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J-pairing interaction, number of states, and nine-*j* sum rules of four identical particles

Y. M. Zhao[1–3,*] and A. Arima[4]

[1]*Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China*
[2]*Cyclotron Center, Institute of Physical Chemical Research (RIKEN), Hirosawa 2-1, Wako-shi, Saitama 351-0198, Japan*
[3]*Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou 730000, China*
[4]*Science Museum, Japan Science Foundation, 2-1 Kitanomaru-koen, Chiyodaku, Tokyo 102-0091, Japan*

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We study the J-pairing Hamiltonian and find that the sum of eigenvalues of spin-I states equals the sum of the norm matrix elements within the pair basis for four identical particles such as four fermions in a single-j shell or four bosons with spin l. We relate the number of states to sum rules of nine-j coefficients. We obtained sum rules for nine-j coefficients $\langle (jj)J,(jj)K : I|(jj)J,(jj)K : I\rangle$ and $\langle (ll)J,(ll)K : I|(ll)J,(ll)K : I\rangle$ summing over (1) even J and even K, (2) even J and odd K, (3) odd J and odd K, and (4) both even and odd values for J and K, where j is a half integer and l is an integer.

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I. INTRODUCTION

The J-pairing Hamiltonian for a single-j shell is an important topic for studying both nuclear structure theory and general many-body systems. For the case of $J=0$ (i.e., the monopole pairing interaction), the famous seniority scheme [1,2] provides exact solutions; for $J=J_{\max}$, the "cluster" picture of Ref. [3] presents an asymptotic classification of states. For other J cases, it was found that pairs with spin J are reasonable building blocks for low-lying states but little is known about exact eigenvalues [4]. In this paper we shall go one step forward along this line by proving that for four identical particles the sum of eigenvalues for the J-pairing interaction is connected with a sum of nine-j coefficients.

The enumeration of spin I states (where the number of spin I states is denoted by D_I in this paper) for fermions in a single-j shell or bosons with spin l (with a convention that j is a half integer and l is an integer) is also a very common practice in nuclear structure theory. D_I is usually obtained by subtracting the number of states with total angular momentum projection $M=I+1$ from that with $M=I$ [5]. Because numbers of states with different M's seem irregular, D_I values are usually tabulated in textbooks, for the sake of convenience. Other methods include Racha's method [1] in terms of the seniority scheme and the generating function method proposed and studied by Katriel *et al.* [6] and Sunko and collaborators [7]. All these works are interesting and important. However, the results are not algebraic. It is therefore desirable to obtain analytical formulas of D_I. For $n=1$ and 2, D_I is known and is understood very well; but the situation becomes complicated when $n \geqslant 3$, except for a few cases with $I \sim I_{\max}$.

Historically, the first interesting formula of D_I was given for the case with $I=0$ and $n=4$ by Ginocchio and Haxton [8]. Their result was revisited by Zamick and Escuderos [9]. In Ref. [10], the authors of the present paper empirically constructed D_I for $n=3$ and 4 and some D_I's for $n=5$. Recently, Talmi suggested a recursion formula of D_I [11] and used this formula to prove D_I formulas obtained empirically for $n=3$ [10]. Talmi's recursion formula is also readily applied to prove the empirical formulas of Ref. [10] for $n=4$. In Ref. [12], we showed that D_I of n particle systems can be enumerated by using the reduction from SU($n+1$) to SO(3), and as an example, D_I for $n=4$ was obtained analytically.

The results of D_I for three identical particles were applied to obtain a number of sum rules for six-j symbols in the appendix of Ref. [3]. One can ask whether the results for $n=4$ can be used similarly to obtain sum rules for nine-j symbols. If the answer is yes, the application would be very interesting, because sum rules of angular momentum couplings are widely applied in many branches of physics, in particular in nuclear structure theory. (Angular momentum coupling-recoupling coefficients and sum rules were compiled in Ref. [13] in 1988.) This paper addresses the following question: Can we obtain sum rules for nine-j symbols based on studies of J-pairing Hamiltonian and number of spin I states for four identical particles? Furthermore, how far can one proceed along this line?

This paper is organized as follows. In Sec. II, we study J-pairing Hamiltonian to obtain summation of all nonzero eigenvalues in the presence of only one J-pairing force. In Sec. III, we present sum rules of nine-j symbols found by using these summations and number of states for $n=4$ obtained in earlier works. In this paper we show that the D_I formulas provide us with a bridge between the J-pairing interaction and sum rules of nine-j symbols for identical particles. A summary and discussion are given in Sec. IV. Appendix A present formulas of nine-j symbols in some special cases. Appendix B discusses the number of matrices involved in our sum-rule calculations.

II. *J*-PARING INTERACTION

In this section we discuss the J-pairing interaction only for identical fermions in a single-j shell. Similar results are readily obtained for bosons with spin l. Our J-pairing Hamiltonian H_J

*Electronic address: ymzhao@sjtu.edu.cn

is defined as follows:

$$H_J = G_J \sum_{M=-J}^{J} A_M^{(J)\dagger} A_M^{(J)},\ A_M^{(J)\dagger} = \frac{1}{\sqrt{2}}[a_j^\dagger \times a_j^\dagger]^{(J)},$$
$$A_M^{(J)} = -(-1)^M \frac{1}{\sqrt{2}}[\tilde{a}_j \times \tilde{a}_j]_{-M}^{(J)}, \tag{1}$$
$$\tilde{A}^{(J)} = -\frac{1}{\sqrt{2}}[\tilde{a}_j \times \tilde{a}_j]^{(J)},$$

where $[]_M^{(J)}$ means coupled to angular momentum J and projection M. We take $G_J = 1$ in this paper.

