_{\sigma NN}(\boldsymbol{\xi} m, m'; \mathbf{k}). \tag{5.11} $$

It then follows from Eqs. (5.3), (5.7), and the relations $\delta^3(r\mathbf{x}) = \delta^3(\mathbf{x})$ and $D_{mm'}^{(1/2)*}(r) = D_{m'm}^{(1/2)}(r^{-1})$ that

$$ \begin{aligned} &{}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|U^{-1}(r) V_a^\pi U(r)|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_a \\ &= {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|V_a^\pi|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_a. \end{aligned} \tag{5.12} $$

Extending Eq. (5.12) to the complete set of basis states, and expanding $U(r)$ to first order in θ, leads to $[V_a^\pi, \mathbf{J}] = \mathbf{0}$. When this is combined with Eqs. (2.1b) and (5.10) we find that

$$ \left[V_a^\pi, \mathbf{S}\right] = \mathbf{0}. \tag{5.13} $$

Clearly the procedure just outlined can be extended to all of the terms in the mass operator interaction V so as to verify that Eq. (5.6) is satisfied, which in turn proves that our mass operator is Poincaré invariant.

VI. THREE-PARTICLE EQUATIONS

To derive integral equations for the various amplitudes, let us start with the equation for the state vector $|\Psi\rangle$, i.e.,

$$ (W - M_0)|\Psi\rangle = V|\Psi\rangle, \tag{6.1} $$

and after inserting Eqs. (4.1b), (4.3), (4.4), and (4.5), contract with the basis states defined by Eqs. (3.4)–(3.6), (3.8), and (3.10). Upon so doing we encounter overlap of the $a = 1$ and $a = 2$ states defined by Eq. (3.6), as well as overlap of the states defined by Eq. (3.8) and those defined by Eq. (3.10) with the same value of a. We determine these inner products by using Eq. (A27) with the isospin indices added. For the overlap of the (3.6) states, we identify π_a, π_b, N with 1, 2, and 3, respectively, to obtain

$$ \begin{aligned} &{}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\rho} tim|\mathbf{p}'(\mathbf{k}'u')\boldsymbol{\rho}'t'i'm'\rangle_b \\ &= \delta^3(\mathbf{p}-\mathbf{p}')\delta_{ut'}\delta_{tu'}\delta_{ii'}\delta^3[\boldsymbol{\rho} - \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')] \\ &\quad\times \delta^3[\boldsymbol{\rho}' - \mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}')] Q_{mm'}(\mathbf{k}, \mathbf{k}'), \quad a \neq b, \end{aligned} \tag{6.2a} $$

$$ \begin{aligned} Q_{mm'}(\mathbf{k}, \mathbf{k}') &= B_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}') B_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}') \\ &\quad\times D_{mm'}^{(1/2)}\left[r_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}') r_{\pi N}^{-1}(\mathbf{k}, -\mathbf{k}-\mathbf{k}')\right]. \end{aligned} \tag{6.2b} $$

The Kroenecker deltas in the isospin indices follow from the fact that u and t' refer to π_a, t and u' refer to π_b, and i and i' refer to N. To determine the overlap of the Eq. (3.8) and Eq. (3.10) states, in Eq. (A27) we identify π_a, N, and σ_b with 1, 2, and 3, respectively, to obtain

$$ \begin{aligned} &{}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\mathbf{p}'(\mathbf{k}'i'm')\boldsymbol{\zeta}'u'\rangle_a \\ &= \delta^3(\mathbf{p}-\mathbf{p}')\delta_{uu'}\delta_{ii'}\delta^3[\boldsymbol{\xi} - \mathbf{f}_{N\sigma}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')] \\ &\quad\times \delta^3[\boldsymbol{\zeta}' - \mathbf{f}_{\pi\sigma}(\mathbf{k}, -\mathbf{k}-\mathbf{k}')] R_{mm'}(\mathbf{k}, \mathbf{k}'), \end{aligned} \tag{6.3a} $$

$$ \begin{aligned} R_{mm'}(\mathbf{k}, \mathbf{k}') &= B_{N\sigma}(\mathbf{k}', -\mathbf{k}-\mathbf{k}') B_{\pi\sigma}(\mathbf{k}, -\mathbf{k}-\mathbf{k}') \\ &\quad\times D_{mm'}^{(1/2)}[r_{\sigma N}(-\mathbf{k}-\mathbf{k}', \mathbf{k}')]. \end{aligned} \tag{6.3b} $$

Here $\mathbf{f}_{ab}(\mathbf{p}_a, \mathbf{p}_b)$ is the three-momentum of particle a in the c.m. frame of particles a and b, which according to the inverse of Eq. (3.3) is given by

$$ \begin{aligned} \mathbf{f}_{ab}(\mathbf{p}_a, \mathbf{p}_b) &= \frac{1}{2}(\mathbf{p}_a - \mathbf{p}_b) - \frac{1}{2}\left[E_a(\mathbf{p}_a) - E_b(\mathbf{p}_b) + \frac{m_a^2 - m_b^2}{\sqrt{p\cdot p}}\right] \\ &\quad\times \frac{\mathbf{p}}{p^0 + \sqrt{p\cdot p}}, \end{aligned} \tag{6.4a} $$

$$ p = (E_a(\mathbf{p}_a) + E_b(\mathbf{p}_b), \mathbf{p}_a + \mathbf{p}_b). \tag{6.4b} $$

The functions $B_{ab}(\mathbf{p}_a, \mathbf{p}_b)$ are defined by

$$ B_{ab}(\mathbf{p}_a, \mathbf{p}_b) = \left\{ \frac{E_a(\mathbf{p}_a) + E_b(\mathbf{p}_b)}{E_a(\mathbf{p}_a)E_b(\mathbf{p}_b)} \times \frac{E_a[\mathbf{f}_{ab}(\mathbf{p}_a, \mathbf{p}_b)] E_b[\mathbf{f}_{ab}(\mathbf{p}_a, \mathbf{p}_b)]}{W_{ab}[\mathbf{f}_{ab}(\mathbf{p}_a, \mathbf{p}_b)]} \right\}^{1/2}. \tag{6.5} $$

The arguments of the SU(2) representatives $D^{(1/2)}$are given by

$$ r_{ab}(\mathbf{p}_a, \mathbf{p}_b) = r_c\left[l_c^{-1}(p_a + p_b), p_b\right], \quad p_a = (E_a(\mathbf{p}_a), \mathbf{p}_a), \tag{6.6} $$

where r_c is a so-called Wigner rotation [11], which for canonical boosts and a general Lorentz transformation a is defined by

$$ r_c(a, p) = l_c^{-1}(ap) a l_c(p). \tag{6.7} $$

Assuming that our state $|\Psi\rangle$ has a total three-momentum $\mathbf{p}'$, we can write for the various components of our state vector the relations

$$ \langle \mathbf{p} im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi(im), \tag{6.8a} $$

$$ {}_a\langle \mathbf{p}(\mathbf{k}u)im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi(\mathbf{k}uim)/\sqrt{2}, \tag{6.8b} $$

$$ {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\rho} tim|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi(\mathbf{k}u\boldsymbol{\rho} tim)/\sqrt{2}, \tag{6.8c} $$

$$ {}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\xi} im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi(\mathbf{k}u\boldsymbol{\xi} im)/\sqrt{2}, \tag{6.8d} $$

$$ {}_a\langle \mathbf{p}(\mathbf{k}im)\boldsymbol{\zeta} u|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi(\mathbf{k}im\boldsymbol{\zeta} u)/\sqrt{2}. \tag{6.8e} $$

Using Eqs. (6.1), (4.2), (4.1b), (4.3), (4.4), and (4.5), and setting $z = W + i\varepsilon$, we find the following coupled integral equations:

$$\psi(im) = \sum_{u'i'm'} \int \frac{U_{\pi NN}(im, \mathbf{k}'u'i'm')}{z - m_N^{(0)}} d^3k' \psi(\mathbf{k}'u'i'm'), \tag{6.9a}$$

$$\begin{aligned}\left[W - W_{\pi N}(\mathbf{k}) - V_N^\pi(k) - V_\pi^N(k)\right]\psi(\mathbf{k}uim) &= \sum_{i'm'} U_{\pi NN}(\mathbf{k}uim, i'm')\psi(i'm') \\ &+ \sum_{t'i'm'} \int V_{\pi NN}(im, \boldsymbol{\rho}'t'i'm'; \mathbf{k}) d^3\rho' \psi(\mathbf{k}u\boldsymbol{\rho}'t'i'm') \\ &+ \sum_{m'} \int V_{\sigma NN}(m, \boldsymbol{\xi}'m'; \mathbf{k}) d^3\xi' \psi(\mathbf{k}u\boldsymbol{\xi}'im') \\ &+ \int V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k}) d^3\zeta\, \psi(-\mathbf{k}, im\boldsymbol{\zeta}u),\end{aligned} \tag{6.9b}$$

$$\begin{aligned}\psi(\mathbf{k}u\boldsymbol{\rho}tim) &= \sum_{i'm'} \frac{V_{\pi NN}(\boldsymbol{\rho}tim, i'm'; \mathbf{k})}{z - W_{\pi\pi N}(\mathbf{k}, \boldsymbol{\rho})} \psi(\mathbf{k}ui'm') \\ &+ \sum_{m'i''m''} \int \frac{\delta^3[\boldsymbol{\rho} - \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k} - \mathbf{k}')] Q_{mm'}(\mathbf{k}, \mathbf{k}')}{z - \omega(\mathbf{k}) - \varepsilon(-\mathbf{k} - \mathbf{k}') - \omega(\mathbf{k}')} d^3k' \\ &\times V_{\pi NN}[\mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k} - \mathbf{k}'), uim', i''m''; \mathbf{k}'] \psi(\mathbf{k}'ti''m''),\end{aligned} \tag{6.9c}$$

$$\begin{aligned}\psi(\mathbf{k}u\boldsymbol{\xi}im) &= \sum_{m'} \frac{V_{\sigma NN}(\boldsymbol{\xi}m, m'; \mathbf{k})}{z - W_{\pi N\sigma}(\mathbf{k}, \boldsymbol{\xi})} \psi(\mathbf{k}uim') \\ &+ \sum_{m'} \int \frac{\delta^3[\boldsymbol{\xi} - \mathbf{f}_{N\sigma}(-\mathbf{k}', -\mathbf{k} + \mathbf{k}')] R_{mm'}(\mathbf{k}, -\mathbf{k}')}{z - \omega(\mathbf{k}) - \sigma(-\mathbf{k} + \mathbf{k}') - \varepsilon(-\mathbf{k}')} d^3k' \\ &\times V_{\sigma\pi\pi}[f_{\pi\sigma}(\mathbf{k}, -\mathbf{k} + \mathbf{k}'); -\mathbf{k}'] \psi(\mathbf{k}'uim'),\end{aligned} \tag{6.9d}$$

$$\begin{aligned}\psi(-\mathbf{k}, im\boldsymbol{\zeta}u) &= \frac{V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k})}{z - W_{N\pi\sigma}(-\mathbf{k}, \boldsymbol{\zeta})} \psi(\mathbf{k}uim) \\ &+ \sum_{m'} \int \frac{\delta^3[\boldsymbol{\zeta} - \mathbf{f}_{\pi\sigma}(\mathbf{k}', \mathbf{k} - \mathbf{k}')] R^*_{m'm}(\mathbf{k}', -\mathbf{k})}{z - \varepsilon(-\mathbf{k}) - \sigma(\mathbf{k} - \mathbf{k}') - \omega(\mathbf{k}')} d^3k' \\ &\times V_{\sigma NN}[f_{N\sigma}(-\mathbf{k}, \mathbf{k} - \mathbf{k}')m', m''; \mathbf{k}'] \psi(\mathbf{k}'uim'').\end{aligned} \tag{6.9e}$$

In deriving (6.9c)–(6.9e) we used the identities

$$\begin{aligned}W_{\pi\pi N}[\mathbf{k}, \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k} - \mathbf{k}')] \\ = \omega(\mathbf{k}) + \varepsilon(-\mathbf{k} - \mathbf{k}') + \omega(\mathbf{k}'),\end{aligned} \tag{6.10a}$$

$$\begin{aligned}W_{\pi N\sigma}[\mathbf{k}, \mathbf{f}_{N\sigma}(-\mathbf{k}', -\mathbf{k} + \mathbf{k}')] \\ = \omega(\mathbf{k}) + \sigma(-\mathbf{k} + \mathbf{k}') + \varepsilon(-\mathbf{k}'),\end{aligned} \tag{6.10b}$$

$$\begin{aligned}W_{N\pi\sigma}[-\mathbf{k}, \mathbf{f}_{\pi\sigma}(\mathbf{k}', \mathbf{k} - \mathbf{k}')] \\ = \varepsilon(-\mathbf{k}) + \sigma(\mathbf{k} - \mathbf{k}') + \omega(\mathbf{k}').\end{aligned} \tag{6.10c}$$

Relation (6.10a), for example, follows from Eq. (3.1d) and the observation that $E_{\pi N}[\mathbf{k}, \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k} - \mathbf{k}')]$ is the energy in the $\pi\pi N$ c.m. frame of a pion of energy $\omega(\mathbf{k}')$ and a nucleon of energy $\varepsilon(-\mathbf{k} - \mathbf{k}')$. Similar considerations lead to Eqs. (6.10b) and (6.10c).

