

Renormalization of the deuteron with one pion exchange

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We analyze the deuteron bound state through the one pion exchange potential. We pay attention to the short distance peculiar singularity structure of the bound state wave functions in coordinate space and the elimination of short distance ambiguities by selecting the regular solution at the origin. We determine the so far elusive amplitude of the converging exponential solutions at the origin. All bound state deuteron properties can then be uniquely deduced from the deuteron binding energy, the pion-nucleon coupling constant, and pion mass. This generates correlations among deuteron properties. Scattering phase shifts and low energy parameters in the 3S_1 - 3D_1 channel are constructed by requiring orthogonality of the positive energy states to the deuteron bound state, yielding an energy independent combination of boundary conditions. We also analyze from the viewpoint of short distance boundary conditions the weak binding regime on the light of long distance perturbation theory and discuss the approach to the chiral limit.

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I. INTRODUCTION

Pion dynamics plays a dominant role in the low energy structure of the nucleon-nucleon interaction, and in particular in the description of light nuclei like the deuteron [1]. The long distance part of the interaction is given by one, two, and higher pion exchanges and the fact that the deuteron is a weakly bound state suggests that many of its properties can indeed be explained in terms of these dynamical degrees of freedom in a model independent way and regardless on the less known short distance interaction. Glendenning and Kramer [2] in the early 60s recognized clear correlations between several deuteron observables generated by truncating the one pion exchange (OPE) potential at a distance $R = 0.4915$ fm and assuming a hard core inside. Tight constraints on deuteron observables were established by Klarsfeld, Martorell, and Sprung [3,4] by integrating the deuteron wave function from infinity down to a cut-off radius using the OPE potential and rigorous inequalities. An accurate determination of the D/S asymptotic ratio was made by Ericson and Rosa-Clot [5,6] based on assuming the OPE correlation between the S and D wave functions and taking realistic potential models to describe the S wave function. (For a review on these developments see, e.g., Ref. [7].) Friar *et al.* use a multipole form factor [8] whereas Ballot *et al.* used separate monopole forms factor for the central and tensor part of the OPE potential mimicking the finite size of nucleons [9,10]. Along a similar line of investigation Sprung *et al.* used a square well potential [11] for the central component and a vanishing potential for the tensor component.

Within the effective field theory (EFT) approach to nuclear physics proposed by Weinberg [12] the situation was revisited from a somewhat different perspective since the OPE potential appears as the lowest order of a perturbative hierarchy based on chiral symmetry [13] (for a review see, e.g., Ref. [14]),

and short distance ambiguities could be eliminated by the renormalization program if the auxiliary regulator is removed from the theory at the end of the calculation. This cut-off independence should occur at *any level* of approximation, no matter how many pions are exchanged. At long distances, renormalization group methods suggest that one is close to an infrared fixed point [15]. The renormalization procedure can be explicitly and analytically carried out within perturbation theory [16]. However, these nice features become a nontrivial numerical problem beyond perturbation theory motivating the use of truncation cut-off schemes. The work of Ref. [17] uses a Gaussian cut-off, in coordinate space to regulate the contact delta interaction, Ref. [18] proposes the use of a subtraction method in momentum space regulating the central part, Ref. [19] uses a sharp momentum cut-off, and in Ref. [20] it was proposed to use a finite short distance cut-off, whereas Ref. [21] puts exponentially suppressed regulators in momentum space. It should be mentioned that in all cases the corresponding coordinate/momentum space cut-off parameter a/Λ is uncomfortably large/small from the viewpoint of renormalization theory. Typically, one has $a \sim 1.4$ fm (see, e.g., Ref. [20]) and $\Lambda = 600$ MeV [17], respectively. So, it is not obvious that according to the basic principles of EFT the short distance ambiguities are, as one might expect, indeed under control. Moreover, the existence of a well behaved finite renormalized limit is never guaranteed *a priori* and one relies mainly on numerics. Actually, the fact that the results on deuteron observables look rather similar, regardless on the particular way how the potential is modeled at short distances, proves that the long distance pion dynamics dominates the physics confirming the findings of Glendenning and Kramer [2] more than 40 years ago, but does not resolve the mathematical problem whether the OPE potential can make unambiguous predictions regardless of any short distance physical scale.

The OPE potential is local in coordinate space where the problem is naturally formulated by the standard Schrödinger framework. Moreover, it is singular at the origin and giving boundary conditions at that point is not a well defined

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procedure for uniquely determining the solution [22] (for a comprehensive review in the one channel case see, e.g., Ref. [23]). There is the added difficulty that we have two coupled second order differential equations. In the deuteron channel one has four independent solutions, which according to their singularity structure correspond to either two regular and two irregular solutions at infinity or three regular and one irregular solution at the origin. The normalizability condition of the deuteron wave functions eliminates all constants for a given deuteron binding energy, which instead of being predicted has to be treated as an independent parameter. One of the advantages of the coordinate space treatment of renormalization is that it can directly be extended to other singular cases such as the two pion exchange (TPE) potential [24,25] which is also finite everywhere except the origin. In contrast, momentum space treatments require an extra regularization of the potential besides the standard cut-off regularization of the Lippmann-Schwinger equation.

The authors of Ref. [11] found a discrete sequence of equivalent short distance cut-off radii having almost the same deuteron properties. In their analysis of the problem one regular solution at the origin with a converging exponential behavior, $\exp(-4(2R/r)^{\frac{1}{2}})$ with $R \sim 1$ fm, was discarded on numerical grounds. The same result was also implicitly used in Ref. [26] and large N_c arguments in favor of it were raised in Ref. [27]. This extra condition would actually *predict* the deuteron binding energy from the OPE potential. As we will show in this paper, the converging exponential is nonvanishing although rather elusive because its contribution to the deuteron wave function only becomes sizable at relatively large distances and accurate numerical work must be done to pin down its value with a certain degree of confidence.

In the present work we show that there is no need to truncate the OPE potential on a physical scale to produce unique and cut-off independent predictions for deuteron properties and scattering observables in terms of the OPE potential parameters and the deuteron binding energy. These might then legitimately be called OPE model independent predictions and paves the way for a systematic investigation on the case where more pions are exchanged and other effects are taken into account [24,25]. After presenting the basic notation in Sec. II A, we discuss in Sec. II B the regular solutions at the origin and establish that the limit when the regulator is removed is finite. For numerical purposes it is useful to define *some* short distance regulator as an auxiliary tool. In Sec. II C we use six different regulators based on boundary conditions of the wave function and check for stability to high precision for all regulators. This procedure generates correlations among deuteron observables if the deuteron binding energy is varied as a free and independent parameter as we do in Sec. III. A particularly interesting situation is provided by the weak binding limit, which can be taken with fixed OPE potential parameters. In such a case the long distance behavior should dominate and one might expect perturbative methods to apply and be compared to the exact OPE calculation. The details of the perturbative calculation are postponed to Appendix A where we present a coordinate space version of the method, in consonance with the exact treatment. A detailed comparison shows that the perturbative argument is too naive

and would only hold in the weak coupling regime as well, due to the appearance of nonanalytical contributions in the πNN coupling constant. Mathematically, we show that it is not possible to go beyond first order since the coefficients of the expansion diverge. Numerically, the disagreement at first order is typically on the 30% level for physical values of the OPE parameters at zero binding. After Ref. [28] the chiral limit in nuclear physics has attracted considerable attention in recent works [29–31] and also the limit of heavy pions in connection with lattice QCD calculations, where the pion mass is still far from its physical value. We study in Sec. IV the correlations among those observables if the pion mass is varied away from its physical value by studying a suitable extension of the Feynman-Hellmann theorem. Another remarkable property of the OPE potential which we deal with in Sec. V is that low energy parameters as well as the scattering phase-shifts can be uniquely determined from the OPE potential parameters and the deuteron binding, due to orthogonality constraints of the bound state and scattering states. In Sec. VI the determination of the nonvanishing coefficient of the converging exponential at the origin is carried out by a short distance expansion to eighth order of the OPE deuteron wave functions.

One of the surprising results in the OPE description of the deuteron has to do with the small asymptotic ratio between the D and S waves, $w(\infty)/u(\infty) = \eta = 0.0256$ coming from a large ratio at short distances of order unity, $w(0)/u(0) = 1/\sqrt{2} = 0.707$. Although this feature is specific to the OPE potential it is somewhat a bit outside the main topic of this work. So we relegate this issue to Appendix C where we show how this can be easily understood if a local rotation of the deuteron wave functions diagonalizing the coupled channel potential is carried out. Obviously, such a transformation cannot simultaneously diagonalize the kinetic terms, but the residual mixing is related to the derivative of a local mixing angle which numerically turns out to be a slowly varying function. Using this as a starting approximation we can determine in a perturbative fashion the asymptotic D/S ratio yielding the exact OPE value with a 1% accuracy.

II. BOUND STATE EQUATIONS AND THEIR SOLUTIONS

A. The OPE deuteron equations

The deuteron coupled channel 3S_1 - 3D_1 set of equations read

$$-u''(r) + U_s(r)u(r) + U_{sd}(r)w(r) = -\gamma^2 u(r), \quad (1)$$

$$-w''(r) + U_{sd}(r)u(r) + \left[U_d(r) + \frac{6}{r^2} \right] w(r) = -\gamma^2 w(r), \quad (2)$$

together with the asymptotic conditions at infinity

$$\begin{aligned} u(r) &\rightarrow A_S e^{-\gamma r}, \\ w(r) &\rightarrow A_D e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \end{aligned} \quad (3)$$

where $\gamma = \sqrt{MB}$ is the deuteron wave number, A_S is the normalization factor and the asymptotic D/S ratio parameter is defined by $\eta = A_D/A_S$. The 3S_1 - 3D_1 coupled channel potential is given by

$$U_s = U_c, \quad U_{sd} = 2\sqrt{2}U_T, \quad U_d = U_C - 2U_T, \quad (4)$$

where the OPE reduced potential ($U = 2\mu V$) is given for $r > 0$ by

$$U_C = -\frac{mMg_A^2}{16\pi f_\pi^2} \frac{e^{-mr}}{r} \quad (5)$$

$$U_T = -\frac{m^2Mg_A^2}{16\pi f_\pi^2} \frac{e^{-mr}}{r} \left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right), \quad (6)$$

where m is the pion mass, $M = 2\mu_{np} = 2M_nM_p/(M_n + M_p)$ twice the reduced proton-neutron mass, g_A the axial nucleon coupling constant, and f_π the pion weak decay constant. Note that we assume this potential to be valid for any strictly positive distance, $r \neq 0$, so the limit $r \rightarrow 0^+$ will be carefully taken, *without* subtracting any contribution in the potential.

It is convenient to define the length scale

$$R = \frac{3g_A^2M}{32\pi f_\pi^2} \quad (7)$$

which value is around 1 fm. For numerical calculations we take $f_\pi = 92.4$ MeV, $M = 938.918$ MeV, $g_A = 1.29$ and hence $R = 1.07764$ fm and $g_{\pi NN} = 13.1083$, i.e., $f_{\pi NN}^2 = 0.07388$. The corresponding pion nucleon coupling constant is $g_{\pi NN} = 13.1083$ according to a phase shift analysis of NN scattering [32]. Nevertheless, after the latest determinations from the GMO sum rule [33] we will also take the value $g_{\pi NN} = 13.3158$. As we will see this variation at the 5% level dominates the uncertainties in the OPE calculations.

B. The short distance regular solutions

We look for normalized functions of the Eq. (2),

$$1 = \int_0^\infty (u(r)^2 + w(r)^2) dr, \quad (8)$$

from which A_S can be determined. The normalization at infinity is guaranteed due to the asymptotic conditions, Eq. (3). However, the coupled channel potential becomes singular at short distances, since $U_T \rightarrow -2R/r^3$. Keeping only this term in Eq. (2) one can decouple the equations through the unitary transformation [11]

$$\begin{aligned} u_A(r) &= \sqrt{\frac{2}{3}}u(r) + \frac{1}{\sqrt{3}}w(r), \\ u_R(r) &= -\frac{1}{\sqrt{3}}u(r) + \sqrt{\frac{2}{3}}w(r), \end{aligned} \quad (9)$$

yielding an attractive singular potential $U_A \rightarrow -4R/r^3$ for u_A and $U_R \rightarrow 8R/r^3$ for u_R . Any solution obtained by integrating from infinity with the Eq. (3) down to the origin has the

asymptotic short distance behavior,¹

$$\begin{aligned} u_R(r) &\rightarrow \left(\frac{r}{R}\right)^{3/4} [C_{1R}e^{+4\sqrt{2}\sqrt{\frac{R}{r}}} + C_{2R}e^{-4\sqrt{2}\sqrt{\frac{R}{r}}}], \\ u_A(r) &\rightarrow \left(\frac{r}{R}\right)^{3/4} [C_{1A}e^{-4i\sqrt{\frac{R}{r}}} + C_{2A}e^{4i\sqrt{\frac{R}{r}}}], \end{aligned} \quad (10)$$

The constants C_{1R} , C_{2R} , C_{1A} , and C_{2A} depend on both γ and η and the OPE potential parameters, $g_{\pi NN}$ and m . Note that the leading short distance r dependence does not involve the pion mass and the deuteron wave number. Higher order corrections to these solutions can be computed systematically to high orders and are presented below in Sec. VI.