For $n = 3$, it was shown in Ref. [3] that there is only one nonzero eigenvalue for H_J when $I \geqslant j - 1$, and all eigenvalues are zero when $I < j - 1$. For $n = 4$ the situation is more complicated, because there can be many nonzero eigenvalues of spin I states for H_J, and most of these eigenvalues are not known except $I = 0$ and $I \simeq I_{\max}$. However, their summation is the trace of the H_J matrix with total spin I and is a constant with respect to any linear transformation. This trace can be obtained by summing the diagonal matrix elements

$$\langle 0|[A^{(J)} \times A^{(K)}]_M^{(I)}[A^{(J)\dagger} \times A^{(K)\dagger}]_M^{(I)}|0\rangle$$
$$= 1 + (-)^I \delta_{JK} - 4(2J+1)(2K+1) \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \tag{2}$$

over K. Here J and K take only even values. This fact can be proved by using two-body coefficients of fractional parentages, which are defined by

$$\langle j^4 \alpha IM|\}j^2(J), j^2(K)I\rangle = \frac{1}{\sqrt{6}}(-)^I \langle j^4 \alpha IM|[A^{(J)\dagger} \times A^{(K)\dagger}]_M^{(I)}|0\rangle.$$

The trace can be calculated as follows:

$$\sum_\alpha \langle j^4 \alpha I|H_J|j^4 \alpha I\rangle = \sum_K \sum_\alpha 6\langle j^2(J), j^2(K)I|\}j^4\alpha I\rangle^2$$
$$= \sum_K \sum_\alpha \langle 0|[A^{(J)} \times A^{(K)}]_M^{(I)}|I\alpha j^4\rangle \times \langle I\alpha j^4|[A^{(J)\dagger} \times A^{(K)\dagger}]_M^{(I)}|0\rangle$$
$$= \sum_K \langle 0|[A^{(J)} \times A^{(K)}]_M^{(I)}[A^{(J)\dagger} \times A^{(K)\dagger}]_M^{(I)}|0\rangle,$$

where $G_J = 1$ is used. This is just the summation of Eq. (2) over even K. One can also regard Eq. (2) as a simple generalization of the result in Ref. [3], where it was shown that the nonzero eigenvalue of H_J for spin I states of three particles is given by the norm $\langle j(j^2)J : I|j(j^2)J : I\rangle$. Note that similar results are applicable to bosons with spin l.

Let us look at nine-j symbols of identical particles under certain conditions. Based on Eq. (2) we easily find the following well-known fact:

$$\begin{Bmatrix} j & j & J \\ j & j & J \\ K & K' & I \end{Bmatrix} = 0 \tag{3}$$

for odd I when $K \neq J$ or $K' \neq J$, based on the fact that two identical pairs produce only even values of I and thus the norm in Eq. (2) equals zero. Here K and K' take even values, or odd values simultaneously. This can be also seen from the permutation symmetry of the nine-j symbol, which requires the left-hand side of Eq. (3) to equal zero unless $K + K' + I$ is even. We note without details that this formula is also applicable to four bosons with spin l [i.e., one can replace j by l in formula (3)]. This is a generalization of the result in Ref. [14], where it was found that

$$\begin{Bmatrix} j & j & 2j-1 \\ j & j & 2j-1 \\ 2j-1 & 2j-3 & 4j-4 \end{Bmatrix} = 0.$$

The norm of Eq. (2) equals zero when $I = 4j-7, 4j-5, 4j-4$, because there are no such states. We thus find

$$\begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} = \frac{1}{4(2J+1)(2K+1)} \tag{4}$$

for $I = 4j-7, 4j-5, 4j-4$ and $J \neq K$ (J, K are even). This is also a generalization of a formula in Ref. [14]:

$$\begin{Bmatrix} j & j & 2j-3 \\ j & j & 2j-1 \\ 2j-3 & 2j-1 & I \end{Bmatrix} = \frac{1}{4(4j-5)(4j-1)}$$

for $I = 4j-7, 4j-5, 4j-4$. Similarly, we have

$$\begin{Bmatrix} j & j & 2j-1 \\ j & j & 2j-1 \\ 2j-1 & 2j-1 & I \end{Bmatrix} = \frac{1}{2(4j-1)^2} \tag{5}$$

for $I = 4j-2, 4j-4$. This formula was also obtained in Ref. [14] and holds for both integer and half-integer values of j. In Appendix A, we present some explicit formulas of nine-j symbols with $J = K = 2j$ or $J = K = 2j-1$. For completeness, we also refer to Refs. [4,13–15], concerning formulas of six-j and nine-j symbols for identical particles.

Now we enumerate the number of matrices of Eq. (2) with different J. This is related to the number of nonzero two-body coefficients of fractional parentage obtained for specific examples in studying regularities of energy centroids in the presence of random interactions [16]. Without going into details we present the results of the number of matrices involved in Eq. (2) with different J.