Substituting Eqs. (6.9a) and (6.9c)–(6.9e) into Eq. (6.9b) leads to an equation for $\psi(\mathbf{k}uim)$, the πN component of the state vector. The first terms on the right-hand sides of Eqs. (6.9c)–(6.9e) lead to the following integrals:

$$\begin{aligned}\sum_{t''i''m''} \int V_{\pi NN}(im, \boldsymbol{\rho}t''i''m''; \mathbf{k}) \\ \times \frac{d^3\rho}{z - W_{\pi\pi N}(k, \rho)} V_{\pi NN}(\boldsymbol{\rho}t''i''m'', i'm'; \mathbf{k}) \\ = \delta_{ii'}\delta_{mm'} \int \frac{d^3\rho F_{\pi NN}(\rho; k)}{z - W_{\pi\pi N}(k, \rho)},\end{aligned} \tag{6.11a}$$

$$F_{\pi NN}(\rho; k) = \int \frac{d\Omega(\boldsymbol{\rho})}{4\pi} \sum_{t''i''m''} |V_{\pi NN}(\boldsymbol{\rho}t''i''m'', im; \mathbf{k})|^2, \tag{6.11b}$$

$$\begin{aligned}\sum_{m''} \int V_{\sigma NN}(m, \boldsymbol{\xi}m''; \mathbf{k}) \frac{d^3\xi}{z - W_{\pi N\sigma}(k, \xi)} V_{\sigma NN}(\boldsymbol{\xi}m'', m'; \mathbf{k}) \\ = \delta_{mm'} \int \frac{d^3\xi\, F_{\sigma NN}(\xi; k)}{z - W_{\pi N\sigma}(k, \xi)},\end{aligned} \tag{6.11c}$$

$$F_{\sigma NN}(\xi; k) = \int \frac{d\Omega(\boldsymbol{\xi})}{4\pi} \sum_{m''} |V_{\sigma NN}(\boldsymbol{\xi}m'', m; \mathbf{k})|^2, \tag{6.11d}$$

$$\begin{aligned}\int V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k}) \frac{d^3\zeta}{z - W_{N\pi\sigma}(k, \zeta)} V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k}) \\ = \int \frac{d^3\zeta\, F_{\sigma\pi\pi}(\zeta; k)}{z - W_{N\pi\sigma}(k, \zeta)},\end{aligned} \tag{6.11e}$$

$$F_{\sigma\pi\pi}(\zeta; k) = \int \frac{d\Omega(\boldsymbol{\zeta})}{4\pi} |V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k})|^2. \tag{6.11f}$$

In writing Eqs. (6.11a) and (6.11c) we have anticipated results obtained in Sec. VII. Combining the above results with the left-hand side of Eq. (6.9b), we are led to define

$$\begin{aligned}d(k, z) = Z_{\pi N}(k)\Bigg[z - W_{\pi N}(k) - V_N^\pi(k) - V_\pi^N(k) \\ - \int d^3\rho \frac{F_{\pi NN}(\rho; k)}{z - W_{\pi\pi N}(k, \rho)} - \int d^3\xi \frac{F_{\sigma NN}(\xi; k)}{z - W_{\pi N\sigma}(k, \xi)} \\ - \int d^3\zeta \frac{F_{\sigma\pi\pi}(\zeta; k)}{z - W_{N\pi\sigma}(k, \zeta)}\Bigg],\end{aligned} \tag{6.12}$$

where we have introduced a function $Z_{\pi N}(k)$, which we will now define. Since $d^{-1}(k, z)$ is a propagator for the πN system, we require that

$$d(k, z) \underset{z \to W_{\pi N}(k)}{\longrightarrow} z - W_{\pi N}(k), \tag{6.13}$$

which leads us to define

$$\begin{aligned}V_N^\pi(k) = -\int d^3\rho \frac{F_{\pi NN}(\rho; k)}{W_{\pi N}(k) - W_{\pi\pi N}(k, \rho)} \\ - \int d^3\xi \frac{F_{\sigma NN}(\xi; k)}{W_{\pi N}(k) - W_{\pi N\sigma}(k, \xi)},\end{aligned} \tag{6.14a}$$

$$V_\pi^N(k) = -\int d^3\zeta \frac{F_{\sigma\pi\pi}(\zeta; k)}{W_{N\pi}(k) - W_{N\pi\sigma}(k, \zeta)}. \tag{6.14b}$$

We see that $V_N^\pi(k)$ and $V_\pi^N(k)$ renormalize the nucleon and the pion in the presence of a spectator pion and nucleon, respectively. Using Eq. (3.1), it is easy to verify that the denominators in Eq. (6.14) do not vanish, so that $V_N^\pi(k)$ and $V_\pi^N(k)$ are real, as they must be to ensure that the interactions (4.4) and (4.5) are Hermitian. Putting Eq. (6.14) into Eq. (6.12), we now have

$$d(k,z) = Z_{\pi N}(k)[z - W_{\pi N}(k)] \times \left\{ 1 + \int d^3\rho \frac{F_{\pi NN}(\rho;k)}{[z - W_{\pi\pi N}(k,\rho)][W_{\pi N}(k) - W_{\pi\pi N}(k,\rho)]} + \int d^3\xi \frac{F_{\sigma NN}(\xi;k)}{[z - W_{\pi N\sigma}(k,\xi)][W_{\pi N}(k) - W_{\pi N\sigma}(k,\xi)]} + \int d^3\zeta \frac{F_{\sigma\pi\pi}(\zeta;k)}{[z - W_{N\pi\sigma}(k,\zeta)][W_{N\pi}(k) - W_{N\pi\sigma}(k,\zeta)]} \right\}. \tag{6.15}$$

A formula for $Z_{\pi N}(k)$ can be found by combining Eqs. (6.13) and (6.15).

The second terms on the right-hand sides of Eqs. (6.9c)–(6.9e), along with Eq. (6.9a), when inserted in Eq. (6.9b), lead to various energy-dependent, effective πN–πN potentials, which are given by

$$B_d(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \sum_{i''m''} \frac{Z_{\pi N}^{1/2}(k) U_{\pi NN}(\mathbf{k}uim, i''m'') U_{\pi NN}(i''m'', \mathbf{k}'u'i'm') Z_{\pi N}^{1/2}(k')}{z - m_N^{(0)}}, \tag{6.16}$$

$$B_c(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \sum_{jnn'} \frac{V_{\pi NN}(im, \boldsymbol{\rho}u'jn; \mathbf{k}) Z_{\pi N}^{1/2}(k) Q_{nn'}(\mathbf{k},\mathbf{k}') Z_{\pi N}^{1/2}(k') V_{\pi NN}(\boldsymbol{\rho}'ujn', i'm'; \mathbf{k}')}{z - \omega(\mathbf{k}) - \varepsilon(-\mathbf{k}-\mathbf{k}') - \omega(\mathbf{k}')}, \tag{6.17a}$$

$$\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}'), \quad \boldsymbol{\rho}' = \mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}'), \tag{6.17b}$$

$$B_e(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \delta_{uu'}\delta_{ii'}[B_\sigma(\mathbf{k}m, \mathbf{k}'m'; z) + B_\sigma^*(\mathbf{k}'m', \mathbf{k}m; z^*)], \tag{6.18a}$$

$$B_\sigma(\mathbf{k}m, \mathbf{k}'m'; z) = \sum_n \frac{V_{\sigma NN}(m, \boldsymbol{\xi}n; \mathbf{k}) Z_{\pi N}^{1/2}(k) R_{nm'}(\mathbf{k}, -\mathbf{k}') Z_{\pi N}^{1/2}(k') V_{\sigma\pi\pi}(\boldsymbol{\zeta}; -\mathbf{k}')}{z - \omega(\mathbf{k}) - \sigma(-\mathbf{k}+\mathbf{k}') - \varepsilon(-\mathbf{k}')} \tag{6.18b}$$

$$\boldsymbol{\xi} = \mathbf{f}_{N\sigma}(-\mathbf{k}', -\mathbf{k}+\mathbf{k}'), \quad \boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k}, -\mathbf{k}+\mathbf{k}'). \tag{6.18c}$$

Here d indicates a direct or s-channel contribution, c a crossed or u-channel contribution, and e an exchange or t-channel contribution.