The regular solution at infinity contains the normalization constant A_S , which is customarily set to one for computational purposes, the deuteron wave number γ and the asymptotic D/S ratio parameter η . The normalizability of the wave function at the origin requires

$$C_{1R}(\gamma, \eta) = 0, \quad (11)$$

which is a relation between η and γ . The other remaining constants are then completely fixed. This means that for the OPE potential, the deuteron binding energy can be used as an independent parameter. Thus, one has three independent variables, γ , the coupling constant with length scale dimension R (or equivalently $g_{\pi NN}$) and the pion mass m . Obviously, this suggests integrating in from infinity and determining η from the regularity condition at the origin (11).

To analyze whether some additional condition arises let us check the selfadjointness of the coupled channel Hamiltonian. The flux at a point r is given by

$$iJ(r) = u^*(r)u'(r) - u^*(r)u'(r) + w^*(r)w'(r) - w^*(r)w'(r), \quad (12)$$

so that current probability conservation at the origin implies

$$|C_{1A}|^2 - |C_{2A}|^2 = 2\sqrt{2}i(C_{1R}^*C_{2R} - C_{2R}^*C_{1R}). \quad (13)$$

Thus, if we set $C_{1R} = 0$ there is no condition on C_{2R} and one has $C_{1A} = C_A e^{i\varphi}$ and $C_{2A} = C_A e^{-i\varphi}$ with C_A and φ real. So, we have three constants, $C_{2R}(\gamma)$, $C_A(\gamma)$, and $\varphi(\gamma)$, characterizing the normalizable solutions at short distances for a given value of the deuteron wave number γ ,

$$\begin{aligned} u_R(r) &\rightarrow C_R(\gamma) \left(\frac{r}{R}\right)^{3/4} e^{-4\sqrt{2}\sqrt{\frac{R}{r}}}, \\ u_A(r) &\rightarrow C_A(\gamma) \left(\frac{r}{R}\right)^{3/4} \sin \left[4\sqrt{\frac{R}{r}} + \varphi(\gamma)\right]. \end{aligned} \quad (14)$$

Actually, if we have any other state, say a scattering state with positive energy, unitarity (i.e., orthogonality) requires that the constant $\varphi(k)$ coincides with the bound state phase $\varphi(\gamma)$. We will come back to this issue later when discussing low energy parameters and scattering solutions in Sec. V. It is natural to expect that some combination of short distance constants is independent on the OPE potential parameters as

¹The solutions for u_A and u_R are written in terms of spherical Bessel functions [11]. We keep the leading short distance behavior only.

they encode short distance physics. In Sec. IV we establish, by demanding the standard Feynman-Hellmann theorem, that specifically the short distance phase φ does not depend on the OPE potential parameters. In Sec. VI we determine the values of the three constants characterizing the three regular solutions by a detailed short distance analysis of the OPE deuteron wave functions.

Note that any additional condition would actually *predict* both γ and η from m and R . This contradicts the claim of Ref. [11] that $C_{2R} = 0$, a conclusion implicitly used in Ref. [26] and supported by the large N_c argument of Ref. [27]. On the other hand, if one takes the experimental values of η and γ as done in Ref. [20] one obtains both nonvanishing C_{1R} and C_{2R} , i.e., the irregular non-normalizable solution, unless a short distance cut-off, $R > 0.8$ fm, is introduced as a physical scale and not as an auxiliary removable regulator.

C. Regularization with boundary conditions

Ideally, one would integrate in the large asymptotic solutions, Eq. (3), and match the short distance behavior of Eq. (11) imposing the regularity condition (11). In practice, however, the converging exponential at the origin is rather elusive since integrated-in solutions quickly run into the diverging exponentials due to round-off errors for $r \sim 0.05$ fm and dominate over the converging exponential. The reason has to do with the fact that the natural scale where both exponentials are comparable is rather large $r = 4\sqrt{2}R \sim 6$ fm, but in that region the lowest order short distance approximation does not hold.

Instead, we will also try putting several short distance boundary conditions corresponding to the choice of regular solutions at the origin,

$$\begin{aligned}
u(a) &= 0 & \text{(BC1),} \\
u'(a) &= 0 & \text{(BC2),} \\
w(a) &= 0 & \text{(BC3),} \\
w'(a) &= 0 & \text{(BC4),} \\
u(a) - \sqrt{2}w(a) &= 0 & \text{(BC5),} \\
u'(a) - \sqrt{2}w'(a) &= 0 & \text{(BC6).}
\end{aligned} \tag{15}$$

The advantage of using this kind of short distance cut-offs based on a boundary condition is that there is only a single scale in the problem as one naturally expects, and that one never needs to declare what is the wave function below the boundary radius. Putting a square well potential as a counter-term [26] with depth U_0 appears natural from standard perturbative experience but needs specification of a further length scale, $1/\sqrt{U_0}$, and moreover, generates multivaluation ambiguities [36,37].

It is convenient to use the superposition principle of boundary conditions to write

$$\begin{aligned}
u(r) &= u_S(r) + \eta u_D(r), \\
w(r) &= w_S(r) + \eta w_D(r),
\end{aligned} \tag{16}$$

where (u_S, w_S) and (u_D, w_D) correspond to the boundary conditions at infinity, Eq. (3) with $A_S = 1$ and $A_D = 0$ and with $A_S = 0$ and $A_D = 1$, respectively. Thus, at the boundary

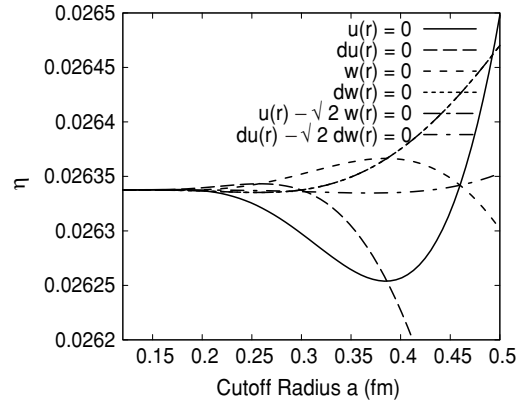


FIG. 1. The dependence of the asymptotic D/S ratio η on the cut-off radius a for several boundary conditions. We use $m = 138.03$ MeV and $R = 1.07764$ fm (corresponding to $g_{\pi NN} = 13.1083$).

we can impose any of the conditions by just eliminating η .² The resulting η value obtained by all these boundary conditions is presented in Fig. 1. Actually, we see that the boundary condition $u'(a) - \sqrt{2}w'(a)$ is about the smoothest condition we can think of, since the u_R combination goes to zero at small distances, its derivative, u'_R , also goes to zero, although a bit less faster since $u'_R/u_R \sim 1/r^{3/2}$. We see that all determinations of η based on any of the proposed cut-offs yield the same value with great accuracy at cut-off radii below 0.2 fm. This is somewhat fortunate since arithmetic precision is outraged typically for $r < 0.06$ fm. Obviously, any short distance cut-off generates finite cut-off effects in the wave functions for distances close to the cut-off radius. In Sec. VI we analyze this problem by matching solutions of the form of Eq. (14) to the integrated in numerical solutions and find that for many practical purposes these finite cut-off effects are negligible. Thus, we will base most of our results on the “smoothest” condition BC6 of Eq. (15).

The resulting deuteron wave functions u and w obtained by integrating in from infinity to the origin the OPE potential are plotted in Fig. 2 where the irregular solutions are obtained with the experimental D/S ratio $\eta_d = 0.0256$ and the regular ones with the OPE D/S ratio $\eta_{\text{OPE}} = 0.026333$. For comparison we also plot the NijmII deuteron wave functions. We emphasize that the value of η is a direct consequence of taking the OPE down to the origin seriously.

²Numerically we find at the cut-off boundary $r = 0.2$ fm

$$\begin{aligned}
u(0.2) &= 1139.23 - 43263.2 \eta \\
w(0.2) &= -1807.33 + 68632.5 \eta \\
u'(0.2) &= -35529.8 + 1.34913 \times 10^6 \eta \\
w'(0.2) &= 55194.3 - 2.09606 \times 10^6 \eta
\end{aligned}$$

These large numbers appear because of the dominance of the diverging exponential at short distances.

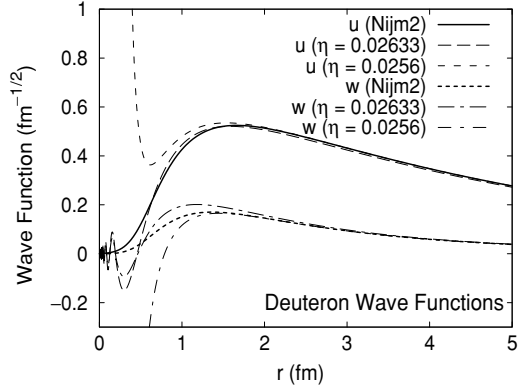


FIG. 2. The deuteron wave functions u and w , obtained by integrating in from infinity to the origin the OPE potential, compared to those obtained with the Nijm II potential [35]. The irregular solutions are obtained with the experimental D/S ratio, $\eta_d = 0.0256$, and the regular ones with the OPE D/S ratio, $\eta_{\text{OPE}} = 0.026333$. We use $m = 138.03$ MeV and $R = 1.07764$ fm (corresponding to $g_{\pi NN} = 13.1083$).

D. Deuteron observables

Once the solutions are known we can determine several properties of interest. The matter radius reads

$$r_m^2 = \frac{1}{4} \langle r^2 \rangle = \frac{1}{4} \int_0^\infty r^2 (u(r)^2 + w(r)^2) dr \quad (17)$$

while potential contribution to the quadrupole moment (without meson exchange currents)

$$Q_d = \frac{1}{20} \int_0^\infty r^2 w(r) [2\sqrt{2}u(r) - w(r)] dr. \quad (18)$$

An important observable is the deuteron inverse radius

$$\langle r^{-1} \rangle = \int_0^\infty dr \frac{u(r)^2 + w(r)^2}{r} \quad (19)$$

TABLE I. Deuteron properties for the OPE compared to the short-range approximation and first order perturbation theory. We use the nonrelativistic relation $\gamma = \sqrt{2\mu_{np}B}$ with $B = 2.224575(9)$ and take $m = 138.03$ MeV and $R = (3/8M)g_{\pi NN}^2/(4\pi) = 1.07764$ fm corresponding to $g_{\pi NN} = 13.1083$ [32] (except the row OPE* where the value $g_{\pi NN} = 13.316$ [33] has been taken). The error is estimated by changing the short distance cut-off in the range $a = 0.1-0.2$ fm.

	$\gamma(\text{fm}^{-1})$	η	$A_S(\text{fm}^{-1/2})$	$r_m(\text{fm})$	$Q_d(\text{fm}^2)$	P_D	$\langle r^{-1} \rangle$	$\alpha_0(\text{fm})$	$\alpha_2(\text{fm}^3)$	$\alpha_2(\text{fm}^5)$	$r_0(\text{fm})$
Short	Input	0	0.6806	1.5265	0	0%	∞	4.3177	0	0	0
OPE(pert)	Input	0.051	0.7373	1.6429	0.4555	0%	∞	4.6089	2.5365	0	0.4831
OPE	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.88(1)%	0.476(3)	5.335(1)	1.673(1)	6.169(1)	1.638(1)
OPE*	Input	0.02687	0.8718(2)	1.9429(6)	0.2826(2)	7.42(1)%	0.471(3)	5.353(1)	1.715(1)	6.4001(1)	1.663(1)
NijmII	Input	0.02521	0.8845(8)	1.9675	0.2707	5.635%	0.4502	5.418	1.647	6.505	1.753
Reid93	Input	0.02514	0.8845(8)	1.9686	0.2703	5.699%	0.4515	5.422	1.645	6.453	1.755
Exp. ^a	0.231605	0.0256(4)	0.8846(9)	1.9754(9)	0.2859(3)	5.67(4)		5.419(7)			1.753(8)

^a(Nonrelativistic) see, e.g., Ref. [39] and references therein.

which appears in low energy pion-deuteron scattering. Finally, the D -state probability is given by³

$$P_D = \int_0^\infty w(r)^2 dr. \quad (20)$$

Both P_D and $\langle r^{-1} \rangle$ are sensitive to the intermediate distance region around 2 fm whereas Q_d and r_m get their contribution from larger distances ~ 4 fm.

The results for the asymptotic S -wave normalization A_S , the matter radius r_m , the quadrupole moment, Q_d , and the D -state probability, P_D are presented in Table I. The errors in the numerical calculation have been assessed by varying the short distance cut-off in the range $a = 0.1-0.2$ fm (in momentum space that would naively correspond to take $\Lambda = 1/a = 2-4$ GeV). As we see, the cut-off uncertainty is smaller than the one induced by variations at the 2% level in the $g_{\pi NN}$ coupling constant in the range between the lowest value (~ 13.1) obtained by a fit to NN phase-shifts [32] and the highest recent value (~ 13.3) determined from the GMO sum rule [33]. Equivalently, this uncertainty corresponds to take $R = 1.0776$ fm and $R = 1.1108$ fm respectively. Our results are generally speaking in agreement with previous determinations where different sorts of cut-off methods have also been implemented.

E. Discussion

At this point it may prove useful to ponder on the previous results from a wider perspective. Let us remind the reader that the basic assumption of an EFT is that the study of long wave length phenomena such as low energy scattering or weakly bound systems do not require a detailed knowledge of short distance physics. This general and widely accepted principle requires some qualification because attractive and repulsive singular potentials behave quite differently in this respect. Singular attractive potentials, $\sim 1/r^n$, generate wave functions vanishing as a power law, $r^{n/4} \sin(r^{-n/2+1} + \varphi)$, and which

³Strictly speaking this is not an observable, but in the nonrelativistic limit it can be related to the deuteron magnetic moment, $\mu_d = (\mu_p + \mu_n)(2 - 3P_D) + 3P_D/2$, see, e.g., Ref. [34].