For $I \geqslant 2j$, the number of matrices with $K = J$ is given by

$$[(4j+2-I)/4] \tag{6}$$

and the number of matrices with $K \neq J$ is

$$[(4j-I)/2]([(4j-I)/2]+1)/2 - [(4j+2-I)/4]. \tag{7}$$

The [] notation in this paper means that one takes the largest integer not exceeding the value inside.

For $I \leqslant 2j-1$, the number of matrices with $J = K$ is always 1 for even I; the case with $J \neq K$ is more complicated and the number of such matrices is given in Appendix B. It is noted that J and K take only even values in this Section.

III. SUM RULES FOR NINE-J SYMBOLS

The procedure to obtain sum rules of nine-j symbols in this paper is straightforward. In Sec. II we obtained summation of eigenvalues for $H = H_J$. From the sum rule of two-body coefficients of fractional parentage, one obtains $n(n-1)/2$ multiplied by the number of spin I states, D_I, if one sums Eq. (2) over even J and even K, namely,

$$\sum_J \sum_\alpha \langle j^4 \alpha I | H_J | j^4 \alpha I \rangle = \sum_{\text{even } J \text{ even } K} \langle 0 | [A^{(J)} \times A^{(K)}]^{(I)}_M [A^{(J)\dagger} \times A^{(K)\dagger}]^{(I)}_M | 0 \rangle = 6 D_I, \quad (8)$$

where the D_I formulas were given in Refs. [10,12]. New sum rules of nine-j symbols now can be obtained by using the D_I formulas, Eq. (2), and Eq. (8).

For realistic systems both J and K are even, as in Eq. (2) of Sec. II and Eq. (8). In this paper we also discuss sum rules of nine-j symbols under other conditions for J and K, such as odd J and odd K, etc. We denote

$$S_I(j^4, \text{condition } X \text{ on } J \text{ and } K) = \sum_X 4(2J+1)(2K+1) \begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix} \quad (9)$$

for sake of simplicity. The condition X of the sum rules for J and K will be one of the following: (1) even J and even K (realistic); (2) even J and odd K or odd J and even K; (3) odd J and odd K; and (4) both even and odd values for J and K. Conditions (2)–(4) are not physical for identical particles in quantum mechanics. We similarly define $S_I(l^4, \text{condition } X \text{ on} J \text{ and } K)$ for l.

First we present our results of $S_I(j^4, \text{requirement} X \text{ on } J \text{ and } K)$, which provides us with rich sum rules of nine-j symbols.

For $I \geqslant 2j$, one obtains

$$S_I(j^4, \text{even } J \text{ even } K) = \frac{1}{2}\left[\frac{4j-I}{2}\right] \times \left[\frac{4j-I+2}{2}\right] \times (-)^I \left[\frac{4j+2-I}{4}\right] - 6D_I, \quad (10)$$

based on Eqs. (2) and (6)–(8). Let us introduce I_0 by the relation $I = I_{\max} - 2I_0$ for even I and $I = I_{\max} - 2I_0 - 3$ for odd I, where $I_{\max} = 4j - 6$. Using I_0 we can rewrite $D_I = D_{I_{\max}-2I_0}$ for even I and $D_I = D_{I_{\max}-2I_0-3}$ for odd I. According to Ref. [12],

$$D_I = 3\left[\frac{I_0}{6}\right]\left(\left[\frac{I_0}{6}\right]+1\right) - \left[\frac{I_0}{6}\right] + \left(\left[\frac{I_0}{6}\right]+1\right)((I_0 \bmod 6)+1) + \delta_{(I_0 \bmod 6),0} - 1.$$

One thus has

$$S_I(j^4, \text{even } J \text{ even } K) = (-)^I\left[\frac{4j+2-I}{4}\right] + \frac{1}{2}\left[\frac{4j-I}{2}\right] \times \left[\frac{4j-I+2}{2}\right] - 18\left[\frac{I_0}{6}\right]\left(\left[\frac{I_0}{6}\right]+1\right) + 6\left(\left[\frac{I_0}{6}\right]+1\right) - 6\left(\left[\frac{I_0}{6}\right]+1\right)((I_0 \bmod 6)+1) - 6\delta_{(I_0 \bmod 6),0}. \quad (11)$$

The behavior of the right-hand side is not easy to see owing to terms such as $(I_0 \bmod 6)$, δ, and $[\frac{I_0}{6}]$. The situation becomes much more transparent when one writes $S_I(j^4, \text{even } J \text{ even } K)$ values explicitly. For $I =$ even, we find the following formulas:

$$S_I(j^4, \text{even } J \text{ even } K) = \begin{cases} 2 & \text{for} \quad I = I_{\max}, \\ 6 & \text{for} \quad I = I_{\max} - 2, \\ 6 & \text{for} \quad I = I_{\max} - 4, \\ 6 & \text{for} \quad I = I_{\max} - 6, \\ 8 & \text{for} \quad I = I_{\max} - 8, \\ 10 & \text{for} \quad I = I_{\max} - 10, \\ \vdots & \qquad \vdots \end{cases} ; \quad (12)$$

for $I =$ odd we can use Eq. (12) to obtain the sum rules $S_I(j^4, \text{even } J \text{ even } K) = S_{I+3}(j^4, \text{even } J \text{ even } K)$. We find that $S_I(j^4, \text{even } J \text{ even } K)$ has a modular behavior:

$$S_I(j^4, \text{even } J \text{ even } K) = S_{((I_{\max}-I) \bmod 12)}(j^4, \text{even } J \text{ even } K) + 6\left[\frac{I_{\max}-I}{12}\right]. \quad (13)$$

For $I = I_{\max} - 1$, one obtains $S_I(j^4, \text{even } JK) = 4$ based on the right-hand side of Eqs. (2) and (8).