Solving Eq. (6.1) has now been reduced to solving the integral equation

$$d(k,z) Z_{\pi N}^{-1/2}(k) \psi(\mathbf{k}uim) = \sum_{u'i'm'} \int B(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \times d^3k' Z_{\pi N}^{-1/2}(k') \psi(\mathbf{k}'u'i'm'), \tag{6.19}$$

with $z = W + i\varepsilon$, and where the complete effective πN–πN potential is given by

$$B(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \sum_{x=d,c,e} B_x(\mathbf{k}uim, \mathbf{k}'u'i'm'; z). \tag{6.20}$$

We now derive an equation for the elastic πN scattering amplitude. We introduce an "initial" state as

$$|\mathbf{p}'\mathbf{k}'u'i'm'\rangle = \frac{Z_{\pi N}^{1/2}(k')}{\sqrt{2}} \sum_{a=1}^{2} |\mathbf{p}'\mathbf{k}'u'i'm'\rangle_a \tag{6.21}$$

and now denote our state vector $|\Psi\rangle$ as $|\Psi(\mathbf{p}'\mathbf{k}'u'i'm')\rangle$. Using Eqs. (6.8b), (A11), and (6.15), we can rewrite Eq. (6.19) as

$$Z_{\pi N}^{-1/2}(k) \psi(\mathbf{k}uim, \mathbf{k}'u'i'm') = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}\delta_{mm'} + \frac{1}{d(k,z)} \sum_{u''i''m''} \int B(\mathbf{k}uim, \mathbf{k}''u''i''m''; z) d^3k'' Z_{\pi N}^{-1/2}(k'') \times \psi(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm'), \tag{6.22}$$

where we have added the initial state labels to the argument of ψ and identified $z = W_{\pi N}(k') + i\varepsilon$. We define the elastic scattering amplitudes as the residue of the pole in $d^{-1}(k,z)$ at $z = W_{\pi N}(k)$, i.e.,

$$X(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \sum_{u''i''m''} \int B(\mathbf{k}uim, \mathbf{k}''u''i''m''; z) \times d^3k'' Z_{\pi N}^{-1/2}(k'') \psi(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm'), \tag{6.23}$$

where upon using Eq. (6.22) we obtain the Lippmann-Schwinger-like equation

$$X(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = B(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) + \sum_{u''i''m''} \int B(\mathbf{k}uim, \mathbf{k}''u''i''m''; z) \times \frac{d^3k''}{d(k'',z)} X(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm'; z). \tag{6.24}$$

We now derive a formula for the production amplitude for the process $\pi N \to \pi\pi N$. From Eqs. (A25), (6.6), and (6.7) we get the relation between a $\pi\pi N$ state in which the two pions are treated in a symmetric way and our states (3.6), i.e.,

$$|\mathbf{p}\mathbf{k}_1 u_1 \mathbf{k}_2 u_2 im\rangle = \sum_{m'} |\mathbf{p}(\mathbf{k}_a u_a)\boldsymbol{\rho}_{bN} u_b im'\rangle_a D^{(1/2)}_{m'm}\{r_c^{-1}[l_c(k_{bN}), \rho_N]\} \times B_{\pi N}(\mathbf{k}_b, -\mathbf{k}_1 - \mathbf{k}_2), \quad (6.25a)$$

$$k_{bN} = (\omega(\mathbf{k}_b) + \varepsilon(-\mathbf{k}_1 - \mathbf{k}_2), -\mathbf{k}_a),$$
$$\rho_N = (\varepsilon(\boldsymbol{\rho}_{bN}), -\boldsymbol{\rho}_{bN}),$$
$$\boldsymbol{\rho}_{bN} = \mathbf{f}_{\pi N}(\mathbf{k}_b, -\mathbf{k}_1 - \mathbf{k}_2), \quad a = 1, 2; \quad b = 1, 2; \quad a \neq b, \quad (6.25b)$$

We have fleshed out the notation by adding subscripts to the $\mathbf{k}$ and $\boldsymbol{\rho}$ that appear in the states of Eq. (3.6) to clarify which particles they refer to. We now contract Eq. (6.1) with the symmetric $\pi\pi N$ state (6.25a). According to Eqs. (4.1b), (4.3), (4.4), and (4.5) only the V_a^π terms contribute. For the V_1^π and V_2^π terms we use Eq. (6.25a) with $a = 1$ and $a = 2$, respectively. With the help of Eq. (6.8b) we find

$$\langle \mathbf{p}\mathbf{k}_1 u_1 \mathbf{k}_2 u_2 im|\Psi(\mathbf{p}'\mathbf{k}'u'i'm')\rangle = \frac{\delta^3(\mathbf{p} - \mathbf{p}')}{z - \widetilde{W}_{\pi\pi N}(\mathbf{k}_1, \mathbf{k}_2)} \frac{1}{\sqrt{2}} \sum_{a=1}^{2} \sum_{ni''m''} D^{(1/2)}_{mn}\{r_c[l_c(k_{bN}), \rho_N]\} \times V_{\pi NN}(\boldsymbol{\rho}_{bN} u_b in, i''m''; k_a)\psi(\mathbf{k}_a u_a i''m'', \mathbf{k}'u'i'm') \times B_{\pi N}(\mathbf{k}_b, -\mathbf{k}_1 - \mathbf{k}_2), \quad (6.26)$$

where the $\pi\pi N$ c.m. energy is given by

$$\widetilde{W}_{\pi\pi N}(\mathbf{k}_1, \mathbf{k}_2) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) + \varepsilon(-\mathbf{k}_1 - \mathbf{k}_2) = W_{\pi\pi N}(\mathbf{k}_1, \boldsymbol{\rho}_{23}) = W_{\pi\pi N}(\mathbf{k}_2, \boldsymbol{\rho}_{13}). \quad (6.27)$$

For the production amplitude we take the residue of the pole in (6.26) at $z = \widetilde{W}_{\pi\pi N}(\mathbf{k}_1, \mathbf{k}_2)$. It is easy to see that the delta-function term in Eq. (6.22) does not contribute, since it is not possible to have $\mathbf{k}_a = \mathbf{k}'$ and $W_{\pi N}(\mathbf{k}') = W_{\pi N}(\mathbf{k}_a) = \widetilde{W}_{\pi\pi N}(\mathbf{k}_1, \mathbf{k}_2)$, so that the production amplitude is given by

$$Y_{\pi\pi N}(\mathbf{k}_1 u_1 \mathbf{k}_2 u_2 im; \mathbf{k}'u'i'm') = \frac{1}{\sqrt{2}} \sum_{a=1}^{2} \sum_{ni''m''} D^{(1/2)}_{mn}\{r_c[l_c(k_{bN}), \rho_N]\} \times V_{\pi NN}(\boldsymbol{\rho}_{b3} u_b in, i''m''; \mathbf{k}_a) Z^{1/2}_{\pi N}(k_a) \times \frac{X(\mathbf{k}_a u_a i''m'', \mathbf{k}'u'i'm'; z)}{d(k_a, z)} B_{\pi N}(\mathbf{k}_b, -\mathbf{k}_1 - \mathbf{k}_2). \quad (6.28)$$

Amplitude (6.28) is of the form commonly used in isobar models. Here the initial pion and nucleon undergo an off-shell scattering, described by X, to an intermediate pion-nucleon state that propagates according to $d^{-1}(k_a, z)$. This is followed by the emission of a pion by the intermediate nucleon.

Using the same techniques that led to Eq. (6.28) we can show that the production amplitude for the process $\pi N \to \pi\sigma N$ is given by

$$Y_{\pi\sigma N}(\mathbf{k}_\pi u \mathbf{k}_N im; \mathbf{k}'u'i'm') = \sum_{nm''} D^{(1/2)}_{mn}\{r_c[l_c(k_{N\sigma}), \xi_N]\} \times V_{\sigma NN}(\boldsymbol{\xi} n, m''; \mathbf{k}_\pi) Z^{1/2}_{\pi N}(k_\pi) \frac{X(\mathbf{k}_\pi uim'', \mathbf{k}'u'i'm'; z)}{d(k_\pi, z)} \times B_{N\sigma}(\mathbf{k}_N, -\mathbf{k}_\pi - \mathbf{k}_N) + V_{\sigma\pi\pi}(\boldsymbol{\zeta}; \mathbf{k}_N) Z^{1/2}_{\pi N}(k_N) \times \frac{X(-\mathbf{k}_N, uim, \mathbf{k}'u'i'm'; z)}{d(k_N, z)} B_{\pi\sigma}(\mathbf{k}_\pi, -\mathbf{k}_\pi - \mathbf{k}_N), \quad (6.29a)$$

$$k_{N\sigma} = (\varepsilon(\mathbf{k}_N) + \sigma(-\mathbf{k}_\pi - \mathbf{k}_N), -\mathbf{k}_\pi), \quad \xi_N = (\varepsilon(\boldsymbol{\xi}), \boldsymbol{\xi}),$$
$$\boldsymbol{\xi} = \mathbf{f}_{N\sigma}(\mathbf{k}_N, -\mathbf{k}_\pi - \mathbf{k}_N), \quad \boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k}_\pi, -\mathbf{k}_\pi - \mathbf{k}_N). \quad (6.29b)$$

Note that the c.m. energy of the $\pi\sigma N$ state is given by

$$\widetilde{W}_{\pi N\sigma}(\mathbf{k}_\pi, \mathbf{k}_N) = \omega(\mathbf{k}_\pi) + \sigma(-\mathbf{k}_\pi - \mathbf{k}_N) + \varepsilon(\mathbf{k}_N) = W_{\pi N\sigma}(\mathbf{k}_\pi, \boldsymbol{\xi}) = W_{N\pi\sigma}(\mathbf{k}_N, \boldsymbol{\zeta}). \quad (6.30)$$

VII. EXPLICIT MODELS FOR VERTEX FUNCTIONS

Here we will use well-known effective Lagrangians to develop explicit models for the various vertex functions that appear in our mass operator. This will lead to models for the pion-nucleon propagator, $d(k, z)$, and for the effective pion-nucleon potential, $B(\mathbf{k}uim, \mathbf{k}'u'i'm'; z)$. By coupling the isospins we will simplify the formulas for the potentials, and for the elastic scattering and production amplitudes.

To develop a model for the $\pi\pi N$ vertex function we use the interaction Hamiltonian, $H_{\pi NN}$, given by Eqs. (3.9)–(3.11) of Ref. [16]. For the process $(\mathbf{p}_1 u) + (\mathbf{p}_2 t) + (\mathbf{p}_N im) \Leftrightarrow (\mathbf{p}'_1 u') + (\mathbf{p}'_N i'm')$ the matrix element of this Hamiltonian is given by

$$\langle \mathbf{p}_1 u, \mathbf{p}_2 t, \mathbf{p}_N im|H_{\pi NN}|\mathbf{p}'_1 u', \mathbf{p}'_N i'm'\rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta^3(\mathbf{p}_1 - \mathbf{p}'_1)\delta_{uu'} \times H_{\pi NN}(\mathbf{p}_2 t, \mathbf{p}_N im; \mathbf{p}'_N i'm') + (\mathbf{p}_1 u \Leftrightarrow \mathbf{p}_2 t), \quad (7.1)$$
$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_N, \quad \mathbf{p}' = \mathbf{p}'_1 + \mathbf{p}'_N,$$

where

$$H_{\pi NN}(\mathbf{p}_2 t, \mathbf{p}_N im; \mathbf{p}'_N i'm') = ig_{\pi NN}(\boldsymbol{\varepsilon}_t^* \cdot \boldsymbol{\tau})_{ii'} C_{\pi N}(\mathbf{p}_2, \mathbf{p}_N, \mathbf{p}'_N) \times \overline{u}(p_N, m)\Gamma(-p_2)u(p'_N, m'), \quad (7.2a)$$

$$\boldsymbol{\varepsilon}_\pm = \mp(1/\sqrt{2})(1, \pm i, 0), \quad \boldsymbol{\varepsilon}_0 = (0, 0, 1), \quad (7.2b)$$

$$\Gamma(q) = \left[\lambda + (1 - \lambda)\frac{\gamma_\mu q^\mu}{2m_N}\right]\gamma_5, \quad (7.2c)$$

$$C_{ab}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}'_b) = m_b/[(2\pi)^3 2E_a(\mathbf{p}_a)E_b(\mathbf{p}_b)E_b(\mathbf{p}'_b)]^{1/2}. \quad (7.3)$$

Here $g_{\pi NN}$ is the pion-nucleon coupling constant. The parameter λ varies between 0 and 1 and determines the mix of pseudoscalar and pseudovector coupling, with $\lambda = 0$ and $\lambda = 1$ corresponding to pure pseudovector and pure pseudoscalar coupling, respectively. In order to get an expression for $V_{\pi NN}(\boldsymbol{\rho} tim, i'm'; \mathbf{k})$, the mass operator matrix element that appears in Eq. (4.4), we evaluate Eq. (7.2a) in the c.m. frame and transform to the basis used in Eq. (4.4). From Eqs. (3.6), (A22), (A20), and (6.5) it follows that

$$\begin{aligned}|\mathbf{p}=\mathbf{0}, (\mathbf{k}u)\boldsymbol{\rho} tim\rangle_1 &= \sum_{m'} |\mathbf{k}=\mathbf{k}_1, u\rangle \otimes |\mathbf{k}_2 t\rangle \otimes |\mathbf{k}_N im'\rangle \\ &\quad\times D^{(1/2)}_{m'm}\{r_c[l_c(k_{2N}), \rho_N]\} B^{-1}_{\pi N}(\mathbf{k}_2, \mathbf{k}_N), \quad (7.4a)\end{aligned}$$

$$\begin{aligned}k_{2N} &= k_2 + k_N = (\omega(\mathbf{k}_2)+\varepsilon(\mathbf{k}_N), \mathbf{k}_2+\mathbf{k}_N) \\ &= (E_{\pi N}(-\mathbf{k}, \boldsymbol{\rho}), -\mathbf{k}), \quad (7.4b)\end{aligned}$$

$$\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}_2, \mathbf{k}_N), \qquad \rho_N = l_c^{-1}(k_{2N})k_N = (\varepsilon(\boldsymbol{\rho}), -\boldsymbol{\rho}). \quad (7.4c)$$