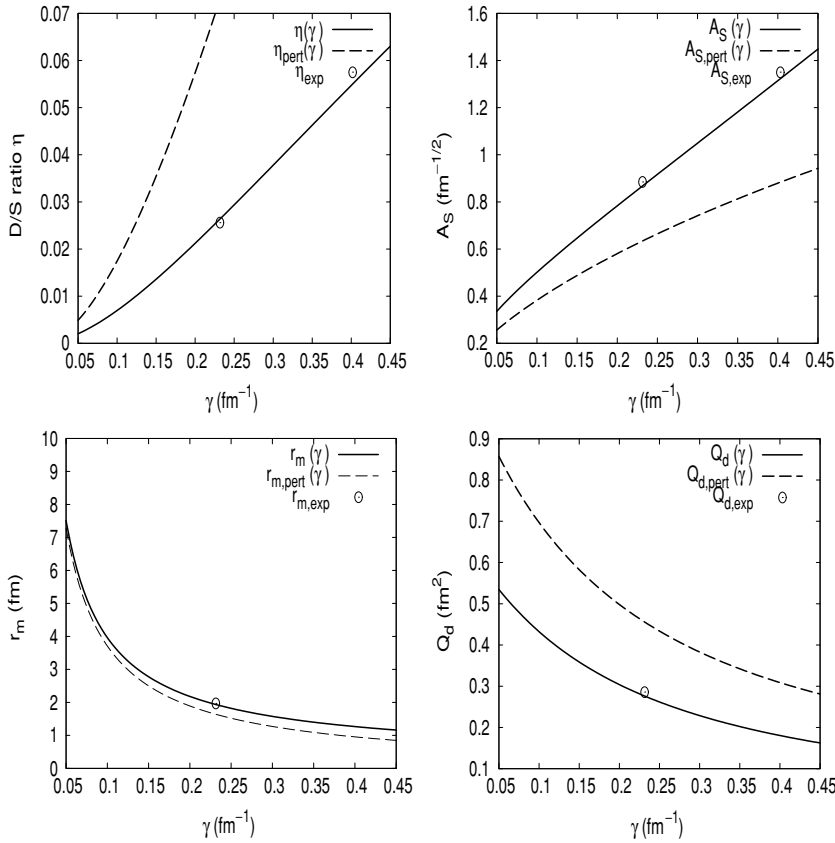


FIG. 3. The dependence of the asymptotic D/S ratio η (upper left panel), the S -wave normalization A_S (in $\text{fm}^{-1/2}$, upper right panel), the matter radius r_m (in fm, lower left panel), and the quadrupole moment Q_d (in fm^2 , lower right panel) on the deuteron wave number γ (in fm^{-1}) for the short-range theory, the first order perturbative result and the exact OPE result. The points represent the experimental values or potential model estimates [39]. We use $m = 138.03$ MeV and $R = 1.07764$ fm.

need a mixed boundary condition to specify the short distance phase φ . Thus, short distance details become less important, regardless on the value of φ . On the contrary, for singular repulsive potentials the wave functions behave as $r^{n/4} e^{\pm r^{-n/2+1}}$ and only for the regular solution short distance details become irrelevant. In the OPE potential, it is precisely the repulsive short distance OPE component which requires a fine tuning of the solutions and eliminates one *a priori* independent parameter like, e.g., the asymptotic D/S ratio η . As we see, if η is treated as an independent variable the short distance behavior of the deuteron wave functions precludes the definition of a normalizable state due to the onset of the *irregular* solution. This short distance insensitivity at low energies could only be implemented by keeping the experimental η value and *ignoring* OPE physics below some scale. The lower limit established in Ref. [20] to obtain a normalizable state was $a \sim 1.3$ fm for the OPE potential. This obviously requires some extension of the wave function below that scale and the pretended model independence becomes a bit obscured. Our point is that the short distance insensitivity materializes automatically for the *regular* OPE deuteron wave functions since they vanish at the origin.

III. OPE CORRELATIONS IN DEUTERON OBSERVABLES

As we have said, in the OPE potential we can use the deuteron wave number as an input of the calculation on the same footing as $g_{\pi NN}$ and the pion mass m . Then, other observables are predicted. We will study now the dependence of these observables on γ , m and R .

A. Dependence on the binding energy

In Fig. 3 we show the dependence of the D/S ratio as a function of the deuteron wave number γ keeping m and R fixed. In the weak binding limit $\gamma \ll m_\pi$, long distances dominate and the finiteness of the wave function at a point $r \gg 1/m$ requires $\eta \sim \gamma^2$. The radius of convergence of such an expansion for the observables is $|\gamma| < m/2$, since the integrals involve the factor $e^{-(2\gamma+m)r}$ at large distances, diverging for $\gamma < -m/2$. The experimental number is not far from $\gamma = m/3$, which is within the domain of analyticity but somewhat close to the convergence radius. So, one expects a slow convergence. As we see in the weak binding limit we have a quadratic behavior $\eta_{\text{OPE}} \sim \gamma^2$ whereas for stronger binding a linear behavior sets in. It is remarkable that the experimental values in the intermediate regime. On the other hand, in the strong binding case $\gamma \gg m_\pi$, short distances dominate and we must have $\eta \sim 1/\sqrt{2}$. Numerically we find for the deuteron observables,

$$\eta^{\text{OPE}} = 0.9638\gamma^2 - 3.46864\gamma^3 + \mathcal{O}(\gamma^4), \quad (21)$$

$$\frac{A_S^{\text{OPE}}}{\sqrt{2}\gamma} = 1 + 1.2455\gamma - 0.4705\gamma^2 + \mathcal{O}(\gamma^3), \quad (22)$$

$$\sqrt{8}\gamma r_m^{\text{OPE}} = 1 + 1.2455\gamma - 0.4705\gamma^2 + \mathcal{O}(\gamma^3), \quad (23)$$

$$Q_d^{\text{OPE}} = 0.6815 - 3.5437\gamma + \mathcal{O}(\gamma^2). \quad (24)$$

Note that we have the weak binding correlation

$$r_m = \frac{A_S}{4\gamma^{3/2}} + \mathcal{O}(\gamma^3) \quad (25)$$

which is compatible at the 2σ confidence level with data; for the experimental value $A_S = 0.8845(8)$ the value $r_m = 1.984(2)$ to be compared with the experimental number $r_m = 1.971(6)$. In the weak binding limit one also has the correlation

$$\frac{\sqrt{2}\gamma^2 Q_d}{\eta_d} = 1 + \mathcal{O}(\gamma) \quad (26)$$

a dependence that one would expect on general grounds by just taking the asymptotic formulas and neglecting the $w(r)^2$ term in the expression for the quadrupole moment. Experimentally this relation is fulfilled with a 15% accuracy. The nonperturbative OPE value is actually closer to potential models.

B. Comparison with perturbation theory

It is instructive to solve the coupled deuteron equations, Eq. (2) in standard perturbation theory for the fixed energy bound state. One of the reasons is to check the correctness of our nonperturbative calculations in the weak binding regime. Another motivation is to establish contact with the perturbative calculations of Ref. [16] where dimensional regularization in the power divergence subtraction (PDS) scheme was implemented. Finally, there is the question of quantitatively assessing the validity of such an approximation. We relegate the calculation to Appendix A. At first order in perturbation theory one gets in the weak binding limit

$$\begin{aligned} \eta_{\text{pert}} &= 1.5497\gamma^2 - 4.15479\gamma^3 + \mathcal{O}(\gamma^4, R^2), \\ \frac{A_{S,\text{pert}}}{\sqrt{2}\gamma} &= 1 - 0.7184\gamma - 2.7394\gamma^2 + \mathcal{O}(\gamma^3, R^2), \\ r_{m,\text{pert}}\sqrt{8}\gamma &= 1 + 0.71843\gamma - 2.7394\gamma^2 + \mathcal{O}(\gamma^3, R^2), \\ Q_{\text{pert}} &= 1.09587 - 5.87576\gamma + \mathcal{O}(\gamma^2, R^2). \end{aligned} \quad (27)$$

The nominally $\mathcal{O}(R^2)$ second order contributions are in fact divergent because the leading order correction to the D -wave component $w(r)$ diverges at the origin (see Appendix A). In general terms we find that the exact OPE results are estimated within 30% by the first order perturbative calculations of Appendix A.

IV. DEPENDENCE ON THE PION MASS AND CHIRAL LIMIT

Recent works [28–31] predict the change of the deuteron binding energy as a function of the pion mass by taking the experimental binding energy at the physical value of the pion mass and making the additional assumption short distance physics to be independent on the pion mass. While it is true that the leading short distance r dependence of the deuteron wave functions are independent on the pion mass, the constants C_A , C_R and φ do in principle depend on the three independent parameters m , $g_{\pi NN}$ and γ . As we have noted γ cannot be predicted for the OPE potential. So the approach pursued in Refs. [28–31] is equivalent to integrate in with the physical pion mass and then integrate out fixing some combination of short distance constants with the unphysical pion mass and searching for the appropriate regular solution at infinity. If one makes the pion lighter long distance effects should dominate,

and one could just use the OPE potential to estimate the chiral limit as a first approximation. It is thus interesting to analyze the pion mass dependence both explicitly (i.e., varying m in the OPE potential) and implicitly (i.e., taking into account the dependence of the OPE coupling R on the pions mass). We will determine the pertinent combination of short distance constants by demanding the Feynman-Hellmann theorem in the OPE potential.

A. Explicit pion mass dependence

To proceed, let us assume that to an infinitesimal change in $m \rightarrow m + \Delta m$ there corresponds a change both in the deuteron wave number $\gamma \rightarrow \gamma + \Delta\gamma$ and in the coupled channel potential matrix $U(r) \rightarrow U(r) + \Delta U(r)$. We can write a Lagrange identity by varying the equation and its adjoint, yielding for a normalized state

$$-\frac{\partial\gamma^2}{\partial m} = \langle \Psi_m | \frac{\partial U}{\partial m} | \Psi_m \rangle + \left[u' \frac{\partial u}{\partial m} - u \frac{\partial u'}{\partial m} + w' \frac{\partial w}{\partial m} - w \frac{\partial w'}{\partial m} \right] \Bigg|_0^\infty. \quad (28)$$

This is an extended Feynman-Hellmann theorem where the second term in the l.h.s. corresponds to the short distance contribution (the term at infinity vanishes for a bound state). One of the advantages of the Feynman-Hellmann theorem is that one could in principle establish comparison theorems, provided the change in the coupled channel potential matrix, ΔU , is a definite quadratic form. Note also that the derivative with respect to m annihilates the centrifugal term, $6/r^2$, and one can diagonalize the coupled channel potential by the unitary transformation Eq. (9) so that the result behaves additively in the attractive and repulsive eigenchannels. Using the leading short distance behavior, Eq. (14), we therefore get

$$-\frac{\partial\gamma^2}{\partial m} = \int_0^\infty dr \left[u_A(r)^2 \frac{\partial U_A}{\partial m} + u_R(r)^2 \frac{\partial U_R}{\partial m} \right] + C_A^2 \frac{d\varphi}{dm}. \quad (29)$$

As we see, assuming as suggested in Ref. [28] that the short distance physics does not depend on the pion mass corresponds to demanding the standard Feynman-Hellmann theorem where only the OPE potential change contributes. For $C_A \neq 0$ one obtains the condition

$$\frac{d}{dm} \varphi(\gamma, m) = \frac{\partial\varphi}{\partial\gamma} \frac{d\gamma}{dm} + \frac{\partial\varphi}{\partial m} = 0, \quad (30)$$

whence a functional relation between the pion mass and the deuteron binding energy follows. However, note that even in this case the sign of the result is indefinite since

$$\frac{\partial\gamma^2}{\partial m} = \int_0^\infty dr \left[u_A(r)^2 \frac{\partial U_A}{\partial m} + u_R(r)^2 \frac{\partial U_R}{\partial m} \right]. \quad (31)$$

So, we have to determine the sign numerically.

The relation (30) has an equivalent formulation in the boundary condition regularization. For instance, if we assume the same condition BC6 of Eq. (42) for all values of the pion

TABLE II. Deuteron properties for the OPE and their dependence on the pion mass. We use the nonrelativistic relation $\gamma = \sqrt{2\mu_{np}B} = 0.231605 \text{ fm}^{-1}$ with $B = 2.224575(9)$ for $m = 138.03$ and $g_{\pi NN} = 13.1083$. $m = 0$ (explicit) means taking $m = 0$ but using $R = 1.07764 \text{ fm}$ (or equivalently $g_{\pi NN} = 13.1083$). $m = 0$ (implicit) means taking $m = 0$ and using $R_0 = 1.06(2) \times R$.

	$\gamma(\text{fm}^{-1})$	η	$A_S(\text{fm}^{-1/2})$	$r_d(\text{fm})$	$Q_d(\text{fm}^2)$	P_D
$m = 138.03 \text{ MeV}$	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.88(1)%
$m = 0$ (explicit)	0.3240(1)	0.09452	0.8444(1)	1.550(1)	0.3006(3)	10.96(2)%
$m = 0$ (implicit)	0.61(10)	0.15(2)	0.48(7)	0.98(10)	0.15(3)	15(1)%

mass we get

$$\frac{d}{dm} \left[\frac{w'(a)}{u(a)\sqrt{2} + w(a)} \right] = 0, \quad (32)$$

where a is taken to be independent of m . In practice, one computes the ratio within the bracket for the physical pion mass and searches for γ such that the ratio for the unphysical pion mass yields the same numerical value. The dependence on the pion mass can be looked up at Fig. 4. If we take the chiral limit we get $B_d(0, g_{\pi NN}) = 4.3539 \text{ MeV}$. This value is very close to the one found in Ref. [26] $B_d(0, g_{\pi NN}) = 4.2 \text{ MeV}$.⁴ Deuteron properties in the explicit $m = 0$ limit are listed in Table II.