For $I \leqslant 2j - 1$, Eq. (8) is less easily simplified [17], owing to the complexity of the D_I formula (see Eq. (3) of Ref. [10]) and number of $J = K$ and $J \neq K$ matrices of Eq. (2). However, by using Eq. (8) of this paper, Eq. (3) of Ref. [10], and the results in Appendix B, one is able to obtain explicitly the sum rules for $I \leqslant 2j - 1$:

$$S_I(j^4, \text{even } J \text{ even } K) = \begin{cases} 2m-2 & \text{for} \quad I = 0, \\ 0 & \text{for} \quad I = 1, \\ 4-2m & \text{for} \quad I = 2, \\ 2m & \text{for} \quad I = 3, \\ 2 & \text{for} \quad I = 4, \\ 4-2m & \text{for} \quad I = 5, \\ 2+2m & \text{for} \quad I = 6, \\ 4 & \text{for} \quad I = 7, \\ 6-2m & \text{for} \quad I = 8, \\ 2+2m & \text{for} \quad I = 9, \\ 6 & \text{for} \quad I = 10, \\ 8-2m & \text{for} \quad I = 11, \\ \vdots & \qquad \vdots \end{cases} \quad (14)$$

which has a modular behavior:

$$S_I(j^4, \text{even } J \text{ even } K) = S_{(I \bmod 12)}(j^4, \text{even } J \text{ even } K) + 6\left[\frac{I}{12}\right]. \quad (15)$$

In Eq. (14), $m = (j - 3/2) \bmod 3$.

In Eqs. (8) and (10)–(15) J and K take only even values. It is interesting to discuss whether there are simple sum rules in which J and K can be both even and odd. For this case $0 \leqslant I \leqslant I_{\max} = 4j$. Starting from Eq. (9.29) of Ref. [2] for $J_1 = J_2 = J_3 = J_4$, $J_{12} = J_{13} = J$, $J_{34} = J_{24} = K$, $J = I$, we multiply $4(2J+1)(2K+1)$ and sum over all JK (i.e., J and K take both even and odd non-negative integers). By using Eq. (9.28) of Ref. [2], we find

$$S_I(j^4, \text{both even and odd values for } J \text{ and } K) = \sum_{J,K=0;\Delta(JKI)}^{2j} (-)^{J+1} = \begin{cases} 4\,[(1+I)/2] & \text{for } I \leqslant 2j+1, \\ 4 + 4\,[(4j-I)/2] & \text{for } I \geqslant 2j, \end{cases} \quad (16)$$

where $\Delta(JKI)$ means that J, K, and I satisfy the triangle relation of angular momentum coupling.

If both J and K are odd values, $0 \leqslant I \leqslant I_{\max} = 4j$. In this case we consider fictitious "bosons" (which are not realistic for identical particles) with a half-integer spin j. According to Ref. [12], the number of states D_I for four bosons with spin j equals that for four fermions in a single-l shell with $2l = 2j + 3$. As D_I for four fermions in a single-l shell was given in Ref. [4], we can derive $S_I(j^4$, odd J odd $K)$ by using Eqs. (2) and (8), together with Eqs. (3)–(5) in Ref. [4]. Similarly to Eqs. (10), (11), and (14), we obtain that, when $I \leqslant 2j$,

$$S_I(j^4, \text{odd } J \text{ odd } K) = \begin{cases} 2-2m & \text{for } I = 0, \\ 0 & \text{for } I = 1, \\ 2m & \text{for } I = 2, \\ 4-2m & \text{for } I = 3, \\ 2 & \text{for } I = 4, \\ 2m & \text{for } I = 5, \\ 6-2m & \text{for } I = 6, \\ 4 & \text{for } I = 7, \\ 2+2m & \text{for } I = 8, \\ 6-2m & \text{for } I = 9, \\ 6 & \text{for } I = 10, \\ 4+2m & \text{for } I = 11, \\ \vdots & \vdots \end{cases} \quad (17)$$

has a modular behavior:

$$S_I(j^4, \text{odd } J \text{ odd } K) = S_{(I \bmod 12)}(j^4, \text{odd } J \text{ odd } K) + 6\left[\frac{I}{12}\right], \quad (18)$$

where $m = (j - 3/2) \bmod 3$ in Eq. (17); and when $I \geqslant 2j$

$$S_I(j^4, \text{odd } J \text{ odd } K) = \begin{cases} 4 & \text{for } I = 4j, \\ 2 & \text{for } I = 4j-2, \\ 4 & \text{for } I = 4j-4, \\ 6 & \text{for } I = 4j-6, \\ 6 & \text{for } I = 4j-8, \\ 6 & \text{for } I = 4j-10, \\ \vdots & \vdots \end{cases} \quad (19)$$

has a modular behavior:

$$S_I(j^4, \text{odd } J \text{ odd } K) = S_{((4j-I) \bmod 12)}(j^4, \text{odd } J \text{ odd } K) + 6\left[\frac{4j-I}{12}\right] \quad (20)$$

for even I, and $S_I(j^4$, odd J odd $K) = S_{I+3}(j^4$, odd J odd $K)$ for odd I. For $I = 4j - 1$ (odd I), $S_I(j^4$, odd J odd $K) = 0$.