Here $\mathbf{k}(=\mathbf{k}_1)$, $\mathbf{k}_2$, and $\mathbf{k}_N$ are the three-momenta of π_1, π_2, and N in their c.m. frame. According to Eq. (3.5a)

$$|\mathbf{p}'=\mathbf{0}, (\mathbf{k}'u')i'm'\rangle_1 = |\mathbf{k}'=\mathbf{k}'_1, u'\rangle \otimes |-\mathbf{k}', i'm'\rangle. \quad (7.5)$$

Here $\mathbf{k}'=\mathbf{k}'_1=-\mathbf{k}'_N=\mathbf{k}$, where $\mathbf{k}'_1$ and $\mathbf{k}'_N$ are the three-momenta of π'_1 and N' in their c.m. frame. Using these relations, we now find that the πNN vertex function that appears in Eq. (4.4) is given by

$$\begin{aligned}V_{\pi NN}(\boldsymbol{\rho} tim, i'm'; \mathbf{k}) &= G_{\pi NN}(\rho)\sum_{m''} B^{-1}_{\pi N}(\mathbf{k}_2, \mathbf{k}_N) D^{(1/2)}_{mm''}\{r_c^{-1}[l_c(k_{2N}), \rho_N]\} \\ &\quad\times H_{\pi NN}(\mathbf{k}_2 t, \mathbf{k}_N im''; -\mathbf{k}, i'm'), \quad (7.6)\end{aligned}$$

where $G_{\pi NN}(\rho)$ is a phenomenological cutoff function that is introduced to provide convergence.

Under a Lorentz transformation a a Dirac spinor transforms according to

$$S(a)u(p,m) = \sum_{m'} u(ap, m') D^{(1/2)}_{m'm}[r_c(a,p)]. \quad (7.7)$$

With the help of this relation and Eq. (6.5), along with the fact that $r_c[l_c^{-1}(k_{2N}), (\varepsilon(-\mathbf{k}), -\mathbf{k})] = 1$, which follows from Eqs. (3.3) and (6.7), we find that

$$\begin{aligned}V_{\pi NN}(\boldsymbol{\rho} tim, i'm'; \mathbf{k}) &= V^*_{\pi NN}(i'm', \boldsymbol{\rho} tim; \mathbf{k}) \\ &= i g_{\pi NN} G_{\pi NN}(\rho)(\boldsymbol{\varepsilon}^*_t \cdot \boldsymbol{\tau})_{ii'}\left[\frac{W_{\pi N}(\boldsymbol{\rho})}{E_{\pi N}(-\mathbf{k}, \boldsymbol{\rho})}\right]^{1/2} \\ &\quad\times C_{\pi N}(\boldsymbol{\rho}, \boldsymbol{\rho}, -\mathbf{k})\overline{u}(\rho_N, m)\Gamma(-\rho_\pi)u(\rho'_N, m'), \quad (7.8a)\end{aligned}$$

$$\begin{aligned}\rho_\pi &= (\omega(\boldsymbol{\rho}), \boldsymbol{\rho}), \;\; \rho_N = (\varepsilon(\boldsymbol{\rho}), -\boldsymbol{\rho}), \\ \rho'_N &= l_c^{-1}(k_{2N})(\varepsilon(-\mathbf{k}), -\mathbf{k}) = (\varepsilon(\boldsymbol{\rho}'), -\boldsymbol{\rho}'), \\ \varepsilon(\boldsymbol{\rho}') &= [E_{\pi N}(-\mathbf{k}, \boldsymbol{\rho})\varepsilon(-\mathbf{k}) - \mathbf{k}^2]/W_{\pi N}(\boldsymbol{\rho}), \\ \boldsymbol{\rho}' &= \mathbf{k}[E_{\pi N}(-\mathbf{k}, \boldsymbol{\rho}) - \varepsilon(-\mathbf{k})]/W_{\pi N}(\boldsymbol{\rho}).\end{aligned} \quad (7.8b)$$

It is straightforward to show that

$$\overline{u}(\rho_N, m)\Gamma(-\rho_\pi)u(\rho'_N, m') = \chi^\dagger_m \boldsymbol{\sigma}\cdot\mathbf{Q}(\boldsymbol{\rho}, \mathbf{k})\chi_{m'}, \quad (7.9a)$$

$$\begin{aligned}\mathbf{Q}(\boldsymbol{\rho}, \mathbf{k}) &= \left[\frac{\varepsilon(\boldsymbol{\rho})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(\boldsymbol{\rho}')+m_N}{2m_N}\right]^{1/2} \\ &\quad\times\left[\frac{\boldsymbol{\rho}}{\varepsilon(\boldsymbol{\rho})+m_N}A_+(\boldsymbol{\rho}) - \frac{\boldsymbol{\rho}'}{\varepsilon(\boldsymbol{\rho}')+m_N}A_-(\boldsymbol{\rho})\right], \quad (7.9b)\end{aligned}$$

$$A_\pm(\boldsymbol{\rho}) = \lambda + (1-\lambda)\frac{\pm W_{\pi N}(\boldsymbol{\rho}) + m_N}{2m_N}. \quad (7.9c)$$

Here χ_m is a two-component, spin-1/2 spinor. Using Eqs. (7.8) and (7.9) it can be verified that Eq. (6.11a) does indeed contain $\delta_{ii'}\delta_{mm'}$ and that $F_{\pi NN}$ is only a function of $\rho = |\boldsymbol{\rho}|$ and $k = |\mathbf{k}|$.

The vertex function $U_{\pi NN}(\mathbf{k}uim, i'm')$ that appears in Eq. (4.3) can be obtained from Eqs. (7.8) and (7.9) by setting $\mathbf{k}=\mathbf{0}$ and letting $\boldsymbol{\rho}\to\mathbf{k}$. We find

$$\begin{aligned}U_{\pi NN}(\mathbf{k}uim, i'm') &= U^*_{\pi NN}(i'm', \mathbf{k}uim) \\ &= (i/\sqrt{3})(\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}})_{mm'}U_{\pi NN}(k), \quad (7.10a)\end{aligned}$$

$$\begin{aligned}U_{\pi NN}(k) &= \sqrt{3}g^{(0)}_{\pi NN}G^{(0)}_{\pi NN}(k)\left[\frac{m_N}{(2\pi)^3 2\omega(k)\varepsilon(k)}\right]^{1/2} \\ &\quad\times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{k}{\varepsilon(k)+m_N}A_+(k). \quad (7.10b)\end{aligned}$$

Here $g^{(0)}_{\pi NN}$ and $G^{(0)}_{\pi NN}(k)$ are a "bare" coupling constant and vertex function. The justification for using different coupling constants and vertex functions in Eqs. (7.8) and (7.10) is given in earlier work by Fuda [29] and Pearce and Afnan [30].

To develop models for the vertex functions $V_{\sigma NN}$ and $V_{\sigma\pi\pi}$ that appear in Eqs. (4.4) and (4.5), we use the Lagrangians given by Eqs. (A12) and (A13) of Ref. [16]. Using the same techniques as those that led to Eqs. (7.8) and (7.9), we find

$$\begin{aligned}V_{\sigma NN}(\boldsymbol{\xi} m, m'; \mathbf{k}) &= V^*_{\sigma NN}(m', \boldsymbol{\xi} m; \mathbf{k}) \\ &= -g_{\sigma NN}G_{\sigma NN}(\xi)\left[\frac{W_{N\sigma}(\boldsymbol{\xi})}{E_{N\sigma}(-\mathbf{k}, \boldsymbol{\xi})}\right]^{1/2} \\ &\quad\times C_{\sigma N}(\boldsymbol{\xi}, \boldsymbol{\xi}, -\mathbf{k})\overline{u}(\xi_N, m)u(\xi'_N, m'), \quad (7.11a)\end{aligned}$$

$$\begin{aligned}\xi_N &= (\varepsilon(\boldsymbol{\xi}), \boldsymbol{\xi}), \\ \xi'_N &= l_c^{-1}(k_{N\sigma})(\varepsilon(-\mathbf{k}), -\mathbf{k}) = (\varepsilon(\boldsymbol{\xi}'), -\boldsymbol{\xi}'), \\ \varepsilon(\boldsymbol{\xi}') &= [E_{N\sigma}(-\mathbf{k}, \boldsymbol{\xi})\varepsilon(-\mathbf{k}) - \mathbf{k}^2]/W_{N\sigma}(\boldsymbol{\xi}), \\ \boldsymbol{\xi}' &= \mathbf{k}[E_{N\sigma}(-\mathbf{k}, \boldsymbol{\xi}) - \varepsilon(-\mathbf{k})]/W_{N\sigma}(\boldsymbol{\xi}).\end{aligned} \quad (7.11b)$$

It is straightforward to show that

$$\begin{aligned}\overline{u}(\xi_N, m)u(\xi'_N, m') &= \left[\frac{\varepsilon(\boldsymbol{\xi})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(\boldsymbol{\xi}')+m_N}{2m_N}\right]^{1/2} \\ &\quad\times\chi^\dagger_m\left\{1+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}')}{[\varepsilon(\boldsymbol{\xi})+m_N][\varepsilon(\boldsymbol{\xi}')+m_N]}\right\}\chi_{m'}. \quad (7.12)\end{aligned}$$

We also find

$$V_{\sigma\pi\pi}(\boldsymbol{\zeta};\mathbf{k}) = -g_{\sigma\pi\pi}G_{\sigma\pi\pi}(\zeta)\left[\frac{W_{\pi\sigma}(\boldsymbol{\zeta})}{E_{\pi\sigma}(-\mathbf{k},\boldsymbol{\zeta})}\right]^{1/2} \times C_{\sigma\pi}(\boldsymbol{\zeta},\boldsymbol{\zeta},-\mathbf{k})\frac{1}{2}\left(1+\frac{\widetilde{g}_{\sigma\pi\pi}}{g_{\sigma\pi\pi}}\frac{\zeta_\pi\cdot\zeta'_\pi}{m_\pi^2}\right), \tag{7.13a}$$

$$\begin{aligned}\zeta_\pi &= (\omega(\boldsymbol{\zeta}),\boldsymbol{\zeta}),\\ \zeta'_\pi &= l_c^{-1}(k_{\pi\sigma})(\omega(-\mathbf{k}),-\mathbf{k}) = (\omega(\boldsymbol{\zeta}'),-\boldsymbol{\zeta}'),\\ \omega(\boldsymbol{\zeta}') &= [E_{\pi\sigma}(-\mathbf{k},\boldsymbol{\zeta})\omega(-\mathbf{k})-\mathbf{k}^2]/W_{\pi\sigma}(\boldsymbol{\zeta}),\\ \boldsymbol{\zeta}' &= \mathbf{k}[E_{\pi\sigma}(-\mathbf{k},\boldsymbol{\zeta})-\omega(-\mathbf{k})]/W_{\pi\sigma}(\boldsymbol{\zeta}).\end{aligned} \tag{7.13b}$$

Using Eqs. (7.11)–(7.13) it can be verified that Eq. (6.11c) does contain $\delta_{mm'}$, $F_{\sigma NN}$ is only a function of $\xi = |\boldsymbol{\xi}|$ and $k = |\mathbf{k}|$, and $F_{\sigma\pi\pi}$ is only a function of $\zeta = |\boldsymbol{\zeta}|$ and $k = |\mathbf{k}\rangle$.