B. Implicit pion mass dependence

To take into account the implicit pion mass dependence we have to take into account the dependence of $R = 3g_A^2 M / 32\pi f_\pi^2$ on the pion mass. In the chiral limit one gets a larger OPE coupling [30]. The value is uncertain and as an educated guess we take $R_0 = 1.06(2)R$. Using the same formulation as in the m dependence, the change in the deuteron binding with respect to the $g_{\pi NN}$ coupling constant or equivalently the scale dimension R we get (assuming as before the short distance angle φ to be independent on m),

$$-R \frac{\partial \gamma^2}{\partial R} = \int_0^\infty dr [u_A(r)^2 U_A + u_R(r)^2 U_R]. \quad (33)$$

Again, the result is indefinite since $U_A < 0$ and $U_R > 0$ and it is not obvious, unlike naive expectations, that a stronger coupling provides stronger binding. The sign depends actually on the details of the wave functions and the particular values of the parameters. Numerically one finds $d\gamma/dR > 0$, a trend that can be understood if the repulsive term in Eq. (33) is neglected or on the basis of the inequality $|u_A| > |u_R|$ which is numerically fulfilled. In any case one has the differential inequality $d\gamma/dR < \gamma/R$.

⁴These authors look for poles of the S -matrix so constructed as to reproduce the physical scattering length $\alpha_0 = 5.42 \text{ fm}$ and effective range $r_0 = 1.75 \text{ fm}$ at the physical value of the pion mass in the 3S_1 eigen channel. By doing so the explicit dependence on $g_{\pi NN}$ becomes rather weak. Actually, they take $g_{\pi NN} = 12.73$ and we would get instead $B_d(0, g_{\pi NN}) = 0.98 \text{ MeV}$ instead. This apparent contradiction is resolved by noting that, as we will see below, for $g_{\pi NN} = 13.1083$ in the OPE we get an scattering length $\alpha_0 = 5.335 \text{ fm}$ an effective range of $r_0 = 1.63$ quite close to the experimental values.

Numerically, we get $\gamma_0 = 0.61(10) \text{ fm}^{-1}$ and hence

$$B_d^0 = 15(5) \text{ MeV}, \quad (34)$$

a value compatible with the analysis of Ref. [30] $B_d^0 = 9.6 \pm 3$ (perhaps with larger errors [31])⁵ but in disagreement with Refs. [26,29] where the deuteron becomes unbound for $m < 90 \text{ MeV}$. In any case we confirm the trend of having a stronger binding of the deuteron in the chiral limit. The corresponding observables can be looked up in Table. II.

V. SCATTERING PROPERTIES IN THE 3S_1 - 3D_1 CHANNEL

A. Orthogonality constraints and phase shifts

For the α and β positive energy scattering states we choose the asymptotic normalization

$$\begin{aligned} u_{k,\alpha}(r) &\rightarrow \frac{\cos \epsilon}{\sin \delta_1} (\hat{j}_0(kr) \cos \delta_1 - \hat{y}_0(kr) \sin \delta_1), \\ w_{k,\alpha}(r) &\rightarrow \frac{\sin \epsilon}{\sin \delta_1} (\hat{j}_2(kr) - \hat{y}_2(kr) \sin \delta_1), \end{aligned} \quad (35)$$

$$\begin{aligned} u_{k,\beta}(r) &\rightarrow -\frac{1}{\sin \delta_1} (\hat{j}_0(kr) \cos \delta_2 - y_0(kr) \sin \delta_2), \\ w_{k,\beta}(r) &\rightarrow \frac{\tan \epsilon}{\sin \delta_1} (\hat{j}_2(kr) \cos \delta_2 - \hat{y}_2(kr) \sin \delta_2), \end{aligned} \quad (36)$$

⁵If we take $R_0 = 1.1R$ we get $B_d^0 = 33 \text{ MeV}$.

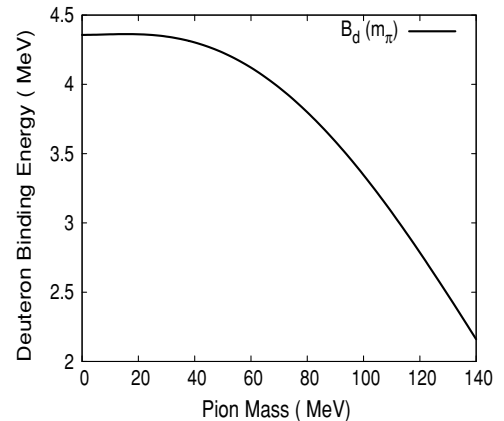


FIG. 4. Explicit pion mass dependence of the deuteron binding energy, keeping $g_{\pi NN} = 13.1083$ fixed. We have $B_d(m) = 2.22457 \text{ MeV}$, $B_d(0) = 4.3539 \text{ MeV}$.

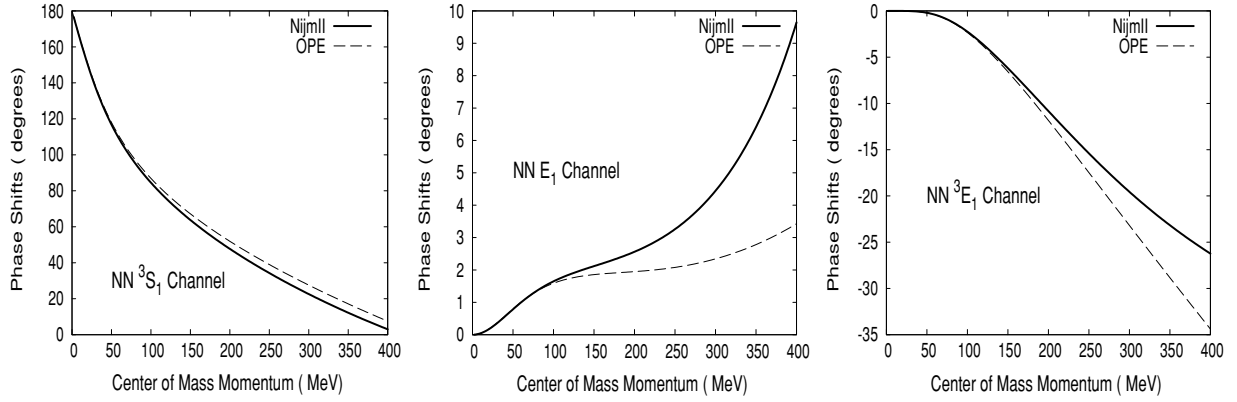


FIG. 5. Eigenphase shifts for the OPE potential as a function of the c.m. np momentum in the triplet 3S_1 - 3D_1 channel compared to the Nijmegen results [40]. The regular scattering wave functions are orthogonal to the regular deuteron bound state wave functions constructed from the OPE with $\gamma = 0.231605 \text{ fm}^{-1}$, $m = 138.03 \text{ MeV}$ and $g_{\pi NN} = 13.1083$.

where $\hat{j}_l(x) = x j_l(x)$ and $\hat{y}_l(x) = x y_l(x)$ are the reduced spherical Bessel functions and δ_1 and δ_2 are the eigenphases in the 3S_1 and 3D_1 channels, and ϵ is the mixing angle E_1 . Again, the general solution at short distances is given by the general Eq. (14), where the constants C_A , C_R , and φ are now different since we have a zero energy state and depend whether we have an α or β state, so we have the short distance constants, $C_{A,\alpha}(k)$, $C_{R,\alpha}(k)$, $\varphi_\alpha(k)$, and $C_{A,\beta}(k)$, $C_{R,\beta}(k)$, $\varphi_\beta(k)$, respectively. This implies certain correlations between δ_1 , δ_2 , and ϵ .

For a regular self-adjoint potential the orthogonality of bound and scattering states comes out automatically. We look now for the consequences of *demanding* this property in the singular OPE potential. Using the standard manipulations to prove orthogonality between states of different energy we get the following relation between α and β states and the bound deuteron state (which we denote by a subscript γ in this section):

$$0 = (\gamma^2 + k^2) \int_0^\infty dr [u_\gamma(r)u_k(r) + w_\gamma(r)w_k(r)] \\ = [u'_\gamma u_k - u_\gamma u'_k + w'_\gamma w_k - w_\gamma w'_k] \Big|_0^\infty. \quad (37)$$

Using the short distance solution, Eq. (14), we get

$$C_{A,i}(k)C_A(\gamma) \sin[\varphi(\gamma) - \varphi_i(k)] = 0, \quad i = \alpha, \beta \quad (38)$$

which yields

$$\varphi(\gamma) = \varphi_\alpha(k) = \varphi_\beta(k). \quad (39)$$

Thus, the short distance phases $\varphi_\alpha(k)$ and $\varphi_\beta(k)$ of the 3S_1 - 3D_1 channel wave functions in the OPE potential at short distances are all determined by deuteron properties. This means, in particular that the low energy parameters and scattering phase shifts are uniquely determined by the deuteron binding energy and the OPE potential parameters.⁶

⁶This property does not hold for other triplet channels with higher partial waves, because there are no bound states in those channels. Nevertheless, it is also true that there is only one independent parameter. This means in practice that one can use one scattering

The previous argument can also be implemented if we have a short distance cut-off at $r = a$, the orthogonality relation of Eq. (37) transforms into the condition

$$u'_\gamma(a)u_{k,i}(a) + w'_\gamma(a)w_{k,i}(a) = u_\gamma(a)u'_{k,i}(a) + w_\gamma(a)w'_{k,i}(a), \quad (40)$$

$$i = \alpha, \beta. \quad (41)$$

Thus, if we impose the same condition on both solutions, Eq. (41) cannot be satisfied unless they are related at the boundary. For instance for the condition BC6 of Eqs. (15), we get the two relations

$$u'_{k,i}(a) = \sqrt{2}w'_{k,i}(a), \quad i = \alpha, \beta \quad (42)$$

The orthogonality relation corresponding to boundary conditions of the form of Eq. (42) implies then the orthogonality constraint

$$\frac{w'_{k,i}(a)}{u_{k,i}(a)\sqrt{2} + w_{k,i}(a)} = \frac{w'_\gamma(a)}{u_\gamma(a)\sqrt{2} + w_\gamma(a)} \quad (43)$$

which is the analog finite cut-off condition of Eq. (39). The remaining conditions in Eqs. (15) generate analogous orthogonality constraints.

The results for the 3S_1 - 3D_1 channel phase shifts using these conditions are presented in Fig. 5. The description is rather satisfactory and it seems to work, as one might expect, up to the vicinity of the c.m. momentum which magnitude coincides with the two pion exchange left cut $k = im$.

B. Low energy parameters

In the low energy limit one has

$$\begin{aligned} \delta_1 &\rightarrow -\alpha_0 k, \\ \delta_2 &\rightarrow -\alpha_2 k^5, \\ \epsilon &\rightarrow \frac{\alpha_{02}}{\alpha_0} k^2 \end{aligned} \quad (44)$$

length out of the three to predict the phase shifts also in other partial waves.

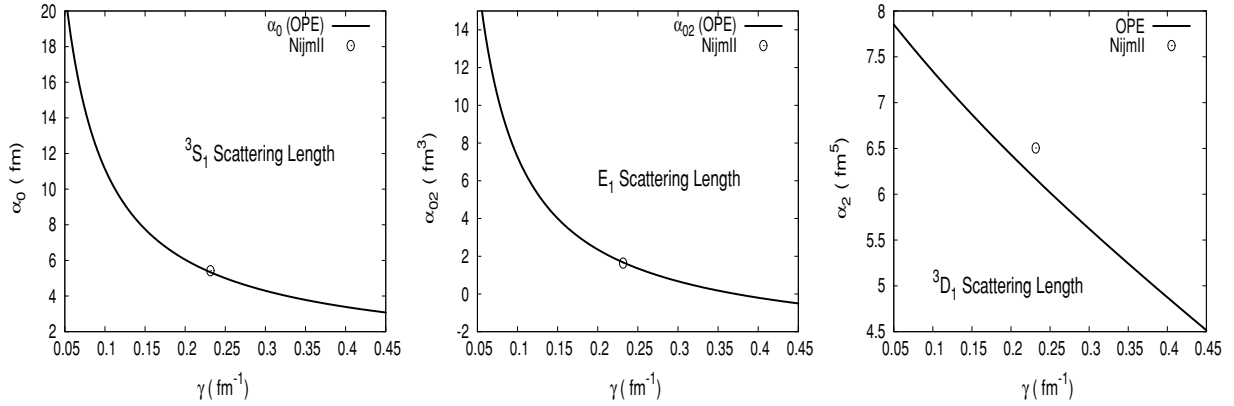


FIG. 6. The dependence of the 3S_1 - 3D_1 scattering lengths α_0 , α_{02} and α_2 on the deuteron wave number γ (in fm^{-1}). The point represents the experimental values or potential model estimates [37,39].

so that the zero energy the wave functions behave asymptotically

$$\begin{aligned}
 u_{0,\alpha}(r) &\rightarrow 1 - \frac{r}{\alpha_0}, \\
 w_{0,\alpha}(r) &\rightarrow \frac{3\alpha_{02}}{\alpha_0 r^2}, \\
 u_{0,\beta}(r) &\rightarrow \frac{r}{\alpha_0}, \\
 w_{0,\beta}(r) &= \frac{3\alpha_2}{\alpha_{02} r^2} - \frac{r^3}{15\alpha_{02}}.
 \end{aligned} \tag{45}$$