For even J and odd K or for odd J and even K, $0 \leqslant I \leqslant I_{\max} = 4j - 1$. For this case

$$S_I(j^4, \text{even } J \text{ odd } K) \equiv S_I(j^4, \text{odd } J \text{ even } K) = (S_I(j^4, \text{both even and odd values for } J \text{ and } K) - S_I(j^4, \text{even } J \text{ even } K) - S_I(j^4, \text{odd } J \text{ odd } K))/2.$$

Using this relation and previous results we find that, when $I \leqslant 2j$,

$$S_I(j^4, \text{even } J \text{ odd } K) \equiv S_I(j^4, \text{odd } J \text{ even } K) = \begin{cases} 2\left[\frac{I}{4}\right] & \text{for even } I, \\ 2 + 2\left[\frac{I}{4}\right] & \text{for odd } I; \end{cases} \quad (21)$$

and, when $I \geqslant 2j$,

$$S_I(j^4, \text{even } J \text{ odd } K) \equiv S_I(j^4, \text{odd } J \text{ even } K) = 2 + \left[\frac{I_{\max} - I}{4}\right]. \quad (22)$$

Similarly, we obtain sum rules by replacing the half-integer j with the integer l. First, let us study the case for even values of J and K. We find that, when $I \leqslant 2l$,

$$S_I(l^4, \text{even } J \text{ even } K) = \sum_{\text{even } J \text{ even } K} 4(2J+1)(2K+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} = \begin{cases} 4-2m & \text{for } I = 0, \\ 0 & \text{for } I = 1, \\ 2m & \text{for } I = 2, \\ 2-2m & \text{for } I = 3, \\ 4 & \text{for } I = 4, \\ 2m & \text{for } I = 5, \\ 6-2m & \text{for } I = 6, \\ 2 & \text{for } I = 7, \\ 4+2m & \text{for } I = 8, \\ 6-2m & \text{for } I = 9, \\ 6 & \text{for } I = 10, \\ 2+2m & \text{for } I = 11, \\ \vdots & \vdots \end{cases} \quad (23)$$

has a modular behavior:

$$S_I(l^4, \text{even } J \text{ even } K) = S_{(I \bmod 12)}(l^4, \text{even } J \text{even} K) + 6\left[\frac{I_{\max} - I}{12}\right], \quad (24)$$

where $m = l \bmod 3$ in Eq. (23), and, when $I \geqslant 2l$,

$$\sum_{\text{even } JK} 4(2J+1)(2K+1)\begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} = \sum_{\text{even } JK}\left(1 + (-)^I \delta_{JK}\right) - 18\left[\frac{I_0}{6}\right]\left(\left[\frac{I_0}{6}\right] + 1\right) + 6\left[\frac{I_0}{6}\right] - 6\left(\left[\frac{I_0}{6}\right] + 1\right)((I_0 \bmod 6) + 1) - 6\delta_{(I_0 \bmod 6),0} + 6$$

$$= \begin{cases} 4 & \text{for} \quad I = I_{\max}, \\ 0 & \text{for} \quad I = I_{\max} - 1, \\ 2 & \text{for} \quad I = I_{\max} - 2, \\ 4 & \text{for} \quad I = I_{\max} - 3, \\ 4 & \text{for} \quad I = I_{\max} - 4, \\ 2 & \text{for} \quad I = I_{\max} - 5, \\ 6 & \text{for} \quad I = I_{\max} - 6, \\ 4 & \text{for} \quad I = I_{\max} - 7, \\ 6 & \text{for} \quad I = I_{\max} - 8, \\ 6 & \text{for} \quad I = I_{\max} - 9, \\ 6 & \text{for} \quad I = I_{\max} - 10, \\ 6 & \text{for} \quad I = I_{\max} - 11, \\ \vdots & \qquad \vdots \end{cases} \quad (25)$$

has a modular behavior:

$$S_I(l^4, \text{even } J \text{ even } K) = S_{((I_{\max} - I) \bmod 12)}(l^4, \text{even } JK) + 6\left[\frac{I_{\max} - I}{12}\right], \quad (26)$$

where $I_{\max} = 4l$.

If J and K take both even and odd values, similarly to the process of obtaining Eq. (16), we find for $I \leqslant 2l$

$$S_I(l^4, \text{both even and odd values for } J \text{ and } K) = \begin{cases} 4 + 4\left[\frac{I}{2}\right] & \text{for even } I, \\ 4\left[\frac{I}{2}\right] & \text{for odd } I; \end{cases} \quad (27)$$

for $I \geqslant 2l$,

$$S_I(l^4, \text{both even and odd values for } J \text{ and } K) = 4 + 4\left[\frac{4l - I}{2}\right]. \quad (28)$$

We note that this sum rule has the same form as Eq. (16) for $I \geqslant 2j$.