We now turn our attention to the effective $\pi N-\pi N$ potentials defined by Eqs. (6.16)–(6.18). We can simplify the various expressions by coupling the isospins and writing

$$\sum_{ui}\sum_{u'i'}\langle 1,1/2,u,i|T,M\rangle A(\mathbf{k}uim,\mathbf{k}'u'i'm';z) \times\langle 1,1/2,u',i'|T'M'\rangle = \delta_{TT'}\delta_{MM'}\chi_m^\dagger A^T(\mathbf{k},\mathbf{k}';z)\chi_{m'},\quad A = B_d, B_c, B_e, B, X. \tag{7.14}$$

Here $\langle 1,1/2,u,i|TM\rangle$ is a Clebsch-Gordon coefficient. A helpful identity in applying Eq. (7.14) is given by [16]

$$(\boldsymbol{\varepsilon}_u^*\cdot\boldsymbol{\tau})_{ii'} = -\sqrt{3}\,\langle 1,1/2,u,i|1/2,i'\rangle. \tag{7.15}$$

From Eqs. (6.16) and (7.10) it follows that the s-channel or direct potential is given by

$$B_d^T(\mathbf{k},\mathbf{k}';z) = \delta_{T,1/2}Z_{\pi N}^{1/2}(k)(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}})\frac{U_{\pi NN}(k)U_{\pi NN}(k')}{z-m_N^{(0)}} \times(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}}')Z_{\pi N}^{1/2}(k'). \tag{7.16}$$

The u-channel or crossed potential is given by Eq. (6.17). To express it in terms of the vertex function defined by Eqs. (7.8) and (7.9), we invert Eq. (7.6) and use Eqs. (7.4b), (7.4c), (6.6), and (6.7) to obtain

$$\sum_{n'}D_{m''n'}^{(1/2)}[r_{\pi N}^{-1}(\mathbf{k},-\mathbf{k}-\mathbf{k}')]B_{\pi N}(\mathbf{k},-\mathbf{k}-\mathbf{k}') \times V_{\pi NN}(\boldsymbol{\rho}'ujn',i'm';\mathbf{k}') = G_{\pi NN}(\rho')H_{\pi NN}(\mathbf{k}u,-\mathbf{k}-\mathbf{k}',jm'';-\mathbf{k}',i'm'). \tag{7.17a}$$

Taking the complex conjugate of this expression and relabeling, we obtain

$$\sum_{n}V_{\pi NN}(im,\boldsymbol{\rho}u'jn;\mathbf{k})B_{\pi N}(\mathbf{k}',-\mathbf{k}-\mathbf{k}') \times D_{nm''}^{(1/2)}[r_{\pi N}(\mathbf{k}',-\mathbf{k}-\mathbf{k}')] = G_{\pi NN}(\rho)H_{\pi NN}^*(\mathbf{k}'u',-\mathbf{k}-\mathbf{k}',jm'';-\mathbf{k},im). \tag{7.17b}$$

Now using Eqs. (6.2b), (7.2), (7.3), (7.4), and (7.15), we find

$$B_c^T(\mathbf{k},\mathbf{k}';z) = (-\delta_{T,1/2}+2\delta_{T,3/2})g_{\pi NN}^2G_{\pi NN}(\rho)Z_{\pi N}^{1/2}(k) \times Z_{\pi N}^{1/2}(k')G_{\pi NN}(\rho')C_{\pi N}(\mathbf{k},-\mathbf{k}-\mathbf{k}',-\mathbf{k}') \times C_{\pi N}(\mathbf{k}',-\mathbf{k}-\mathbf{k}',-\mathbf{k}) \times\frac{[\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k},\mathbf{k}')][\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k}',\mathbf{k})]}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')}, \tag{7.18a}$$

$$\mathbf{v}(\mathbf{k},\mathbf{k}') = \left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}{2m_N}\right]^{1/2} \times\left\{[1-\Lambda(\mathbf{k},\mathbf{k}')]\frac{\mathbf{k}}{\varepsilon(\mathbf{k})+m_N} - [1+\Lambda(\mathbf{k},\mathbf{k}')]\frac{\mathbf{k}+\mathbf{k}'}{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}\right\}, \tag{7.18b}$$

$$\Lambda(\mathbf{k},\mathbf{k}') = (1-\lambda)\frac{\omega(\mathbf{k}')+\varepsilon(-\mathbf{k}-\mathbf{k}')-\varepsilon(-\mathbf{k})}{2m_N}, \tag{7.18c}$$

$$\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}',-\mathbf{k}-\mathbf{k}'),\quad \boldsymbol{\rho}' = \mathbf{f}_{\pi N}(\mathbf{k},-\mathbf{k}-\mathbf{k}'). \tag{7.18d}$$

We now turn our attention to the σ-exchange potential given by Eq. (6.18). Using techniques similar to those that led to Eq. (7.17), we can show that

$$\sum_n V_{\sigma NN}(m,\boldsymbol{\xi}n;\mathbf{k})B_{N\sigma}(-\mathbf{k}',-\mathbf{k}+\mathbf{k}') \times D_{nm'}^{(1/2)}[r_{\sigma N}(-\mathbf{k}+\mathbf{k}',-\mathbf{k}')] = -g_{\sigma NN}G_{\sigma NN}(\xi)C_{\sigma N}(-\mathbf{k}+\mathbf{k}',-\mathbf{k}',-\mathbf{k}) \times\chi_m^\dagger\Delta(\mathbf{k},\mathbf{k}')\chi_{m'}, \tag{7.19a}$$

$$\Delta(\mathbf{k},\mathbf{k}') = \left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(-\mathbf{k}')+m_N}{2m_N}\right]^{1/2} \times\left\{1-\frac{(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{k}')}{[\varepsilon(-\mathbf{k})+m_N][\varepsilon(-\mathbf{k}')+m_N]}\right\}, \tag{7.19b}$$

$$\boldsymbol{\xi} = \mathbf{f}_{\pi N}(-\mathbf{k}',-\mathbf{k}+\mathbf{k}'). \tag{7.19c}$$

We can also show that

$$B_{\pi\sigma}(\mathbf{k},-\mathbf{k}+\mathbf{k}')V_{\sigma\pi\pi}(\boldsymbol{\zeta};-\mathbf{k}') = -g_{\sigma\pi\pi}G_{\sigma\pi\pi}(\zeta)C_{\sigma\pi}(-\mathbf{k}+\mathbf{k}',\mathbf{k},\mathbf{k}')\Omega(\mathbf{k},\mathbf{k}'), \tag{7.20a}$$

$$\Omega(\mathbf{k},\mathbf{k}') = \frac{1}{2}\left[1+\frac{\widetilde{g}_{\sigma\pi\pi}}{g_{\sigma\pi\pi}}\frac{\omega(\mathbf{k})\omega(\mathbf{k}')-\mathbf{k}\cdot\mathbf{k}'}{m_\pi^2}\right], \tag{7.20b}$$

$$\boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k},-\mathbf{k}+\mathbf{k}'). \tag{7.20c}$$

Using Eqs. (7.19) and (7.20) in Eq. (6.18), along with Eqs. (6.3b) and (7.14), we find

$$B_e^T(\mathbf{k},\mathbf{k}';z) = B_\sigma(\mathbf{k},\mathbf{k}';z) + B_\sigma^\dagger(\mathbf{k}',\mathbf{k};z^*), \tag{7.21a}$$

$$B_\sigma(\mathbf{k},\mathbf{k}';z) = g_{\sigma NN}G_{\sigma NN}(\xi)Z^{1/2}_{\pi N}(k)Z^{1/2}_{\pi N}(k')g_{\sigma\pi\pi}G_{\sigma\pi\pi}(\zeta) \times C_{\sigma N}(-\mathbf{k}+\mathbf{k}',-\mathbf{k}',-\mathbf{k})C_{\sigma\pi}(-\mathbf{k}+\mathbf{k}',\mathbf{k},\mathbf{k}') \times \frac{\Delta(\mathbf{k},\mathbf{k}')\Omega(\mathbf{k},\mathbf{k}')}{z-\omega(\mathbf{k})-\sigma(-\mathbf{k}+\mathbf{k}')-\varepsilon(-\mathbf{k}')}, \quad (7.21b)$$

$$\boldsymbol{\xi} = \mathbf{f}_{N\sigma}(-\mathbf{k}',-\mathbf{k}+\mathbf{k}'), \quad \boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k},-\mathbf{k}+\mathbf{k}'). \quad (7.21c)$$

Combining Eqs. (6.24), (6.20), and (7.14), we find that the off-shell amplitude for πN–πN elastic scattering is the solution of the matrix-integral equation

$$X^T(\mathbf{k},\mathbf{k}';z) = B^T(\mathbf{k},\mathbf{k}';z) + \int B^T(\mathbf{k},\mathbf{k}'';z)\frac{d^3k''}{d(k'',z)}X^T(\mathbf{k}'',\mathbf{k}';z), \quad (7.22a)$$

$$B^T(\mathbf{k},\mathbf{k}';z) = \sum_{x=d,c,e} B^T_x(\mathbf{k},\mathbf{k}';z). \quad (7.22b)$$

With the techniques developed in Ref. [16] this three-dimensional equation can be reduced to a set of uncoupled, one-dimensional integral equations whose solutions lead to the partial-wave, $\pi N \to \pi N$ scattering amplitudes.

Let us now turn our attention to the production amplitudes. Using Eq. (7.17a) in Eq. (6.28), along with Eqs. (6.6) and (6.7), we find

$$Y_{\pi\pi N}(\mathbf{k}_1u_1\mathbf{k}_2u_2im;\mathbf{k}'u'i'm') = -\frac{ig_{\pi NN}}{\sqrt{2}}\sum_{a=1}^{2}\sum_{i''m''}G_{\pi NN}(\rho_{bN})(\boldsymbol{\varepsilon}^*_{u_b}\cdot\boldsymbol{\tau})_{ii''} \times C_{\pi N}(\mathbf{k}_b,-\mathbf{k}_1-\mathbf{k}_2,-\mathbf{k}_a)\chi^\dagger_m[\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k}_a,\mathbf{k}_b)]\chi_{m''} \times Z^{1/2}_{\pi N}(k_a)\frac{X(\mathbf{k}_au_ai''m'',\mathbf{k}'u'i'm';z)}{d(k_a,z)}, \quad (7.23a)$$

$$\boldsymbol{\rho}_{bN} = \mathbf{f}_{\pi N}(\mathbf{k}_b,-\mathbf{k}_1-\mathbf{k}_2), \ a\neq b. \quad (7.23b)$$