Using these zero energy solutions one can determine the effective range. The 3S_1 effective range parameter is given by

$$r_0 = 2 \int_0^\infty \left[\left(1 - \frac{r}{\alpha_0}\right)^2 - u_\alpha(r)^2 - w_\alpha(r)^2 \right] dr. \tag{46}$$

In the zero energy case, the vanishing of the diverging exponentials at the origin imposes a condition on the α and β states which generate a correlation between α_0 , α_{02} , and α_2 . Using the superposition principle of boundary conditions we may write the solutions in such a way that

$$\begin{aligned}
 u_{0,\alpha}(r) &= u_1(r) - \frac{1}{\alpha_0} u_2(r) + \frac{3\alpha_{02}}{\alpha_0} u_3(r), \\
 w_{0,\alpha}(r) &= w_1(r) - \frac{1}{\alpha_0} w_2(r) + \frac{3\alpha_{02}}{\alpha_0} w_3(r), \\
 u_\beta(r) &= \frac{1}{\alpha_0} u_2(r) + \frac{3\alpha_2}{\alpha_{02}} u_3(r) - \frac{1}{15\alpha_{02}} u_4(r), \\
 w_\beta(r) &= \frac{1}{\alpha_0} w_2(r) + \frac{3\alpha_2}{\alpha_{02}} w_3(r) - \frac{1}{15\alpha_{02}} w_4(r),
 \end{aligned}$$

where the functions $u_{1,2,3,4}$ and $w_{1,2,3,4}$ are independent on α_0 , α_{02} , and α_2 and fulfill suitable boundary conditions. As a consequence we get a linear correlation between $1/\alpha_0$, α_{02}/α_0 and also a linear correlation between α_2/α_{02} and $1/\alpha_{02}$. This means in turn that according to the OPE potential both α_{02} and α_2 depends linearly with α_0 . Numerically we get the following

correlations,

$$\begin{aligned}
 \alpha_{02} &= 0.963571370240\alpha_0 - 3.467616391389, \\
 \alpha_2 &= 3.467616391389 \frac{\alpha_{02}}{\alpha_0} + 5.080264230656.
 \end{aligned} \tag{47}$$

These relations are cut-off independent and unique consequences of the OPE potential. On the other hand, the orthogonality between the bound state and the scattering state yields

$$\begin{aligned}
 \alpha_0 &= 1.037805911852\alpha_{02} + 3.598712446758, \\
 \alpha_{02} &= 0.288382561043\alpha_0\alpha_2 - 1.465059639612\alpha_0
 \end{aligned} \tag{48}$$

The provided high accuracy is indeed needed. The four equations, Eqs. (47) and (48), overdetermine the values of the three scattering lengths and could be solved in triplets yielding four different solutions. Actually, there are only two independent solutions which differences are compatible within our numerical uncertainties. The scattering lengths and effective range are presented in Table I and compared to their perturbative value (see Appendix A) and to the high quality Nijmegen potential models [35].⁷ As we see, the agreement with the high quality potentials is at the few percent level. Perturbation theory does not account for most of the contribution to the effective range since the orthogonality constraints preclude a short distance contribution to r_0 and also to the deuteron matter radius r_m . This means in practice that the counterterm named C_2 in Refs. [16,38] must vanish (see Appendix A for a detailed discussion). The dependence of the scattering lengths α_0 , α_{02} and α_2 on the deuteron wavenumber γ can be seen in Fig. 6, where γ dependent generalizations of the correlations, Eqs. (47) and (49) hold.

The phase shifts look very similar to previous work [26] using an energy expansion of a square well potential as a counterterm and adjusting the depth of the two lowest orders

⁷The values of α_0 and r_0 have been determined in Ref. [39], whereas α_{02} and α_2 have been determined by us in Ref. [37]. See also Ref. [41] for a extensive determination in all partial waves.

to reproduce the 3S_1 scattering length α_0 and effective range r_0 as independent parameters or our variable phase approach with non-trivial initial conditions in Ref. [37] where the full coupled channel S -matrix was tailored to reproduce the effective range expansion to any order treating all parameters as independent. Roughly speaking, both approaches could be mapped to an energy dependent boundary condition with no *a priori* orthogonality constraints.⁸ The fact that the orthogonality constrained boundary condition generates the bulk of the low energy threshold parameters with only *one parameter* naturally explains the similarity between the present phase shifts and those in previous works [26] and suggests that there is perhaps no need to make the short distance boundary condition energy dependent if the short distance cut-off is removed.

VI. SHORT DISTANCE SOLUTIONS AND DETERMINATION OF THE COEFFICIENTS

In this section we determine the coefficients of the OPE deuteron wave functions appearing at short distances in Eq. (11). In particular, we compute the energy independent and OPE potential parameters independent short distance phase φ . Let us remind that any choice of φ corresponds to a different choice of short distance physics; given φ and the OPE potential all deuteron and scattering properties are uniquely determined. However, the leading asymptotic form cannot directly be used to match the numerical solution obtained by integrating in the large distance solution. On the one hand, if we use cut-off approaches to determine the regular solution, there are short distance cut-off effects when the distance gets close to the cut-off radius. On the other hand, the fact that the diverging exponential dominates over the converging one provides too weak a signal for the corresponding coefficient. To remedy the situation we improve on the short distance solution to provide a reliable approximation at larger distances (~ 1 fm) where the diverging exponential is less dominant, and look for plateaus in the matching radius. It turns out (see below) that one should go at eight order in this expansion for a robust determination of the short distance coefficients. Actually, one can then directly match the short distance improved wave functions to the numerical solution without no reference to cutoffs. We will try the two methods and see that they yield to compatible results for the short distance coefficients.

In the limit $r \rightarrow 0$ the solutions to the coupled equations can be written in an expansion of the form⁹

$$\begin{aligned} u(r) &= u_0 \left(\frac{r}{R}\right)^{a_1} e^{a_0\sqrt{\frac{R}{r}}} f(r), \\ w(r) &= w_0 \left(\frac{r}{R}\right)^{a_2} e^{a_0\sqrt{\frac{R}{r}}} g(r) \end{aligned} \quad (49)$$

⁸The main difference in this regard has to do with the multi-valuation problem of the potential counter-term in Ref. [26] typical of inverse scattering problems. The approach of Ref. [37] does not have this problem.

⁹This expansion looks similar to a coupled channel WKB expansion although here the mode conversion problem does not show up.

with

$$\begin{aligned} f(r) &= \sum_{n=0}^{\infty} A_n \left(\frac{r}{R}\right)^{n/2}, \\ g(r) &= \sum_{n=0}^{\infty} B_n \left(\frac{r}{R}\right)^{n/2}. \end{aligned} \quad (50)$$

At leading order we get the equations

$$\begin{aligned} u_0 a_0^2 + 16\sqrt{2}w_0 &= 0, \\ 16\sqrt{2}u_0 + (a_0^2 - 16)w_0 &= 0 \end{aligned} \quad (51)$$

which have the four nontrivial solutions,

$$(1A), \quad a_0 = -4i, \quad w_0 = \frac{u_0}{\sqrt{2}}, \quad (52)$$

$$(2A), \quad a_0 = +4i, \quad w_0 = \frac{u_0}{\sqrt{2}}, \quad (53)$$

$$(2R), \quad a_0 = -4\sqrt{2}, \quad w_0 = -\sqrt{2}u_0, \quad (54)$$

$$(1R), \quad a_0 = +4\sqrt{2}, \quad w_0 = -\sqrt{2}u_0. \quad (55)$$

The next to leading order equation becomes compatible only if

$$a_1 = a_2 = 3/4. \quad (56)$$

For any solution in Eq. (52) we may then solve for the remaining coefficients. One peculiar feature of this expansion is that if one wants to determine the solution to a given order, one has to compute the coefficients at a higher order. The reason is that strictly speaking a truncation of the expansion involves also nondiagonal elements, and one has the freedom to choose between solving u or w to a given accuracy. The explicit result to eight order is presented in Appendix B. The general short distance solution is written as a linear combination of the four independent solutions,

$$\begin{aligned} u(r) &= \frac{1}{\sqrt{3}} \left(\frac{r}{R}\right)^{3/4} \left[-C_{1R} f_{1R}(r) e^{+4\sqrt{2}\sqrt{\frac{R}{r}}} \right. \\ &\quad - C_{2R} f_{2R}(r) e^{-4\sqrt{2}\sqrt{\frac{R}{r}}} + \sqrt{2} C_{1A} f_{1A}(r) e^{-4i\sqrt{\frac{R}{r}}} \\ &\quad \left. + \sqrt{2} C_{2A} f_{2A}(r) e^{4i\sqrt{\frac{R}{r}}} \right], \\ w(r) &= \frac{1}{\sqrt{3}} \left(\frac{r}{R}\right)^{3/4} \left[\sqrt{2} C_{1R} g_{1R}(r) e^{+4\sqrt{2}\sqrt{\frac{R}{r}}} \right. \\ &\quad + \sqrt{2} C_{2R} g_{2R}(r) e^{-4\sqrt{2}\sqrt{\frac{R}{r}}} + C_{1A} g_{1A}(r) e^{-4i\sqrt{\frac{R}{r}}} \\ &\quad \left. + C_{2A} g_{2A}(r) e^{4i\sqrt{\frac{R}{r}}} \right]. \end{aligned} \quad (57)$$

This expansion converges rather fast for each solution up to distances of about $r \sim 0.6 - 0.9$ fm. That is about what one needs, since that is sufficiently far above the cut-off radius $a \sim 0.1$ fm. Matching u , w , u' , and w' at some point in this region we get a linear relation between C_{1R} , C_{2R} , C_{1A} , C_{2A} , and η . Actually, we find that the signal of the converging exponential is about hundred to thousand times that of the

diverging exponential in the range between 0.6 and 1 fm.¹⁰ Matching directly the integrated in solution to the short distance solution with a vanishing coefficient of the diverging exponential $C_{1R} = 0$, we get at the scale $0.7 \text{ fm} < r < 0.9 \text{ fm}$

$$\begin{aligned} C_{2R} &= -0.47(1), & \eta &= 0.0263333(1) \\ \bar{C}_{1A} &= 0.1327(3), & \bar{C}_{2A} &= 0.2277(5). \end{aligned} \quad (58)$$

We can instead determine $\eta = 0.0263332$ from the boundary condition BC6 in Eq. (15) at $r = 0.2 \text{ fm}$ and deduce the remaining constants yielding

$$\begin{aligned} |C_{1R}| &< 10^{-7}, & C_{2R} &= -0.47(1) \\ \bar{C}_{1A} &= 0.1327(3), & \bar{C}_{2A} &= 0.2277(5). \end{aligned} \quad (59)$$

The errors have been estimated by varying the matching point in the region $0.7 \text{ fm} < r < 0.9 \text{ fm}$. Note that although the coefficient of the diverging exponential C_{1R} is six orders of magnitude larger than the one of the converging exponential, the solution through the matching condition is about eight orders of magnitude smaller (may even change sign). So that the result provides a sizable signal for the converging exponential. With these values we show in Fig. 7 the short distance wave functions compared to the integrated in numerical ones when the matching is undertaken at $r = 0.8 \text{ fm}$. To improve on the short distance side we have taken $C_{1R} = 0$. The error in the region $0.7 \text{ fm} < r < 0.9 \text{ fm}$ never exceeds a 0.01%. We have checked that setting the constant $C_{2R} = 0$ introduces a larger deviation from the numerical solution as compared to the computed value in the region above 1 fm. Finally, the corresponding short distance angle reads

$$\varphi = -\tan^{-1} \frac{\bar{C}_{2A}}{\bar{C}_{1A}} = -59.7(1)^\circ. \quad (60)$$

The discussion in this section explicitly shows that contrary to the findings in Ref. [11] the coefficient of the converging exponential does not vanish.

VII. CONCLUSIONS AND OUTLOOK

In this paper we have reanalyzed the OPE potential in the triplet 3S_1 - 3D_1 channel both for bound and scattering states.

¹⁰For instance at $r = 0.8 \text{ fm}$ we get

$$\begin{aligned} u &= 1.9683 - 59.4526\eta \\ &= 1.14054\bar{C}_{1A} - 52.3866 C_{1R} + 1.09959\bar{C}_{2A} - 0.00169556 C_{2R}, \\ w &= -4.00531 + 159.28\eta \\ &= -0.667631\bar{C}_{1A} - 551.52C_{1R} + 1.67549\bar{C}_{2A} + 0.0177228C_{2R} \\ u' &= -6.84126 + 287.992\eta \\ &= -1.34949\bar{C}_{1A} + 273.671C_{1R} + 4.02872\bar{C}_{2A} - 0.00925419C_{2R} \\ w' &= 16.2726 - 607.14\eta \\ &= -0.667631\bar{C}_{1A} - 551.52C_{1R} + 1.67549\bar{C}_{2A} + 0.0177228 C_{2R} \end{aligned}$$

where the l.h.s. corresponds to the numerical solution and the r.h.s. to the short distance approximation, and the barred coefficients $\bar{C}_{1A} = (C_{1A} + C_{2A})/2$ and $\bar{C}_{2A} = (-C_{1A} + C_{2A})/2i$ have been introduced.