For odd J and odd K values, $0 \leqslant I \leqslant I_{\max} = 4l - 2$. We find that, when $I \leqslant 2l$,

$$S_I(l^4, \text{odd } J \text{ odd } K) = \begin{cases} 2m & \text{for} \quad I = 0, \\ 0 & \text{for} \quad I = 1, \\ 4 - 2m & \text{for} \quad I = 2, \\ -2 + 2m & \text{for} \quad I = 3, \\ 4 & \text{for} \quad I = 4, \\ 4 - 2m & \text{for} \quad I = 5, \\ 2 + 2m & \text{for} \quad I = 6, \\ 2 & \text{for} \quad I = 7, \\ 8 - 2m & \text{for} \quad I = 8, \\ 2 + 2m & \text{for} \quad I = 9, \\ 6 & \text{for} \quad I = 10, \\ 6 - 2m & \text{for} \quad I = 11, \\ \vdots & \qquad \vdots \end{cases} \quad (29)$$

has a modular behavior:

$$S_I(l^4, \text{odd } J \text{ odd } K) = S_{(I \bmod 12)}(l^4, \text{odd } J \text{ odd } K) + 6\left[\frac{I}{12}\right], \quad (30)$$

where $m = l \bmod 3$ in Eq. (29); and, when $I \geqslant 2l$,

$$S_I(l^4, \text{odd } J \text{ odd } K) = \begin{cases} 2 & \text{for} \quad I_{\max} - I = 0, \\ 4 & \text{for} \quad I_{\max} - I = 2, \\ 2 & \text{for} \quad I_{\max} - I = 4, \\ 6 & \text{for} \quad I_{\max} - I = 6, \\ 6 & \text{for} \quad I_{\max} - I = 8, \\ 6 & \text{for} \quad I_{\max} - I = 10, \\ \vdots & \qquad \vdots \end{cases} \quad (31)$$

has a modular behavior:

$$S_I(l^4, \text{odd } J \text{ odd } K) = S_{((I_{\max} - I) \bmod 12)}(l^4, \text{odd } J \text{ odd } K) + 6\left[\frac{I_{\max} - I}{12}\right]. \quad (32)$$

For odd $I \geqslant 2l$, $S_I = S_{I+3}$ with $S_{I_{\max}-1} = 0$.

If we take odd J and even K values or we take even J and odd K values, $0 \leqslant I \leqslant I_{\max} = 4l - 1$. For this case

$$S_I(l^4, \text{even } J \text{ odd } K) \equiv S_I(l^4, \text{odd } J \text{ even } K) = \left(S_I(l^4, \text{both even and odd values for } J \text{ and } K) - S_I(l^4, \text{even } J \text{ even } K) - S_I(l^4, \text{odd } J \text{ odd } K)\right)/2 = \begin{cases} 2\left[\frac{I+2}{4}\right] & \text{for} \quad I \leqslant 2l, \\ 2 + \left[\frac{I_{\max} - I}{4}\right] & \text{for} \quad I \geqslant 2l. \end{cases} \quad (33)$$

IV. SUMMARY AND DISCUSSION

In summary, in this paper we first showed that the sum of eigenvalues of spin I states for the J-pairing interaction is given by

$$\sum_K \left(1+(-)^I \delta_{JK}\right) - 4\sum_K (2J+1)(2K+1)\begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix}$$

for fermions and by

$$\sum_K \left(1+(-)^I \delta_{JK}\right) - 4\sum_K (2J+1)(2K+1)\begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix}$$

for bosons. Then we related them with number of spin I states to obtain nine-j sum rules. We studied

$$4(2J+1)(2K+1)\begin{Bmatrix} j & j & J \\ j & j & K \\ J & K & I \end{Bmatrix}$$

and

$$4(2J+1)(2K+1)\begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix},$$

summing over J and K under the following situations: (1) all J and K are even; (2) J and K can be both even and odd; (3) all J and K are odd; (4) J is even and K is odd. We also obtained formulas for special J, K, and I values, based on the physical meaning of the norm in Eq. (2).

Sum rules in Eqs. (A1) and (A2) of Ref. [3] can be obtained as a special case of the results in this paper: $I = 0$ for Eqs. (14) and (23). This work is therefore a generalization of some of our earlier results. We use the J-pairing interaction as a tool to obtain the sum rules but these results are independent of the interaction.

In Ref. [12], it was found that the number of spin I states D_I for four bosons with spin l and that for four fermions in a single-j shell are the same when $2l = 2j - 3$. This produces the same value of the right-hand side in Eq. (8) for fermions and bosons. Unfortunately, the numbers of J and K for these two cases are different (the number of J for bosons is $l + 1 = j - 1/2$ whereas that for fermions is $j + 1/2$), which present different sum rules of the case with even values for both J and K.

One may ask how far one can proceed along this line, that is, to construct sum rules of angular momentum coupling by using formulas of D_I. As n increases, the D_I formulas and the sum of eigenvalues of spin I states become more and more complicated. The situation is already complicated for $n = 4$. For $n = 5$ there are D_I formulas for only $I \sim 0$ or $\sim I_{\max}$. Therefore, it is difficult to obtain D_I formulas and new sum rules of angular momentum couplings in which more particles ($n \geqslant 5$) are involved, except for a few cases with $I \sim I_{\max}$ where the D_I is given by a fixed number series [10,11].