Coupling the isospins according to

$$\sum_{u_1M''}\sum_{u_2i}\sum_{i''}\sum_{u'i'}\langle 1,T'',u_1,M''|T,M\rangle\langle 1,1/2,u_2,i|T'',M''\rangle Y_{\pi\pi N}(\mathbf{k}_1u_1\mathbf{k}_2u_2im;\mathbf{k}'u'i'm')\langle 1,1/2,u',i'|T',M'\rangle = \delta_{TT'}\delta_{MM'}\chi^\dagger_m Y^{T''T}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}')\chi_{m'}, \quad (7.24)$$

where

$$Y^{T''T}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}') = i\sqrt{\frac{3}{2}}g_{\pi NN}\Bigg\{\delta_{T'',1/2}G_{\pi NN}(\rho_{2N}) \times C_{\pi N}(\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2,-\mathbf{k}_1)[\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k}_1,\mathbf{k}_2)] \times Z^{1/2}_{\pi N}(k_1)\frac{X^T(\mathbf{k}_1,\mathbf{k}';z)}{d(k_1,z)} + (-1)^{T''+1/2}\sqrt{2(2T''+1)}\begin{Bmatrix}1 & 1/2 & T\\ 1 & T'' & 1/2\end{Bmatrix}G_{\pi NN}(\rho_{1N}) \times C_{\pi N}(\mathbf{k}_1,-\mathbf{k}_1-\mathbf{k}_2,-\mathbf{k}_2)[\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k}_2,\mathbf{k}_1)] \times Z^{1/2}_{\pi N}(k_2)\frac{X^T(\mathbf{k}_2,\mathbf{k}';z)}{d(k_2,z)}\Bigg\}. \quad (7.25)$$

We now use Eqs. (6.29), (7.17), and (7.18), and write for the σ production amplitude

$$\sum_{ui}\sum_{u'i'}\langle 1,1/2,u,i|T,M\rangle Y_{\pi\sigma N}(\mathbf{k}_\pi u\mathbf{k}_N im;\mathbf{k}'u'i'm') \times\langle 1,1/2,u',i'|T',M'\rangle = \delta_{TT'}\delta_{MM'}\chi^\dagger_m Y^T_{\pi\sigma N}(\mathbf{k}_\pi,\mathbf{k}_N;\mathbf{k}')\chi_{m'}, \quad (7.26)$$

which leads to

$$Y^T_{\pi\sigma N}(\mathbf{k}_\pi,\mathbf{k}_N;\mathbf{k}') = -g_{\sigma NN}G_{\sigma NN}(\xi)C_{\sigma N}(-\mathbf{k}_\pi-\mathbf{k}_N,\mathbf{k}_N,-\mathbf{k}_\pi) \times\Delta(\mathbf{k}_N,-\mathbf{k}_\pi)Z^{1/2}_{\pi N}(k_\pi)\frac{X^T(\mathbf{k}_\pi,\mathbf{k}';z)}{d(k_\pi,z)} - g_{\sigma\pi\pi}G_{\sigma\pi\pi}(\zeta)C_{\sigma\pi}(-\mathbf{k}_\pi-\mathbf{k}_N,\mathbf{k}_\pi,-\mathbf{k}_N) \times\Omega(\mathbf{k}_\pi,-\mathbf{k}_N)Z^{1/2}_{\pi N}(k_N)\frac{X^T(-\mathbf{k}_N,\mathbf{k}';z)}{d(k_N,z)}, \quad (7.27a)$$

$$\boldsymbol{\xi} = \mathbf{f}_{N\sigma}(\mathbf{k}_N,-\mathbf{k}_\pi-\mathbf{k}_N), \quad \boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k}_\pi,-\mathbf{k}_\pi-\mathbf{k}_N). \quad (7.27b)$$

VIII. UNITARITY

We now show that the solutions of Eq. (7.22) in combination with the production amplitudes, Eqs. (7.25) and (7.27), satisfy the correct unitarity relation. Let us introduce an operator notation according to

$$\langle\overrightarrow{\mathbf{k}}|X^T(z)|\mathbf{k}'\rangle = X^T(\mathbf{k},\mathbf{k}';z), \quad \langle\overrightarrow{\mathbf{k}}|B^T(z)|\mathbf{k}'\rangle = B^T(\mathbf{k},\mathbf{k}';z), \quad \langle\overrightarrow{\mathbf{k}}|t(z)|\mathbf{k}'\rangle = \delta^3(\mathbf{k}-\mathbf{k}')/d(k,z), \quad (8.1)$$

which allows us to rewrite Eq. (7.22) in the form

$$X^T(z) = B^T(z) + B^T(z)t(z)X^T(z). \quad (8.2)$$

Using the techniques employed in Sec. VI of Ref. [31], it is a matter of straightforward algebra to show that

$$\Delta X^T = \Delta B^T + X^T(\mp)t(\mp)\Delta B^T + \Delta B^T t(\pm)X^T(\pm) + X^T(\mp)[\Delta t + t(\mp)\Delta B^T t(\pm)]X^T(\pm), \quad (8.3)$$

where $(\pm) = (W\pm i\varepsilon)$, $\Delta X^T = X^T(+) - X^T(-)$, etc. From Eqs. (7.15) and (7.21) we see that the operator $B^T(z)$ has a discontinuity that arises from factors of the form $1/(z-W')$, where W' is the c.m. energy of the relevant intermediate state. We have

$$\frac{1}{W+i\varepsilon-W'} - \frac{1}{W-i\varepsilon-W'} = -2\pi i\delta(W-W'). \quad (8.4)$$

From Eq. (7.18) we encounter the delta function $\delta[W-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')]$. If $W=W_{\pi N}(\mathbf{k})$ or $W=W_{\pi N}(\mathbf{k}')$, the argument of the delta function cannot vanish, and similarly for the delta function that arises from Eq. (7.21). As a result of this the discontinuity in the elastic scattering amplitude is given by the relation

$$X^T(\mathbf{k},\mathbf{k}';+) - X^T(\mathbf{k},\mathbf{k}';-) = \int X^T(\mathbf{k},\mathbf{k}'';-)d^3k''\left[\frac{1}{d(k'',+)} - \frac{1}{d(k'',-)}\right] \times X^T(\mathbf{k}'',\mathbf{k}';+) + \int \frac{X^T(\mathbf{k},\mathbf{k}'';-)}{d(k'',-)}d^3k'' \times [B^T(\mathbf{k}'',\mathbf{k}''';+) - B^T(\mathbf{k}'',\mathbf{k}''';-)]d^3k'''\frac{X^T(\mathbf{k}''',\mathbf{k}';+)}{d(k''',+)},$$
$$W = W_{\pi N}(\mathbf{k}) = W_{\pi N}(\mathbf{k}'). \tag{8.5}$$

We find from Eq. (6.15) that the discontinuity in the πN propagator is given by

$$\frac{1}{d(k,+)} - \frac{1}{d(k,-)} = -2\pi i\delta[W - W_{\pi N}(\mathbf{k})] - \frac{d(k,+)-d(k,-)}{d(k,-)d(k,+)}. \tag{8.6}$$

From Eq. (6.12) we find the discontinuity

$$d(k,+) - d(k,-) = 2\pi i Z_{\pi N}(k)\left\{\int d^3\rho\, F_{\pi NN}(\rho;k)\delta[W - W_{\pi\pi N}(k,\rho)] + \int d^3\xi\, F_{\sigma NN}(\xi;k)\delta[W - W_{\pi N\sigma}(k,\xi)] + \int d^3\zeta\, F_{\sigma\pi\pi}(\zeta;k)\delta[W - W_{\pi N\sigma}(k,\zeta)]\right\}. \tag{8.7}$$

When we use this result in Eqs. (8.6) and (8.5) we encounter integration elements of the type $d^3kd^3\rho$, $d^3kd^3\xi$, and $d^3kd^3\zeta$. We can convert these elements to integration elements in terms of the individual particle momenta by introducing the appropriate Jacobian. The Jacobians can by derived by inserting Eqs. (A25) into (A26) and comparing the result with the completeness relation that follows from Eq. (A24). We find

$$d^3k_1d^3\rho_{23} = d^3k_1d^3k_2B^2_{23}(\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2), \quad \boldsymbol{\rho}_{23} = \mathbf{f}_{23}(\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2). \tag{8.8}$$

Using these results along with Eqs. (6.30) (6.11), (6.27), (6.30), (7.2), and (7.17)–(7.20), we can show that

$$-\int X^T(\mathbf{k},\mathbf{k}'';-)d^3k''\frac{d(k'',+)-d(k'',-)}{d(k'',-)d(k'',+)}X^T(\mathbf{k}'',\mathbf{k}';+) = -2\pi i\left\{\int \frac{X^T(\mathbf{k},\mathbf{k}_1;-)}{d(k_1,-)}Z^{1/2}_{\pi N}(k_1)d^3k_1d^3k_2 3g^2_{\pi NN} \times G^2_{\pi NN}(\rho)C^2_{\pi N}(\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2,-\mathbf{k}_1)[\boldsymbol{\sigma}\cdot\mathbf{v}(\mathbf{k}_1,\mathbf{k}_2)]^2 \times \delta[W - \widetilde{W}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2)]Z^{1/2}_{\pi N}(k_1)\frac{X^T(\mathbf{k}_1,\mathbf{k}';+)}{d(k_1,+)} + \int \frac{X^T(\mathbf{k},\mathbf{k}_\pi;-)}{d(k_\pi,-)}Z^{1/2}_{\pi N}(k_\pi)d^3k_\pi d^3k_N g^2_{\sigma NN}G^2_{\sigma NN}(\xi) \times C^2_{\sigma N}(-\mathbf{k}_\pi-\mathbf{k}_N,\mathbf{k}_N,-\mathbf{k}_\pi)\Delta^\dagger(\mathbf{k}_N,-\mathbf{k}_\pi)\Delta(\mathbf{k}_N,-\mathbf{k}_\pi) \times \delta[W - \widetilde{W}_{\pi N\sigma}(\mathbf{k}_\pi,\mathbf{k}_N)]Z^{1/2}_{\pi N}(k_\pi)\frac{X^T(\mathbf{k}_\pi,\mathbf{k}';+)}{d(k_\pi,+)} + \int \frac{X^T(\mathbf{k},-\mathbf{k}_N;-)}{d(k_N,-)}Z^{1/2}_{\pi N}(k_N)d^3k_Nd^3k_\pi g^2_{\sigma\pi\pi}G^2_{\sigma\pi\pi}(\zeta) \times C^2_{\sigma\pi}(-\mathbf{k}_\pi-\mathbf{k}_N,\mathbf{k}_\pi,-\mathbf{k}_N)\Omega^2(\mathbf{k}_\pi,-\mathbf{k}_N) \times \delta[W - \widetilde{W}_{\pi N\sigma}(\mathbf{k}_\pi,\mathbf{k}_N)]Z^{1/2}_{\pi N}(k_N)\frac{X^T(-\mathbf{k}_N,\mathbf{k}';+)}{d(k_N,+)}, \tag{8.9a}$$

$$\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2),$$
$$\boldsymbol{\xi} = \mathbf{f}_{N\sigma}(\mathbf{k}_N,-\mathbf{k}_\pi-\mathbf{k}_N), \quad \boldsymbol{\zeta} = \mathbf{f}_{\pi\sigma}(\mathbf{k}_\pi,-\mathbf{k}_\pi-\mathbf{k}_N). \tag{8.9b}$$

We can easily obtain the expression for the discontinuity in $B^T(\mathbf{k},\mathbf{k}';z)$ from Eqs. (7.18), (7.21), (7.22b), and (8.4). With these results in hand it is a matter of straightforward algebra to verify that it follows from Eqs. (8.5) and (8.9) that the discontinuity in the elastic scattering amplitude is given by

$$X^T(\mathbf{k},\mathbf{k}';+) - X^T(\mathbf{k},\mathbf{k}';-) = -2\pi i\int X^T(\mathbf{k},\mathbf{k}_\pi;-)d^3k_\pi\delta[W - W_{\pi N}(\mathbf{k}_\pi)] \times X^T(\mathbf{k}_\pi,\mathbf{k}';+) - 2\pi i\sum_{T''}\int Y^{T''T\dagger}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k})d^3k_1d^3k_2 \times \delta[W - \widetilde{W}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2)]Y^{T''T}_{\pi\pi N}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}') - 2\pi i\int Y^{T\dagger}_{\pi\sigma N}(\mathbf{k}_\pi,\mathbf{k}_N;\mathbf{k})d^3k_\pi d^3k_N \times \delta[W - \widetilde{W}_{\pi N\sigma}(\mathbf{k}_\pi,\mathbf{k}_N)]Y^T_{\pi\sigma N}(\mathbf{k}_\pi,\mathbf{k}_N;\mathbf{k}'). \tag{8.10}$$

This is the correct unitarity relation.