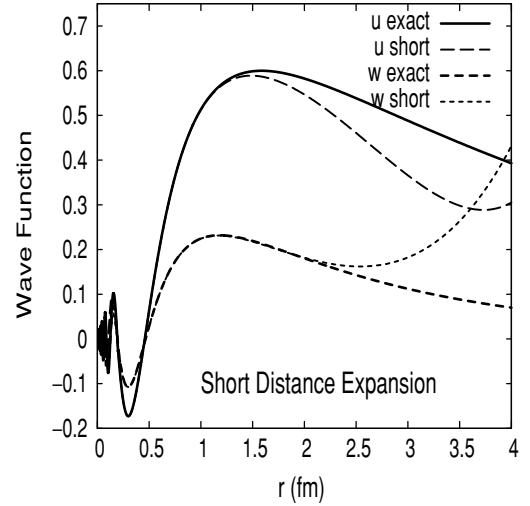


FIG. 7. The short distance expansion for the deuteron wave functions matched to the numerical solution at a distance about $r = 1 \text{ fm}$. We take $C_{1A} = 0.1328$, $C_{2A} = 0.2277$, $C_{2R} = -0.46$, and $C_{1R} = 0$. The numerical solutions are normalized by taking $A_S = 1$.

Rather than modeling the interaction below some finite short distance we have adopted the viewpoint of taking the potential seriously down to the origin. This must be carefully done and in a way as to get rid of any short distance ambiguities. In addition, this procedure proves crucial to be able to disentangle the OPE contribution from other contributions, like TPE and higher, electromagnetic effects and relativistic corrections to deuteron and NN scattering observables.

The OPE coupled channel potential is singular at short distances and additional conditions need to be specified on the wave functions at the origin. Actually, the singular eigenpotentials at short distances are attractive and repulsive and while in the attractive case a mixed boundary condition specifies the corresponding short distance eigenfunction, in the repulsive case one must impose a standard homogeneous boundary condition. This only leaves one free parameter, which we have chosen to be the deuteron binding energy and which cannot be determined from the OPE potential. All remaining deuteron observables come out for free. For the scattering states in the 3S_1 - 3D_1 channel, we have demanded orthogonality constraints between all states of different energy. This condition is actually an additional requirement for singular potentials, since the orthogonality relation carries information on the peculiar short distance behavior of the wave functions, and is not necessarily satisfied. The most obvious example where orthogonality constraints are violated corresponds to energy dependent potentials and energy dependent boundary conditions in coordinate space. A less trivial but significant example is the case of dimensional regularization in the PDS scheme as a perturbative analysis in coordinate space of both bound and scattering states reveals. The power of the orthogonality constraints for singular potentials is that all scattering properties are then predicted from the OPE potential parameters and the deuteron binding energy.

In our analysis it turns out that the short distance form of all wave functions is characterized by some short distance

constants. We have clarified the role played by the exponentially suppressed regular solution by determining its non-vanishing value numerically using short distance expansions to high order, to explore the region below 0.1 fm, not accessible to standard numerical integration methods. Another relevant constant is given by a short distance phase φ which plays the role of a fundamental dimensionless constant in the OPE problem. It does not depend on the energy nor on the OPE parameters, but it is related to the form of the OPE potential in the chiral limit. The closeness of this phase to $\pi/3$ is mysterious and suggestive and requires further investigation.

It is remarkable that indeed the bulk of the experimental results both for the bound state as well the scattering observables are accounted for at the 2–3% level by the OPE potential taken from zero to infinity. We interpret this success as a confirmation on the validity of our choice of regular solutions and the use of orthogonality constraints. The discrepancies can legitimately be attributed to other effects such as TPE, electromagnetic and relativistic corrections. Many of the methods and results obtained in this paper can be generalized in a straightforward manner to take these effects into account and to the study of higher partial waves without any substantial modifications. In particular, the number of independent constants in a given channel depends on the short distance behavior of the long range potential. The bonus of such a program would be the complete elimination of short distance ambiguities in the study of the NN interaction with known long distance forces as determined by chiral symmetry. In our view this an indispensable prerequisite to assess the relevance of chiral symmetry in nuclear physics in a model independent way. The systematic study of these effects will be reported elsewhere [24,25].

ACKNOWLEDGMENTS

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APPENDIX A: PERTURBATIVE SOLUTIONS

1. Bound state

In this appendix we solve the coupled deuteron equations, Eq. (2) in standard perturbation theory for the fixed negative energy bound state. A somewhat related approach looking for the equivalence with the PDS scheme of Ref. [38] in the one-channel positive energy case can be looked up in Ref. [42]. The problem of orthogonality was not discussed. The requirement of normalizability of the deuteron state requires the D wave component to vanish. Thus, at lowest order we have the normalizable solutions

$$\begin{aligned} u_\gamma^{(0)}(r) &= e^{-\gamma r}, \\ w_\gamma^{(0)}(r) &= 0. \end{aligned} \quad (\text{A1})$$

At first order we have to solve the equations

$$\begin{aligned} -u_\gamma^{(1)''}(r) + \gamma^2 u_\gamma^{(1)}(r) &= -U_s(r)e^{-\gamma r}, \\ -w_\gamma^{(1)''}(r) + \left[\frac{6}{r^2} + \gamma^2 \right] w_\gamma^{(1)}(r) &= -U_{sd}(r)e^{-\gamma r}. \end{aligned} \quad (\text{A2})$$

Using the regular and irregular solutions at the origin

$$\begin{aligned} u_{\text{reg}}(r) &= 2 \frac{\sinh(\gamma r)}{\gamma r}, \\ w_{\text{reg}}(r) &= 2 \left(1 + \frac{3}{(\gamma r)^2} \right) \sinh(\gamma r) - \frac{6}{\gamma r} \cosh(\gamma r), \\ u_{\text{irreg}}(r) &= e^{-\gamma r}, \end{aligned} \quad (\text{A3})$$

$$w_{\text{irreg}}(r) = e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \quad (\text{A4})$$

we get

$$u_\gamma^{(1)}(r) = \int_0^\infty G_s(r, r') U_s(r') e^{-\gamma r'} dr', \quad (\text{A5})$$

$$w_\gamma^{(1)}(r) = \int_0^\infty G_d(r, r') U_{sd}(r') e^{-\gamma r'} dr', \quad (\text{A6})$$

where G_s and G_d are the corresponding Green functions. Explicit calculation yields

$$\begin{aligned} u_\gamma^{(1)}(r) &= e^{-\gamma r} \frac{m^2 R \Gamma(0, mr + 2r\gamma) - \Gamma(0, mr)}{3\gamma} - \frac{2m^2 R \text{Ei}(-mr - 2r\gamma) \sinh(r\gamma)}{3\gamma}, \\ w_\gamma^{(1)}(r) &= e^{-r\gamma} \left(1 + \frac{3}{r^2 \gamma^2} + \frac{3}{r\gamma} \right) \left[\frac{m^2 R (3m^2 - 4\gamma^2) \Gamma(0, mr + 2r\gamma) - \Gamma(0, mr)}{6\sqrt{2}\gamma^3} \right. \\ &\quad + \frac{R \left(-6m^3 \gamma + 6m^2 \gamma^2 + 4\gamma^4 + (3m^4 - 4m^2 \gamma^2) \log \left(1 + \frac{2\gamma}{m} \right) \right)}{6\sqrt{2}\gamma^3} \\ &\quad + \frac{e^{-mr-2r\gamma} R (6 + 6mr + m^2 r^2 - m^3 r^3 + 4r\gamma + 4mr^2 \gamma + 2m^2 r^3 \gamma)}{2\sqrt{2}r^4 \gamma^3} \\ &\quad \left. - \frac{R e^{-mr} (6 + 6mr + m^2 r^2 - m^3 r^3 - 8r\gamma - 8mr^2 \gamma + 4r^2 \gamma^2 + 4mr^3 \gamma^2)}{2\sqrt{2}r^4 \gamma^3} \right] \\ &\quad + \left(2 \left(1 + \frac{3}{\gamma^2 r^2} \right) \sinh(\gamma r) - \frac{6}{\gamma r} \cosh(\gamma r) \right) \end{aligned} \quad (\text{A7})$$

$$\times \left[\frac{e^{-mr-2r\gamma} R(6 + 6mr + m^2 r^2 - m^3 r^3 + 4r\gamma + 4mr^2 \gamma + 2m^2 r^3 \gamma)}{2\sqrt{2}r^4 \gamma^3} - \frac{m^2 R(3m^2 - 4\gamma^2)\text{Ei}(-mr - 2r\gamma)}{6\sqrt{2}\gamma^3} \right], \quad (\text{A8})$$

where $\Gamma(0, z)$ and $\text{Ei}(z)$ are the standard incomplete Gamma function and the exponential integral function, respectively,

$$\Gamma(0, z) = \int_z^\infty dt \frac{e^{-t}}{t}, \quad (\text{A9})$$

$$\text{Ei}(z) = -P \int_{-z}^\infty dt \frac{e^{-t}}{t}. \quad (\text{A10})$$

At asymptotically large distances we have

$$u_\gamma^{(1)}(r) \rightarrow c_{\text{pert}} e^{-\gamma r}, \quad (\text{A11})$$

$$w_\gamma^{(1)}(r) \rightarrow \eta_{\text{pert}} e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \quad (\text{A12})$$

where

$$c_{\text{pert}} = \int_0^\infty U_s(r) u_{\text{reg}}(r) e^{-\gamma r} dr, \quad (\text{A13})$$

$$\eta_{\text{pert}} = \int_0^\infty U_{sd}(r) w_{\text{reg}}(r) e^{-\gamma r} dr. \quad (\text{A14})$$

Explicit calculation yields

$$c_{\text{pert}} = \frac{Rm^2}{3\gamma} \log \left(1 + \frac{2\gamma}{m} \right), \quad (\text{A15})$$

$$\begin{aligned} \eta_{\text{pert}} &= \frac{R}{6\sqrt{2}m\gamma^3} \left[4\gamma^4 + 6m^2\gamma^2 - 6m^3\gamma \right. \\ &\quad \left. + (3m^4 - 4m^2\gamma^2) \log \left(1 + \frac{2\gamma}{m} \right) \right] \\ &= \frac{32\sqrt{2}R}{45m} \gamma^2 - \frac{2\sqrt{2}R\gamma^3}{3m^2} + \dots \end{aligned} \quad (\text{A16})$$

The numerical value we get is $\eta_{\text{pert}} = 0.0510$ almost twice the exact OPE result. Taking this perturbative value for η we show in Fig. 8 the perturbative deuteron wave functions as compared to the exact ones.

Unfortunately, if one wants to improve on this first order calculation going to second order perturbation theory there is a problem since the behavior of the perturbative wave functions at short distances is given by

$$u_\gamma^{(1)}(r) = -\frac{Rm^2}{3\gamma} \log \left(1 + \frac{2\gamma}{m} \right) + \dots, \quad (\text{A17})$$

$$w_\gamma^{(1)}(r) = \frac{\sqrt{2}R}{r} - \frac{2}{3}\sqrt{2}R\gamma + \dots \quad (\text{A18})$$

making the wave function non-normalizable, unlike the exact regular wave function. This divergence at short distances

actually precludes going to higher orders in perturbation theory.

The normalization at first order is given by

$$\frac{1}{A_S^2} = \int_0^\infty (e^{-2\gamma r} + 2u_\gamma^{(1)}(r)e^{-\gamma r}) \quad (\text{A19})$$

and hence

$$\begin{aligned} \frac{A_S}{\sqrt{2}\gamma} &= 1 - \frac{2Rm^2}{3(m+2\gamma)} + \frac{Rm^2}{3\gamma} \log \left(1 + \frac{2\gamma}{m} \right) \\ &= 1 - \frac{2R\gamma}{3} - \frac{16R\gamma^2}{9m} + \dots \end{aligned} \quad (\text{A20})$$

The deuteron matter radius is given by

$$r_{m, \text{pert}}^2 = \frac{1}{4} A_S^2 \int_0^\infty r^2 (e^{-2\gamma r} + 2u_\gamma^{(1)}(r)e^{-\gamma r}) \quad (\text{A21})$$

and hence to first order one has

$$r_{m, \text{pert}}^2 = \frac{1}{8\gamma^2} + \frac{m^2 R(3m + 10\gamma)}{18\gamma(m + 2\gamma)^3} + \dots \quad (\text{A22})$$

yielding in the weak binding regime

$$\sqrt{8}\gamma r_{m, \text{pert}} = 1 + \frac{2R\gamma}{3} - \frac{16R\gamma^2}{9m} + \dots \quad (\text{A23})$$

Finally, the quadrupole moment at first order is given by

$$Q_{\text{pert}} = \frac{\sqrt{2}}{10} \int_0^\infty r^2 w_\gamma^{(1)}(r) e^{-\gamma r} dr. \quad (\text{A24})$$

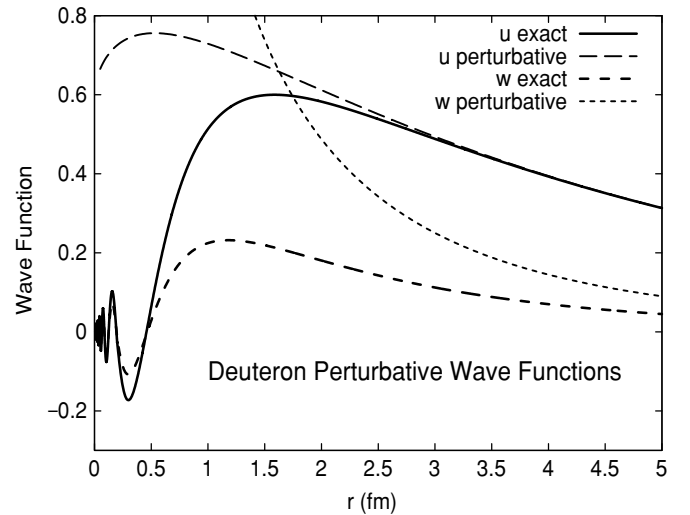


FIG. 8. Perturbative deuteron wave functions compared to the exact ones as a function of the distance (in fm). The exact ones are normalized by taking $A_S = 1$.