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APPENDIX A: FORMULAS OF SPECIAL NINE-J SYMBOLS

In this Appendix we present formulas for the nine-j symbol

$$\begin{Bmatrix} j & j & J \\ j & j & J \\ J & J & I \end{Bmatrix}, \tag{A1}$$

where $J = 2j$ or $2j - 1$, based on its expansion in terms of six-j symbols. The value of j in this Appendix can be either a half integer or an integer. One sees that the nine-j symbol of Eq. (A1) equals zero if I is odd, because a phase factor $(-)^{4j+4J+I} = (-)^I$ appears if one exchanges the first and the second rows in Eq. (A1). From this one obtains that the nine-j symbol of Eq. (A1) vanishes unless I is even. In the following we discuss the nine-j symbols of Eq. (A1), with I being even and $J = 2j$ or $2j - 1$.

We define

$$f'_m = \begin{Bmatrix} j & j & 2j \\ j & j & 2j \\ 2j & 2j & 4j-m \end{Bmatrix} \tag{A2}$$

and obtain following formulas:

$$f'_0 = \frac{1}{(4j+1)^2},$$

$$f'_2 = \frac{-1}{2(4j+1)^2(4j-1)},$$

$$f'_4 = \frac{3(2j-1)}{2(16j^2-1)^2(4j-3)},$$

$$f'_6 = \frac{-3\times 5(2j-2)}{4(4j-5)(4j-3)(4j-1)^2(4j+1)^2},$$

$$f'_8 = \frac{3\times 5\times 7(2j-2)(2j-3)}{4(4j-7)(4j-5)(4j-3)^2(4j-1)^2(4j+1)^2},$$

$$f'_{10} = \frac{-3\times 5\times 7\times 9}{8(4j-3)^2(4j-1)^2(4j+1)^2} \times \frac{(2j-4)(2j-3)}{(4j-9)(4j-7)(4j-5)},$$

$$f'_{12} = \frac{3\times 5\times 7\times 9\times 11}{8(4j-5)^2(4j-3)^2(4j-1)^2(4j+1)^2)} \times \frac{(2j-3)(2j-4)(2j-5)}{(4j-11)(4j-9)(4j-7)}.$$

We define

$$f_I = \begin{Bmatrix} j & j & 2j \\ j & j & 2j \\ 2j & 2j & I \end{Bmatrix}$$

and obtain following formulas:

$$f_0 = (-)^{2j} \frac{[(2j-1)!]^2}{(4j+1)^2(4j-1)!} \frac{1}{2}(2j),$$

$$f_2 = -(-)^{2j} \frac{[(2j-1)!]^2}{(4j+1)^2(4j-1)!} \frac{1}{2} \frac{(2j)(2j+1)}{(4j-1)},$$

$$f_4 = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{3}{4} \frac{(2j)(2j+1)(2j+2)}{(4j-3)(4j-1)},$$

$$f_6 = -(-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{5}{4} \times \frac{(2j)(2j+1)(2j+2)(2j+3)}{(4j-5)(4j-3)(4j-1)},$$

$$f_8 = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{7\times 5}{16} \times \frac{(2j)(2j+1)(2j+2)(2j+3)(2j+4)}{(4j-7)(4j-5)(4j-3)(4j-1)},$$

$$f_{10} = -(-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{9\times 7}{16} \times \frac{(2j)(2j+1)\cdots(2j+5)}{(4j-9)(4j-7)\cdots(4j-3)(4j-1)},$$

$$f_{12} = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \times \frac{11\times 9\times 7/3}{32} \times \frac{(2j)(2j+1)\cdots(2j+6)}{(4j-11)(4j-9)\cdots(4j-3)(4j-1)},$$

$$f_{14} = -(-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{13\times 11\times 9/3}{32} \times \frac{(2j)(2j+1)\cdots(2j+7)}{(4j-13)(4j-11)\cdots(4j-3)(4j-1)},$$

$$f_{16} = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{15\times 13\times 11\times 9/3}{256} \times \frac{(2j)(2j+1)\cdots(2j+8)}{(4j-15)(4j-13)\cdots(4j-3)(4j-1)},$$

$$f_{18} = -(-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{17\times 15\times 13\times 11/3}{256} \times \frac{(2j)(2j+1)\cdots(2j+9)}{(4j-17)(4j-15)\cdots(4j-3)(4j-1)},$$

$$f_{20} = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{\frac{19\times 17\times 15\times 13\times 11}{5\times 3}}{512} \times \frac{(2j)(2j+1)\cdots(2j+10)}{(4j-19)(4j-17)\cdots(4j-3)(4j-1)},$$

$$f_{22} = -(-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{\frac{21\times 19\times 17\times 15\times 13}{5\times 3}}{512} \times \frac{(2j)(2j+1)\cdots(2j+11)}{(4j-21)(4j-19)\cdots(4j-3)(4j-1)},$$

$$f_{24} = (-)^{2j} \frac{[(2j-1)!]}{(4j+1)^2(4j-1)!} \frac{\frac{23\times 21\times 19\times 17\times 15\times 13}{5\times 3\times 3}}{512} \times \frac{(2j)(2j+1)\cdots(2j+12)}{(4j-23)(4j-21)\cdots(4j-3)(4j-1)}.$$