IX. SUMMARY AND DISCUSSION

We have succeeded in developing a practical method for constructing relativistic, three-particle models of the pion-nucleon system. The models will have some of the character of a quantum field theory in that the basic interactions are vertex interactions, and moreover renormalization effects are present. In constrast to a quantum field theory, the Hilbert space is restricted to a few types of state; e.g., here we have $|N\rangle$, $|\pi N\rangle$, $|\pi\pi N\rangle$, and $|\pi\sigma N\rangle$ states.

The fact that the various subspaces of the model are coupled by vertex interactions is what makes the model tractable. As we saw, the N-, $\pi\pi N$-, and $\pi\sigma N$-components of a state vector of the system can be expressed rather simply in terms of the πN-components, which makes it possible to reduce solving the model to solving Eq. (6.19), an equation that involves only πN components. The way we have coupled the various subspaces through the vertex interactions is consistent with quantum field theory. In fact it is interesting to observe that the results obtained here give credence to the Tamm-Dancoff method [32] for solving a quantum field theory. In this method the field theory state vector is expanded in a limited set of Fock space states, and a set of coupled integral equations for the components of the state vector is derived. This set of equations is similar to the coupled equations given by Eq. (6.9). The Tamm-Dancoff method [32] is often dismissed, since it appears to be inconsistent with Poincaré invariance and unitarity. The results obtained here show that with a little care the Tamm-Dancoff method can yield results that are consistent with special relativity and unitarity. Our method also provides justification for constructing models for the pion-nucleon system by summing the diagrams of time-ordered perturbation theory [9,34]. As with the Tamm-Dancoff procedure, this method appears to have problems with Poincaré invariance and unitarity. However a little thought shows that the equations we have obtained for the elastic scattering and production amplitudes can be thought of as summing subsets of time-ordered, perturbation theory diagrams, and therefore indirectly we have demonstrated that summing such diagrams can lead to results that satisfy the requirements of special relativity and unitarity.

With the method developed here the elastic πN-scattering amplitudes are obtained by putting the solutions of Eq. (7.22) on the energy shell. The production amplitudes are given by Eqs. (7.25) and (7.27), which are expressed directly in terms of the half-off-shell solutions of Eq. (7.22). These equations are quite similar to the Amado-Lovelace equations [3,4] that were popularized in the 1960's. The model presented here can be viewed as an extension of the Aaron, Amado, and Young (AAY) model [7] of pion-nucleon scattering, an extension in that here s-channel nucleon exchange and t-channel sigma exchange have been added to the AAY's u-channel nucleon exchange. It should be noted, however, that the pion-nucleon propagator given by Eq. (6.15), i.e., $1/d(k;z)$, differs from the one in the AAY model, and in such a way that it avoids a difficulty with the AAY propagator that was pointed out by Garcilazo and Mathelisch [33]. Garcilazo and Mathelisch [33] presented an alternative to the AAY propagator, however their propagator has a problem with clustering, which Eq. (6.15) manages to avoid.

It is clear that the present model can be extended so as to provide a realistic model of the pion-nucleon system. An appealing feature of the method developed here is that it makes it relatively straightforward to extend existing exchange models of the pion-nucleon system [16,34,35] to include three-particle channels in such a way that Poincaré invariance and unitarity are satisfied. Baryons such as the $\Delta(1232)$ and the $P_{11}(1440)$ can be handled in the three-particle framework by following the procedure used here for the nucleon. Just as with the nucleon, these baryons lead to s- and u-channel exchange processes. The t-channel ρ-meson exchange process can be treated in exact analogy to the present treatment of the σ-meson. The extension of the author's exchange model of the pion-nucleon system [16] to include three-particle channels is currently underway.

APPENDIX A: BASIS STATES

Here we define our basis states and determine their inner products. We also develop relations between various basis states.

We define single-particle states for a particle with spin s by

$$|\mathbf{p}m\rangle = U[l_c(p)]|\mathbf{0}m\rangle N^{1/2}(|\mathbf{p}|), \quad p = (E(\mathbf{p}), \mathbf{p}), \quad \text{(A1)}$$

where $l_c(p)$ is the canonical boost defined by Eq. (3.3), with p the four-momentum of the particle. N is a normalization parameter, which is determined by the inner product of the states. We assume that the rest-frame state $|\mathbf{0}m\rangle$ is an SU(2) basis state that rotates according to

$$U(r)|\mathbf{0}m\rangle = \sum_{m'} |\mathbf{0}m'\rangle D^{(s)}_{m'm}(r), \quad \text{(A2)}$$

where $D^{(s)}(r)$ is a standard SU(2) matrix representative of the rotation r. We can determine the action of the unitary operator $U(a)$ on state (A1) by writing

$$U(a)|\mathbf{p}m\rangle = U[l_c(ap)]U[r_c(a, p)]|\mathbf{0}m\rangle N^{1/2}(|\mathbf{p}|), \quad \text{(A3)}$$

where $r_c(a, p)$ is the Wigner rotation defined by Eq. (6.7). From here and Eq. (A2) it follows that

$$U(a)|\mathbf{p}m\rangle = \sum_{m'} |\mathbf{p}'m'\rangle D^{(s)}_{m'm}[r_c(a, p)][N(|\mathbf{p}|)/N(|\mathbf{p}'|)]^{1/2}, \quad p' = ap. \quad \text{(A4)}$$

When a is a three-rotation r, the Wigner rotation simplifies, i.e.,

$$r_c(r, p) = r. \quad \text{(A5)}$$

This can be verified by expressing the canonical boost in the form

$$l_c(p) = \exp(-i\omega\hat{\mathbf{p}} \cdot \mathbf{k}), \quad \omega = \tanh^{-1}[|\mathbf{p}|/E(\mathbf{p})], \quad \hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|, \quad \text{(A6)}$$

and using the fact that the generator $\mathbf{k}$ is a three-vector operator under rotations, which implies that $rl_c(p)r^{-1} = l_c(rp)$. As a result of Eq. (A5) the single-particle states rotate according to

$$U(r)|\mathbf{p}m\rangle = \sum_{m'} |r\mathbf{p}, m'\rangle D^{(s)}_{m'm}(r). \quad \text{(A7)}$$

We define an n-particle state by

$$|\mathbf{p}\{\mathbf{k}\}\{m\}\rangle = U[l_c(p)]|\mathbf{k}_1 m_1\rangle \otimes |\mathbf{k}_2 m_2\rangle \otimes \cdots \otimes |\mathbf{k}_n m_n\rangle N^{1/2}(\mathbf{p}, \{\mathbf{k}\}), \quad \text{(A8a)}$$

$$\{\mathbf{k}\} = \{\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{n-1}\}, \quad \text{(A8b)}$$

$$\{m\} = \{m_1, m_2, \ldots, m_n\}, \quad \text{(A8c)}$$

$$\sum_{i=1}^{n} \mathbf{k}_i = \mathbf{0}, \quad \text{(A8d)}$$

where $\mathbf{p}$ is the total three-momentum of the particles, the $\mathbf{k}_i$'s are their c.m. three-momenta, and the m_j's are their spin components. The total c.m. energy W and the total four-momentum p of this state are given by

$$W(\{\mathbf{k}\}) = \sum_{i=1}^{n} E_i(\mathbf{k}_i), \quad \text{(A9a)}$$

$$p = (E(\mathbf{p}, \{\mathbf{k}\}), \mathbf{p}) = ([\mathbf{p}^2 + W^2(\{\mathbf{k}\})]^{1/2}, \mathbf{p}). \quad \text{(A9b)}$$

Note that the normalization factor N has the property

$$N(\mathbf{0}, \{\mathbf{k}\}) = 1, \quad \text{(A10)}$$

so that when $\mathbf{p} = \mathbf{0}$ the n-particle state reduces to a simple direct product of the single-particle states. For the orthogonality relation of the states we assume

$$\langle \mathbf{p}\{\mathbf{k}\}\{m\}|\mathbf{q}\{\mathbf{l}\}\{n\}\rangle = \delta^3(\mathbf{p}-\mathbf{q})\prod_{i=1}^{n-1}\delta^3(\mathbf{k}_i-\mathbf{l}_i)\prod_{j=1}^{n}\delta_{m_j n_j}, \quad \text{(A11)}$$

which determines the normalization factor N, as we now show. Using Eqs. (6.7) and (A7), we find that

$$U(a)|\mathbf{p}\{\mathbf{k}\}\{m\}\rangle = \sum_{\{m'\}}|\mathbf{p}'\{r_c(a,p)\mathbf{k}\}\{m'\}\rangle D_{\{m'\}\{m\}}[r_c(a,p)] \times [N(\mathbf{p},\{\mathbf{k}\})/N(\mathbf{p}',\{r_c(a,p)\mathbf{k}\})]^{1/2},$$
$$p' = ap, \quad \text{(A12a)}$$

$$D_{\{m'\}\{m\}} = D^{(s_1)}_{m'_1 m_1}D^{(s_2)}_{m'_2 m_2}\cdots D^{(s_n)}_{m'_n m_n}. \quad \text{(A12b)}$$

Inserting $U^\dagger[l_c^{-1}(p)]U[l_c^{-1}(p)]$ between the two states in Eq. (A11) and using Eqs. (A8a), (A12), and (A10), we find

$$\delta^3(\mathbf{p}-\mathbf{q})\prod_{i=1}^{n-1}\delta^3(\mathbf{k}_i-\mathbf{l}_i)\prod_{j=1}^{n}\delta_{m_j n_j} = [N(\mathbf{p},\{\mathbf{k}\})N(\mathbf{q},\{\mathbf{l}\})]^{1/2}\delta^3(\mathbf{0}-\mathbf{q}') \times \prod_{i=1}^{n-1}\delta^3\{\mathbf{k}_i - r_c[l_c^{-1}(p),q]\mathbf{l}_i\}D_{\{m\}\{n\}}\{r_c[l_c^{-1}(p),q]\},$$
$$q' = l_c^{-1}(p)q. \quad \text{(A13)}$$

Because of $\delta^3(\mathbf{0}-\mathbf{q}')$ we can let $q' \to (W(\{\mathbf{l}\}),\mathbf{0})$ in $r_c[l_c^{-1}(p), q]$, and we can also let $q = l_c(p)q' \to pW(\{\mathbf{l}\})/W(\{\mathbf{k}\})$. From Eq. (6.7) $r_c[l_c^{-1}(p),q] = l_c^{-1}(q')l_c^{-1}(p)l_c(q)$, while from Eq. (A6) $l_c^{-1}(q') \to 1$ and $l_c(q) \to l_c(p)$, so we find

$$r_c[l_c^{-1}(p),q]|_{\mathbf{q}'=\mathbf{0}} = 1. \quad \text{(A14)}$$

Since $E(\mathbf{p},\{\mathbf{k}\})\delta^3(\mathbf{p}-\mathbf{q})$ is a Lorentz invariant form, we can write

$$E(\mathbf{p},\{\mathbf{k}\})\delta^3(\mathbf{p}-\mathbf{q}) = W(\{\mathbf{k}\})\delta^3(\mathbf{0}-\mathbf{q}'), \quad \text{(A15)}$$

which when put into Eq. (A13) along with Eq. (A14) leads to the result

$$N(\mathbf{p},\{\mathbf{k}\}) = W(\{\mathbf{k}\})/E(\mathbf{p},\{\mathbf{k}\}). \quad \text{(A16)}$$