The integral can be evaluated to give

$$\begin{aligned} Q_{\text{pert}} &= \frac{8R(4m^2 + 9\gamma m + 6\gamma^2)}{45(m + 2\gamma)^3} \\ &= \frac{32R}{45m} - \frac{8R\gamma}{3m^2} + \frac{128R\gamma^2}{15m^3} + \dots \end{aligned} \quad (\text{A25})$$

yielding $Q_{\text{pert}} = 0.4555 \text{ fm}^2$.

Our perturbative expressions for A_S , r_m , and Q coincide with those of Kaplan, Savage, and Wise [16] provided one takes in their expression for r_m the renormalization scale in the PDS scheme to be $\mu = \gamma$, instead of taking $\mu = m$ as they do or else taking the C_2 counterterm identically equal to zero. Actually, the C_2 counterterm can be mapped into a short distance contribution to the effective range parameter r_0 in the 3S_1 channel. The value of η was not given in that reference but can be deduced from the off-diagonal scattering amplitude in the 3S_1 - 3D_1 channel given in their previous work [38] by evaluating the residue at the deuteron pole. The result also agrees with the calculation presented here.

2. Low energy parameters

To check the identification $C_2 = 0$ further let us compute the S -wave effective range r_0 . For our purposes it is sufficient to analyze the zero energy scattering state. The lowest order solution is given by an α state

$$\begin{aligned} u_{0,\alpha}^{(0)}(r) &= \left(1 - \frac{r}{\alpha_0}\right) \\ w_{0,\alpha}^{(0)}(r) &= 0. \end{aligned} \quad (\text{A26})$$

At zeroth order in the OPE coupling the orthogonality constraint yields

$$\begin{aligned} 0 &= \int_0^\infty u_{0,\alpha}^{(0)}(r) u_\gamma^{(0)}(r) dr \\ &= \frac{1}{\gamma^2} \left[-\frac{1}{\alpha_0} + \gamma \right] \end{aligned} \quad (\text{A27})$$

which yields the scattering length to lowest order,

$$\alpha_0^{(0)} = \frac{1}{\gamma}. \quad (\text{A28})$$

At first order we use the regular solution $u_{\text{reg}}(r) = r$ and the irregular solution $u_{\text{irreg}}(r) = (1 - r/\alpha_0)$ and get similarly to the bound state case the first order correction to the α state,

$$\begin{aligned} u_{0,\alpha}^{(1)}(r) &= \frac{2R}{3\alpha_0} e^{-mr} (1 - \alpha_0 m) - \frac{2m^2 r R}{3} \text{Ei}(-mr) \\ w_{0,\alpha}^{(1)}(r) &= \frac{e^{-mr} R}{15\sqrt{2}\alpha_0 m^2 r^2} \\ &\times (120 - 64\alpha_0 m + 120 m r - 34\alpha_0 m^2 r \\ &+ 40 m^2 r^2 - 2\alpha_0 m^3 r^2 + \alpha_0 m^4 r^3 - \alpha_0 m^5 r^4) \\ &+ \frac{R(-120 + 64\alpha_0 m + \alpha_0 m^6 r^5 \Gamma(0, mr))}{15\sqrt{2}\alpha_0 m^2 r^2}. \end{aligned} \quad (\text{A29})$$

Note that asymptotically the first order correction to the S -wave vanishes exponentially and hence cannot contribute to the scattering length. On the other hand, the orthogonality relation to first order reads

$$0 = \int_0^\infty dr [u_\gamma^{(0)} u_{0,\alpha}^{(0)} + u_\gamma^{(1)} u_{0,\alpha}^{(0)} + u_\gamma^{(1)} u_{0,\alpha}^{(1)}] \quad (\text{A30})$$

and after computing the integrals one gets

$$\begin{aligned} 0 &= -\frac{1}{\alpha_0} + \gamma + \frac{R}{3\alpha_0\gamma} \left[m^2(1 + \alpha_0\gamma) \log\left(1 + \frac{2\gamma}{m}\right) \right. \\ &\quad \left. - 2\gamma(m - \gamma + \alpha_0 m\gamma) \right]. \end{aligned} \quad (\text{A31})$$

Solving perturbatively for α_0 we get at first order

$$\alpha_{0,\text{pert}} = \frac{1}{\gamma} - \frac{2m^2 R}{3\gamma^2} \left[\frac{\gamma(\gamma - 2m)}{m^2} + \log\left(1 + \frac{2\gamma}{m}\right) \right] + \dots \quad (\text{A32})$$

Numerically one gets $\alpha_{0,\text{pert}} = (4.3177 + 0.2912 + \dots) \text{ fm}$ to be compared with the full OPE result $\alpha_0 = 5.34$ and the experimental value $\alpha_0 = 5.42 \text{ fm}$. The E_1 scattering length α_{02} can be read off from the D -wave, using the asymptotic condition in Eq. (45)

$$\begin{aligned} \alpha_{02} &= \frac{4\sqrt{2}R(15 - 8\alpha_0 m)}{45m^2} \\ &= \frac{4\sqrt{2}R(15\gamma - 8m)}{45\gamma m^2} \end{aligned} \quad (\text{A33})$$

in the second line we have substituted the perturbative relation $\alpha_0 = 1/\gamma + \mathcal{O}(R)$. Note the linear correlation $\alpha_{02} = 1.5499\alpha_0 - 4.1530$ to be compared with the exact OPE relation in Eq. (47). The numerical value one gets for the first and second lines taking the experimental values of $\alpha_0 = 5.42$ and γ are $\alpha_{02} = 4.24 \text{ fm}^3$ and $\alpha_{02} = 2.53 \text{ fm}^3$, respectively, to be compared with the experimental $\alpha_{02} = 1.64 \text{ fm}^3$. In the weak binding limit one obtains

$$\gamma \alpha_{0,\text{pert}} = 1 + \frac{2R\gamma}{3} - \frac{16R\gamma^2}{9m} + \dots \quad (\text{A34})$$

In this limit we have the perturbative linear correlation between the scattering length and the deuteron matter radius

$$r_m = \frac{\alpha_0}{2\sqrt{2}} + \mathcal{O}(\gamma^3, R^2) \quad (\text{A35})$$

which yields the value $r_m = 1.92$ for the experimental scattering length $\alpha_0 = 5.42 \text{ fm}$. The linear correlation was established empirically with realistic potentials in Refs. [43,44].

To first order the effective range in the 3S_1 eigenchannel is given by

$$r_0 = -4 \int_0^\infty dr u_{0,\alpha}^{(0)}(r) u_{0,\alpha}^{(1)}(r) \quad (\text{A36})$$

yielding

$$\begin{aligned} r_{0,\text{pert}} &= \frac{4R(3m^2 - 8\gamma m + 6\gamma^2)}{9m^2} \\ &= 1.4369 - 5.4789\gamma + 5.8758\gamma^2 \\ &= 0.4831 \text{ fm}, \end{aligned} \quad (\text{A37})$$

a result much smaller than the full OPE result (1.64 fm) and the experimental number (1.75 fm). Again, our result corresponds to a theory where the short distance contribution to the effective range vanishes, i.e., $C_2 = 0$. A nonvanishing value of C_2 was needed to fit the experimental values of both the matter radius and the effective range. Our calculation shows that the scheme developed in Refs. [16] and [38] does not fulfill perturbatively the orthogonality constraints.

APPENDIX B: SHORT DISTANCE EXPANSION

For the $f(r)$ function we get (we use $x = r/R$)

$$f_{1A} = 1 - \frac{35i}{32}\sqrt{x} - \frac{1811x}{6144} + \frac{2441i}{65536}x^{\frac{3}{2}} - \frac{34805x^2}{8388608} + x^3 \left(\frac{9873675}{17179869184} + \frac{m^2 R^2}{36} - \frac{m^3 R^3}{32} - \frac{3R^2 \gamma^2}{64} \right) + x^{\frac{7}{2}} \left(\frac{193405905i}{549755813888} + \frac{353i}{24192}m^2 R^2 - \frac{709i}{92160}m^3 R^3 + \frac{i}{28}m^4 R^4 - \frac{709i}{61440}R^2 \gamma^2 \right) + x^{\frac{5}{2}} \left(\frac{-333725i}{268435456} - \frac{i}{15}m^3 R^3 - \frac{i}{10}R^2 \gamma^2 \right), \quad (\text{B1})$$

$$f_{2A} = f_{1A}^*, \quad (\text{B2})$$

$$f_{2R} = 1 + \frac{67\sqrt{x}}{32\sqrt{2}} + \frac{7763x}{12288} + \left(\frac{8873}{131072\sqrt{2}} - \frac{m^2 R^2}{3\sqrt{2}} \right) x^{\frac{3}{2}} + \left(-\frac{105845}{33554432} - \frac{55m^2 R^2}{192} \right) x^2 + \left(\frac{881405}{1073741824\sqrt{2}} - \frac{10807m^2 R^2}{184320\sqrt{2}} + \frac{m^3 R^3}{15\sqrt{2}} + \frac{R^2 \gamma^2}{10\sqrt{2}} \right) x^{\frac{5}{2}} + \left(-\frac{23360715}{137438953472} - \frac{332899m^2 R^2}{11796480} + \frac{47m^3 R^3}{960} + \frac{m^4 R^4}{36} + \frac{47R^2 \gamma^2}{640} \right) x^3 + \left(\frac{419268465}{4398046511104\sqrt{2}} + \frac{30559591m^2 R^2}{31708938240\sqrt{2}} + \frac{2141m^3 R^3}{1290240\sqrt{2}} + \frac{229m^4 R^4}{8064\sqrt{2}} + \frac{2141R^2 \gamma^2}{860160\sqrt{2}} \right) x^{\frac{7}{2}}$$

$$f_{1R} = f_{2R}(x \rightarrow e^{2\pi i} x), \quad (\text{B3})$$

and for the $g(r)$ function one has

$$g_{1A} = 1 - \frac{35i}{32}\sqrt{x} - \frac{4883x}{6144} + \frac{82075i}{196608}x^{\frac{3}{2}} + \frac{1245195x^2}{8388608} + \left(\frac{-5136285i}{268435456} - \frac{i}{15}m^3 R^3 - \frac{i}{10}R^2 \gamma^2 \right) x^{\frac{5}{2}} + \left(\frac{42237195}{17179869184} - \frac{m^2 R^2}{18} - \frac{m^3 R^3}{32} - \frac{3R^2 \gamma^2}{64} \right) x^3$$

$$+ \left(\frac{494999505i}{549755813888} - \frac{65i}{12096}m^2 R^2 + \frac{2363i}{92160}m^3 R^3 + \frac{i}{28}m^4 R^4 + \frac{2363i}{61440}R^2 \gamma^2 \right) x^{\frac{7}{2}} + \mathcal{O}(x^4) \quad (\text{B4})$$

$$g_{2A} = g_{1A}^* \quad (\text{B5})$$

$$g_{2R} = 1 + \frac{67\sqrt{x}}{32\sqrt{2}} + \frac{13907x}{12288} + \left(\frac{307195}{393216\sqrt{2}} - \frac{m^2 R^2}{3\sqrt{2}} \right) x^{\frac{3}{2}} + \left(\frac{5075595}{33554432} - \frac{55m^2 R^2}{192} \right) x^2 + \left(\frac{19661565}{1073741824\sqrt{2}} - \frac{41527m^2 R^2}{184320\sqrt{2}} + \frac{m^3 R^3}{15\sqrt{2}} + \frac{R^2 \gamma^2}{10\sqrt{2}} \right) x^{\frac{5}{2}} + \left(-\frac{143137995}{137438953472} - \frac{128033m^2 R^2}{3932160} + \frac{47m^3 R^3}{960} + \frac{m^4 R^4}{36} + \frac{47R^2 \gamma^2}{640} \right) x^3 + \left(\frac{1476620145}{4398046511104\sqrt{2}} - \frac{45736601m^2 R^2}{31708938240\sqrt{2}} + \frac{45149m^3 R^3}{1290240\sqrt{2}} + \frac{229m^4 R^4}{8064\sqrt{2}} + \frac{45149R^2 \gamma^2}{860160\sqrt{2}} \right) x^{\frac{7}{2}} + \mathcal{O}(x^4)$$

$$g_{1R} = g_{2R}(x \rightarrow e^{2\pi i} x). \quad (\text{B6})$$

APPENDIX C: LOCAL DIAGONALIZATION AND PERTURBATIVE MIXING

One of the puzzles one encounters in the description of the deuteron with the OPE potential is that while the dimensionless D/S ratio parameter is rather small at long distances $w/u \rightarrow \eta = 0.0256$, it actually comes from a strong mixing at short distances where $w/u \rightarrow 1/\sqrt{2} \sim 0.707$. Actually, the analog question for scattering states is that there seems to be a natural hierarchy for the phase shifts in the 3S_1 - 3D_1 channel, namely, $\delta_{3S1} \gg \delta_{3D1} \gg \epsilon_1$ even though the threshold behavior of the D -wave is more suppressed than that of the mixing angle. The question is whether one can think of an expansion in terms of the η parameter. There are two obvious situations where the mixing does not occur. One is the absence of tensor force. In the OPE potential that would also eliminate the D -wave. Another situation is dropping the mixing terms in the OPE potential, which is questionable since they are actually larger than the diagonal terms. It is possible, however, to write the equations in a form that the mixing is manifestly small at all distances. To this end we make a local rotation of the deuteron wave functions

$$\begin{pmatrix} u(r) \\ w(r) \end{pmatrix} = \begin{pmatrix} \cos \theta(r) & \sin \theta(r) \\ -\sin \theta(r) & \cos \theta(r) \end{pmatrix} \begin{pmatrix} u_A(r) \\ u_R(r) \end{pmatrix} \quad (\text{C1})$$