We define

$$g'_m = \begin{Bmatrix} j & j & 2j-1 \\ j & j & 2j-1 \\ 2j-1 & 2j-1 & 4j-m \end{Bmatrix}$$

and obtain $g'_2 = g'_4 = 1/[2(4j-1)^2]$, [see Eq. (5) of Sec. II]. For g'_m with larger m we obtain

$$g'_6 = -\frac{3(2j-2)(16j-15)}{2(4j-5)(4j-3)^2(4j-1)^2},$$

$$g'_8 = \frac{15(2j-3)(6j-7)}{2(4j-7)(4j-5)(4j-3)^2(4j-1)^2},$$

$$g'_{10} = -\frac{7\times 5\times 3}{(4j-5)(4j-3)(4j-1)} \times \frac{(2j-4)(2j-3)(32j-45)}{4(4j-9)(4j-7)\cdots(4j-1)},$$

$$g'_{12} = \frac{9\times 7\times 5\times 3}{(4j-5)(4j-3)(4j-1)} \times \frac{(2j+4)(2j-5)(20j-33)}{4(4j-11)(4j-9)\cdots(4j-1)}.$$

We define

$$g_I = \begin{Bmatrix} j & j & 2j-1 \\ j & j & 2j-1 \\ 2j-1 & 2j-1 & I \end{Bmatrix}$$

and obtain

$$g_0 = (-)^{2j} \frac{j(4j-3)\,[(2j-1)!]^2}{(4j-1)(4j-1)!},$$

$$g_2 = -(-)^{2j} \frac{j(8j^2-6j-3)\,[(2j-1)!]^2}{(4j-3)(4j-1)(4j-1)!},$$

$$g_4 = (-)^{2j} \frac{3j(2j+1)(4j^2-3j-5)\,[(2j-1)!]^2}{(4j-5)(4j-3)(4j-1)(4j-1)!},$$

$$g_6 = -(-)^{2j} \frac{j(j+1)(2j+1)\,[(2j-1)!]^2}{(4j-1)!} \times \frac{5(8j^2-6j-21)}{(4j-7)(4j-5)(4j-3)(4j-1)},$$

$$g_8 = (-)^{2j} \frac{j(j+1)(2j+1)(2j+3)\,[(2j-1)!]^2}{(4j-1)!} \times \frac{35(4j^2-3j-18)}{2(4j-9)(4j-7)(4j-5)(4j-3)(4j-1)},$$

$$g_{10} = -(-)^{2j} \frac{j(j+1)(j+2)(2j+1)(2j+3)}{(4j-1)!} \times \frac{63(8j^2-6j-55)\,[(2j-1)!]^2}{2(4j-11)(4j-9)\cdots(4j-1)},$$

$$g_{12} = (-)^{2j} \frac{j(j+1)(j+2)(2j+1)(2j+3)(2j+5)}{(4j-1)!} \times \frac{231(4j^2-3j-39)\,[(2j-1)!]^2}{2(4j-13)(4j-11)\cdots(4j-1)}.$$

Some of these g_m were also obtained for fermions in a single-j shell in Ref. [4], where j is a half integer, whereas here j can be either an integer or a half integer.

APPENDIX B: NUMBER OF MATRICES WITH $K \neq J$ FOR $I \leqslant 2j$

The number of matrices with $J = K$ is always 1 for an even value of I, which contribute $2 \times (j + 1/2)$ on the left-hand side of Eq. (8), whereas that (denoted by F_J here) with $J \neq K$ is rather complicated:

for $I \leqslant [j]$ with $J > 2[(I-1)/2]$ and $J < 2j - 1 - 2\left[\frac{I}{2}\right]$, $F_J = 2\left[\frac{I}{2}\right]$;

for $I \leqslant [j]$ with $J < 2[(I-1)/2]$, $F_J = J$;

for $I \leqslant [j]$ with $J \geqslant 2j - 1 - 2\left[\frac{I}{2}\right]$,
$F_J = \left[\frac{I}{2}\right] + \left[\frac{2j-1-J}{2}\right]$;

for $[j] \leqslant I \leqslant 2j$ with $J < 2j - 1 - 2\left[\frac{I}{2}\right]$, $F_J = J$;

for $[j] \leqslant I \leqslant 2j$ with $J \geqslant 2j - 1 - 2\left[\frac{I}{2}\right]$ and $J < 2\left[\frac{I}{2}\right]$,
$F_J = 2j - 1 - 2\left[\frac{I}{2}\right] + \left[\frac{J - \left(2j-1-2\left[\frac{I}{2}\right]\right)}{2}\right]$;

for $[j] \leqslant I \leqslant 2j$ with $J > 2\left[\frac{I}{2}\right]$,
$F_J = (2j - 1 - J)/2 + \left[\frac{I}{2}\right]$.

The complexity in this classification makes it tedious to show $\sum_{JK}(1 + (-)^I)$ by one formula, because one must simplify many terms such as [], which means taking the largest integer not exceeding the value inside. However, one can obtain explicit sum rules by writing down their value and studying their individual modular behavior, as shown in this paper.

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