With this result the complete definition of our n-particle states becomes

$$|\mathbf{p}\{\mathbf{k}\}\{m\}\rangle = U[l_c(p)]|\mathbf{k}_1 m_1\rangle\otimes|\mathbf{k}_2 m_2\rangle\otimes\cdots\otimes \times|\mathbf{k}_n m_n\rangle[W(\{\mathbf{k}\})/E(\mathbf{p},\{\mathbf{k}\})]^{1/2}. \quad \text{(A17)}$$

Since $N(\mathbf{p},\mathbf{k})$ depends only on the magnitudes of the vectors, we can replace Eq. (A12a) with

$$U(a)|\mathbf{p}\{\mathbf{k}\}\{m\}\rangle = \sum_{\{m'\}}|\mathbf{p}'\{r_c(a,p)\mathbf{k}\}\{m'\}\rangle D_{\{m'\}\{m\}}[r_c(a,p)] \times [E(\mathbf{p}',\{\mathbf{k}\})/E(\mathbf{p},\{\mathbf{k}\})]^{1/2},$$
$$p' = ap. \quad \text{(A18)}$$

Our states (A17) can be expressed in terms of the simple direct product states defined by

$$|\mathbf{p}_1 m_1, \mathbf{p}_2 m_2, \cdots, \mathbf{p}_n m_n\rangle = |\mathbf{p}_1 m_1\rangle\otimes|\mathbf{p}_2 m_2\rangle\otimes\cdots\otimes|\mathbf{p}_n m_n\rangle. \quad \text{(A19)}$$

If we apply Eq. (A18) to a single-particle state, the equation simplifies to

$$U(a)|\mathbf{p}m\rangle = \sum_{m'}|\mathbf{p}'m'\rangle D^{(s)}_{m'm}[r_c(a,p)][E(\mathbf{p}')/E(\mathbf{p})]^{1/2},$$
$$p' = ap, \quad \text{(A20)}$$

which when used in Eq. (A17) leads to

$$|\mathbf{p}\{\mathbf{k}\}\{m\}\rangle = [W(\{\mathbf{k}\})/E(\mathbf{p},\{\mathbf{k}\})]^{1/2} \times \prod_{i=1}^{n}\sum_{m'_i}|\mathbf{p}_i m'_i\rangle D^{(s_i)}_{m'_i m_i}\{r_c[l_c(p),k_i]\} \times [E_i(\mathbf{p}_i)/E_i(\mathbf{k}_i)]^{1/2}. \quad \text{(A21)}$$

Convenient three-particle states for constructing matrix elements of interactions can be obtained by replacing the state $|\mathbf{k}_1 m_1\rangle\otimes|\mathbf{k}_2 m_2\rangle\otimes|\mathbf{k}_3 m_3\rangle$ with $|\mathbf{k}_1 m_1\rangle\otimes|\mathbf{k}_{23}\boldsymbol{\rho}_{23}m_2 m_3\rangle$, where $|\mathbf{k}_{23}\boldsymbol{\rho}_{23}m_2 m_3\rangle$ is obtained from Eq. (A21) by making the replacements $\mathbf{p}\to\mathbf{k}_{23}$, $\{\mathbf{k}\}\to\boldsymbol{\rho}_{23}$, and $\{m\}\to m_2 m_3$. This second state is given explicitly by

$$|\mathbf{k}_{23}\boldsymbol{\rho}_{23}m_2 m_3\rangle = \sum_{m'_2 m'_3}|\mathbf{k}_2 m'_2\rangle\otimes|\mathbf{k}_3 m'_3\rangle \times D^{(s_2)}_{m'_2 m_2}\{r_c[l_c(k_{23})\rho_2]\}D^{(s_3)}_{m'_3 m_3}\{r_c[l_c(k_{23})\rho_3]\} \times\left[\frac{E_2(\mathbf{k}_2)E_3(\mathbf{k}_3)}{E_{23}(\mathbf{k}_{23},\boldsymbol{\rho}_{23})}\frac{W_{23}(\boldsymbol{\rho}_{23})}{E_2(\boldsymbol{\rho}_{23})E_3(\boldsymbol{\rho}_{23})}\right]^{1/2}, \quad \text{(A22a)}$$

$$k_{23} = k_2 + k_3 = (E_2(\mathbf{k}_2)+E_3(\mathbf{k}_3), \mathbf{k}_2+\mathbf{k}_3) = (E_{23}(\mathbf{k}_{23},\boldsymbol{\rho}_{23}), \mathbf{k}_2+\mathbf{k}_3), \quad \text{(A22b)}$$

$$\rho_2 = l_c^{-1}(k_{23})k_2 = (E_2(\boldsymbol{\rho}_{23}),\boldsymbol{\rho}_{23}),$$
$$\rho_3 = l_c^{-1}(k_{23})k_3 = (E_3(\boldsymbol{\rho}_{23}),-\boldsymbol{\rho}_{23}), \quad \text{(A22c)}$$

where W_{23} and E_{23} are defined by Eqs. (3.1b) and (3.1c), respectively. It should be noted that ρ_2 and ρ_3 are obtained by applying an inverse canonical boost, $l_c^{-1}(k_{23})$, to k_2 and k_3, respectively, so that $\boldsymbol{\rho}_{23}$ can be interpreted as the three-momentum of particle 2, or the negative of the three-momentum of particle 3, in a 2,3 c.m. frame obtained by an inverse canonical boost from the 1,2,3 c.m. frame. We now boost the state $|\mathbf{k}_1 m_1\rangle\otimes|\mathbf{k}_{23}\boldsymbol{\rho}_{23}m_2 m_3\rangle$ from the three-particle c.m. frame to a frame in which the 3 particles have a total three-momentum $\mathbf{p}$ and define

$$|\mathbf{p}\mathbf{k}_1\boldsymbol{\rho}_{23}m_1 m_2 m_3\rangle = U[l_c(p)]|\mathbf{k}_1 m_1\rangle\otimes|-\mathbf{k}_1,\boldsymbol{\rho}_{23}m_2 m_3\rangle \times[W_{123}(\mathbf{k}_1,\boldsymbol{\rho}_{23})/E_{123}(\mathbf{p},\mathbf{k}_1,\boldsymbol{\rho}_{23})]^{1/2}, \quad \text{(A23)}$$

where W_{123} and E_{123} are given by Eqs. (3.1d) and (3.1e), respectively. Carrying out an analysis similar to the one that led to Eq. (A17), it can be verified that

$$\langle\mathbf{p}\mathbf{k}_1\boldsymbol{\rho}_{23}m_1 m_2 m_3|\mathbf{p}'\mathbf{k}'_1\boldsymbol{\rho}'_{23}m'_1 m'_2 m'_3\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}_1-\mathbf{k}'_1)\delta^3(\boldsymbol{\rho}_{23}-\boldsymbol{\rho}'_{23})\delta_{m_1 m'_1}\delta_{m_2 m'_2}\delta_{m_3 m'_3}. \quad \text{(A24)}$$

We can express the three-particle states $|\mathbf{p}\mathbf{k}_1\mathbf{k}_2 m_1 m_2 m_3\rangle$ as a linear combination of the states (A23) by the following steps: insert Eq. (A22a) into Eq. (A23), use Eq. (A17) and the identities $W(\mathbf{k}_1, \mathbf{k}_2) = W_{123}(\mathbf{k}_1, \boldsymbol{\rho}_{23})$ and $E(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) = E_{123}(\mathbf{p}, \mathbf{k}_1, \boldsymbol{\rho}_{23})$, invert the relation between the states by using the orthogonality of the rotation matrices, use Eqs. (A22c) and (6.5)–(6.7). Realizing that we can, of course, interchange the roles of 1 and 2, we find

$$\begin{aligned} |\mathbf{p}\mathbf{k}_1\mathbf{k}_2 m_1 m_2 m_3\rangle &= \sum_{m_b' m_3'} |\mathbf{p}\mathbf{k}_a \boldsymbol{\rho}_{b3} m_a m_b' m_3'\rangle D^{(s_b)}_{m_b' m_b}[r_{3b}(\mathbf{k}_3, \mathbf{k}_b)] \\ &\quad \times D^{(s_3)}_{m_3' m_3}[r_{b3}(\mathbf{k}_b, \mathbf{k}_3)] B_{b3}(\mathbf{k}_b, \mathbf{k}_3), \\ &\quad a = 1, 2; \quad b = 1, 2; \quad a \neq b. \end{aligned} \tag{A25}$$

We now determine the inner product $\langle \mathbf{p}\mathbf{k}_1 \boldsymbol{\rho}_{23} m_1 m_2 m_3 | \mathbf{p}'\mathbf{k}_2' \boldsymbol{\rho}_{13}' m_2' m_1' m_3'\rangle$. We begin by inserting the completeness relation for the states (A25) between the states in the inner product. According to Eq. (A11) this completeness relation is

$$1 = \sum_{m_1 m_2 m_3} \int |\mathbf{p}\mathbf{k}_1\mathbf{k}_2 m_1 m_2 m_3\rangle d^3p\, d^3k_1\, d^3k_2 \langle \mathbf{p}\mathbf{k}_1\mathbf{k}_2 m_1 m_2 m_3|. \tag{A26}$$

For the state on the left in Eq. (A26) we use Eq. (A25) with $a = 1$ and $b = 2$, while for the state on the right we use $a = 2$ and $b = 1$. We also use $\boldsymbol{\rho}_{b3} = \mathbf{f}_{b3}(\mathbf{k}_b, \mathbf{k}_3)$ where $\mathbf{f}_{b3}$ is given by Eq. (6.4). We find

$$\begin{aligned} &\langle \mathbf{p}\mathbf{k}_1 \boldsymbol{\rho}_{23} m_1 m_2 m_3 | \mathbf{p}'\mathbf{k}_2' \boldsymbol{\rho}_{13}' m_2' m_1' m_3'\rangle \\ &\quad = \delta^3(\mathbf{p} - \mathbf{p}')\delta^3[\boldsymbol{\rho}_{23} - \mathbf{f}_{23}(\mathbf{k}_2', -\mathbf{k}_1 - \mathbf{k}_2')] \\ &\qquad \times \delta^3[\boldsymbol{\rho}_{13}' - \mathbf{f}_{13}(\mathbf{k}_1, -\mathbf{k}_1 - \mathbf{k}_2')] B_{23}(\mathbf{k}_2', -\mathbf{k}_1 - \mathbf{k}_2') \\ &\qquad \times B_{13}(\mathbf{k}_1, -\mathbf{k}_1 - \mathbf{k}_2') D^{(s_1)}_{m_1 m_1'}\left[r_{31}^{-1}(-\mathbf{k}_1 - \mathbf{k}_2', \mathbf{k}_1)\right] \\ &\qquad \times D^{(s_2)}_{m_2 m_2'}[r_{32}(-\mathbf{k}_1 - \mathbf{k}_2', \mathbf{k}_2')] \\ &\qquad \times D^{(s_3)}_{m_3 m_3'}\left[r_{23}(\mathbf{k}_2', -\mathbf{k}_1 - \mathbf{k}_2') r_{13}^{-1}(\mathbf{k}_1, -\mathbf{k}_1 - \mathbf{k}_2')\right]. \end{aligned} \tag{A27}$$

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