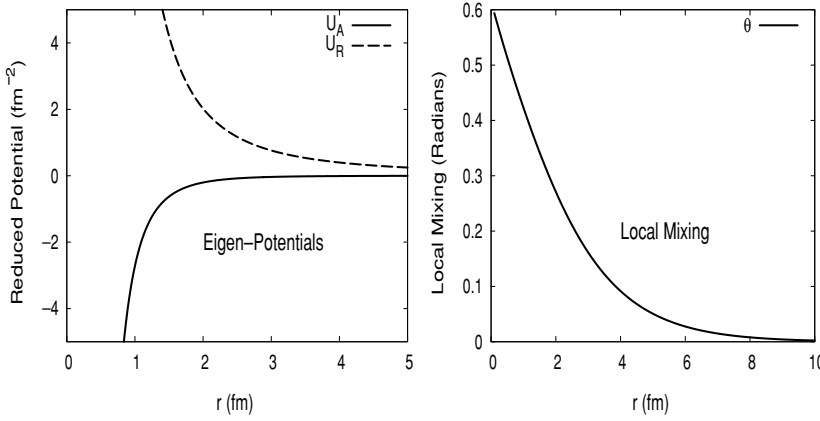


FIG. 9. (Left) Reduced eigenpotentials (in fm^{-2}) $U_A(r)$ and $U_R(r)$ as a function of the distance r (in fm). (Right) Local mixing angle $\theta(r)$ as a function of the distance r (in fm).

in such a way as to diagonalize the potential we have

$$\begin{pmatrix} U_s & U_{sd} \\ U_{sd} & U_d + \frac{6}{r^2} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} U_A & 0 \\ 0 & U_R \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (\text{C2})$$

The deuteron equations for the OPE potentials read after the local rotation

$$\begin{aligned} -u_A''(r) + [U_A(r) + \theta'(r)^2]u_A(r) + \gamma^2 u_A(r) &= [2\theta'(r)u_R'(r) + \theta''(r)u_R(r)], \\ -u_R''(r) + [U_R(r) + \theta'(r)^2]u_R(r) + \gamma^2 u_R(r) &= -[2\theta'(r)u_A'(r) + \theta''(r)u_A(r)]. \end{aligned} \quad (\text{C3})$$

In the coupled channel space these equations can be visualized as a particle with spin in the presence of a gauge potential $\theta'(r)$. At long distances we have the expansions

$$\theta = \frac{2\sqrt{2}R}{9r} e^{-mr} (m^2 r^2 + 3mr + 3) + \dots, \quad (\text{C4})$$

$$U_A = -\frac{2m^2 R}{3r} e^{-mr} + \dots, \quad (\text{C5})$$

$$U_R = \frac{6}{r^2} + \frac{2R}{3r^3} (m^2 r^2 + 6mr + 6) e^{-mr} + \dots, \quad (\text{C6})$$

whereas at short distances we have the behavior

$$\theta = \cos^{-1} \sqrt{\frac{2}{3} - \frac{r}{3\sqrt{2}R} + \frac{r^2}{18\sqrt{2}R^2} - \frac{(18m^2 R^2 - 5)r^3}{324\sqrt{2}R^3} + \dots} \quad (\text{C7})$$

$$U_A = -\frac{4R}{r^3} + \frac{2}{r^2} - \frac{2}{3rR} + \dots \quad (\text{C8})$$

$$U_R = \frac{8R}{r^3} + \frac{4}{r^2} + \frac{2 - 6m^2 R^2}{3rR} + \dots \quad (\text{C9})$$

Note that in the locally rotated basis the mixing is related to the derivative of the mixing angle, θ' which is small at all distances (see Fig. 9). Actually, at asymptotically large distances we have

$$u_A(r) \rightarrow u(r) \quad u_R(r) \rightarrow w(r). \quad (\text{C10})$$

If we neglect the mixing term in Eq. (C3) the equations decouple and, actually, there is no nontrivial solution for the repulsive eigenchannel, since the energy is fixed arbitrarily. Hence in the absence of mixing we have $u_R = 0$. At this level of approximation we then get

$$u(r) = \cos \theta(r) u_A(r), \quad (\text{C11})$$

$$w(r) = \sin \theta(r) u_A(r). \quad (\text{C12})$$

In Fig. 10 we show the solutions of the decoupled equations compared to the exact ones. As we see, the difference in the

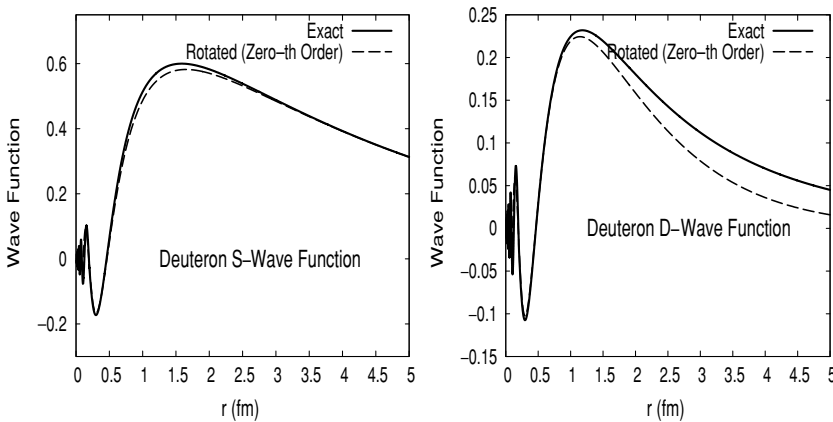


FIG. 10. The rotated eigenfunctions at zeroth order compared to the exact ones. They correspond to take $\eta = 0$.

TABLE III. Deuteron properties for the OPE potential. We use the nonrelativistic relation $\gamma = \sqrt{2\mu_{np}B}$ with $B = 2.224575(9)$. We compare the η -expansion at leading order (LO), with standard perturbation theory at next to leading order (NLO) and the exact OPE result.

	$\gamma(\text{fm}^{-1})$	η	$A_S(\text{fm}^{-1/2})$	$r_m(\text{fm})$	$Q_d(\text{fm}^2)$	P_D
OPE- η (LO)	Input	0	0.8752	1.9423	0.1321	6%
OPE-pert (NLO)	Input	0.051	0.7373	1.6429	0.4555	0
OPE-exact	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.88(1)%
NijmII	Input	0.0253(2)	0.8845(8)	1.968(1)	0.271(1)	5.67(4)%
Exp. (nonrel.)	0.231605	0.0256(4)	0.8846(9)	1.971(6)	0.2859(3)	5.67(4)%

wave functions and hence the D/S mixing is indeed small. Note that this is *not* the same as to neglect the tensor force. The results for the deuteron observables are presented in Table III. As we see, the quality of the zeroth η approximation is rather good.

Actually, we can check *a posteriori* that the mixing is indeed small for the zeroth order solutions. The inhomogeneous term at short distances behaves as

$$2\theta'(r)u'_A(r) \rightarrow -\frac{2}{3\sqrt{2}R^2} \left(\frac{r}{R}\right)^{-3/4} \times C_A \sin(4\sqrt{R/r} + \alpha) \quad (\text{C13})$$

which compared to the remaining terms in Eq. (C3) can indeed be considered small. Under these circumstances, the mixing can then be included perturbatively, yielding

$$u_R(r) = \int_0^\infty dr' G_R(r, r') [2\theta'(r')u'_A(r') + \theta''(r')u_A(r')] \quad (\text{C14})$$

with $G_R(r, r')$ the Green function of the homogeneous equation in the repulsive eigenchannel,

$$G_R(r, r') = w_{\text{reg}}(r)w_{\text{irreg}}(r')\theta(r' - r) + w_{\text{reg}}(r')w_{\text{irreg}}(r)\theta(r - r') \quad (\text{C15})$$

with Wronskian equal to unity and $w_{\text{reg}}(r)$ and $w_{\text{irreg}}(r)$ the regular solution and irregular solutions at the origin, respectively. Asymptotically one has,

$$\begin{aligned} w_{\text{reg}}(r) &\rightarrow C \left(\frac{r}{R}\right)^{3/4} e^{-4\sqrt{2}\sqrt{R/r}} & (r \rightarrow 0) \\ w_{\text{reg}}(r) &\rightarrow e^{+\gamma r} \left(1 - \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right) & (r \rightarrow \infty) \\ w_{\text{irreg}}(r) &\rightarrow C \left(\frac{r}{R}\right)^{3/4} e^{+4\sqrt{2}\sqrt{R/r}} & (r \rightarrow 0) \\ w_{\text{irreg}}(r) &\rightarrow e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right) & (r \rightarrow \infty). \end{aligned} \quad (\text{C16})$$

To get in practice the coefficient C we start with $C = 1$ at short distances and build the ratio to the asymptotic form at a sufficiently large distance. With these conditions the solution $u_R(r)$ at large distances behaves as

$$u_R(r) \rightarrow \eta e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right) \quad (r \rightarrow \infty) \quad (\text{C17})$$

with

$$\eta = \int_0^\infty dr w_{\text{reg}}(r) [2\theta'(r)u'_A(r) + \theta''(r)u_A(r)]. \quad (\text{C18})$$

At short distances we get, from Eq. (C14) and using the asymptotic forms of Eqs. (C16) and (C8), the result

$$u_R(r) \rightarrow CC_A \left(\frac{r}{R}\right)^{7/4} \cos(4\sqrt{R/r} + \alpha) \quad (\text{C19})$$

in agreement with the leading short distance behavior of the full solution for the combination $u(r) - \sqrt{2}w(r)$ (see Sec. VI). The perturbative value for the asymptotic D/S ratio we get is

$$\eta_{\text{pert}} = 0.0261 \quad (\text{C20})$$

quite close to the OPE exact one, $\eta_{\text{OPE}} = 0.0263$.

APPENDIX D: LONG DISTANCE SOLUTIONS

As a complement to the perturbative treatment of Appendix A we analyze the bound solutions at long distances. The asymptotic deuteron wave functions for the OPE potential can be written in the form

$$u(r) = e^{-\gamma r} \left[\sum_k F_k(r) e^{-kmr} \right], \quad (\text{D1})$$

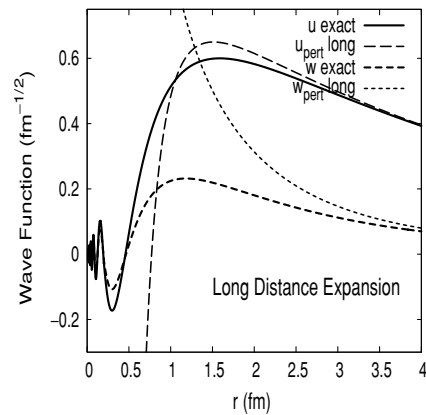


FIG. 11. The long distance expansion for the deuteron wave functions matched. We take $\eta^{\text{OPE}} = 0.2633$, and $A_S = 1$ for the norm of the numerical solutions.

$$w(r) = \eta e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right) \left[\sum_k G_k(r) e^{-kmr} \right]. \quad (\text{D2})$$

The first order solution can be evaluated analytically, yielding

$$F_1(r) = \frac{R e^{-mr} (m^2 r^2 - 2(1 + r\gamma) - 2mr(1 + r\gamma)) \eta}{\sqrt{2} r^3 \gamma^2} + \frac{m^2 R (3\sqrt{2}m^2 \eta + \gamma^2 (4 - 4\sqrt{2}\eta)) \text{Ei}(-mr)}{12\gamma^3} + \frac{e^{2r\gamma} m^2 R (-3\sqrt{2}m^2 \eta + 4\gamma^2 (-1 + \sqrt{2}\eta)) \text{Ei}(-mr - 2r\gamma)}{12\gamma} \quad (\text{D3})$$

$$G_1(r) = \frac{m^2 R (-4\gamma^2 (\sqrt{2} - 2\eta) + 3m^2 (\sqrt{2} - \eta)) \text{Ei}(-mr)}{12\gamma^3 \eta} + \frac{R e^{-mr} (3m^3 r (\sqrt{2} - \eta) + m^2 r\gamma (-3 + r\gamma) (\sqrt{2} - \eta) + 2\gamma^2 (1 + mr) (2\eta + r\gamma (-\sqrt{2} + \eta)))}{2r\gamma^2 (3 + 3r\gamma + r^2 \gamma^2) \eta} + \frac{e^{2r\gamma} m^2 R (3 - 3r\gamma + r^2 \gamma^2) (4\gamma^2 (\sqrt{2} - 2\eta) - 3m^2 (\sqrt{2} - \eta)) \text{Ei}(-mr - 2r\gamma)}{12\gamma^3 (3 + 3r\gamma + r^2 \gamma^2) \eta}. \quad (\text{D4})$$

The second order can also be evaluated but the expression is too long to be presented here. In Fig. 11 we present the first order solutions compared to the exact ones. The perturbative solutions of Appendix A are obtained from the requirement that the S -wave u , becomes normalizable when extended down to the origin. This can only happen

in the D/S asymptotic ratio, η takes the value given by Eq. (A16). This illustrates the fact that perturbation theory can always be applied at long distances but fails at short distances.

Note that here one treats the coupling constant R and the mixing parameter η as independent variables.

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