Three-body problem at finite temperature and density

A. N. Kvinikhidze[1,2] and B. Blankleider[2]

[1]*A. Razmadze Mathematical Institute, Georgian Academy of Sciences, Aleksidze Str. 1, Tbilisi 0193, Georgia*
[2]*Department of Physics, Flinders University, Bedford Park, South Australia 5042, Australia*

We derive practical three-body equations for the equal-time three-body Green's function in matter. Our equations describe both bosons and fermions at finite density and temperature and take into account all possible two-body subprocesses allowed by the underlying Hamiltonian.

I. INTRODUCTION

Three-body correlations play an important role in describing many aspects of many-body systems. In early studies of nuclear matter, in which the main focus of interest was its binding energy, three-body correlations were found to play a small but significant part [1–4]. More recently, it has been the study of in-matter three-body systems themselves that has been of main interest. Indeed, the in-matter three-body problem plays an important role in describing a large variety of interesting phenomena in many-body systems. For example, for understanding the formation of bound states in heavy-ion collisions, three-body calculations are needed for studying the modification of the binding energy and wave function of a three-nucleon bound state because of nuclear matter of finite density and temperature [5,6]. Similarly, studies of the binding energy of three quarks are of relevance to the understanding of color superconductivity and phase transitions in quark matter [7,8]. Three-body calculations are also needed to describe nonequilibrium processes of cluster formation in an interacting many-body system [9,10] and play a fundamental role in determining the two-particle–one-hole (pph) and two-hole-one-particle (hhp) contributions to the self-energy of the single-particle propagator [11].

Our goal in this paper is to formulate three-dimensional (3D) equations for the finite-temperature in-matter three-body problem that take into account all possible two-body subprocesses allowed by the underlying Hamiltonian. To put this goal into context, it is worthwhile to briefly review the progress made so far on this subject. From the very beginning, it was recognized that Faddeev's approach [12] provided a powerful tool in the description of few-body properties in quantum mechanics. It is therefore natural that, not long after its formulation, this approach was also applied to quantum field theory, first within a four-dimensional (4D) formulation [13] and then within a 3D one that are obtains by equating times in 4D Green's functions [14]. These early formulations were for three particles in vacuum. With the application of quantum-field-theoretical (QFT) methods to statistical physics [15,16], it became possible to apply Fadeev's approach also to the field-theoretic description of three particles within a many-body environment. However, one major obstacle in formulating a practical description in this way is the hole contribution to the single-particle propagator in the form of an advanced part [see, for example, Eq. (4)], which is not

present in the quantum-mechanical description of particles in vacuum. The presence of this hole contribution makes the field-theoretic description inherently 4D, even in the nonrelativistic case. The first steps in applying the Faddeev approach to the many-body environment avoided this problem by use of a Bethe-Goldstone type of modification of the Faddeev equations, which involves of a simple momentum cutoff that restricts the intermediate-state particles to be above the Fermi surface [1]. Although such modified equations can be treated with the Faddeev method, they do not take into account the hole contributions that reside in the advanced parts of the single-particle propagators. The way beyond this approximation was proposed by Schuck, Villars, and Ring [17], who used equal-time Green's functions to obtain a 3D field-theoretic description. To derive their equation for the zero-temperature equal-time three-body wave function, they approximated the effective pair-interaction kernels by terms linear in the physical two-body potentials. Because the exact expression for the effective pair-interaction kernel involves an infinite series of higher-order terms as well [see Eq. (48)], the linear approximation cannot be considered as satisfactory for the strong coupling case, e.g., when two-body bound states are possible. The current state-of-the-art formulation [18], which has been used extensively for calculations [5–10,18], can be considered as the model of Ref. [17] extended to finite temperatures, with the extension being performed with the imaginary-time formalism of perturbation theory [16].

In this context, one goal in this paper is to formulate practical field-theoretic three-body equations, valid at finite temperature and density, that take into account the *whole* of the just-mentioned series for the effective pair interaction. As in Refs. [17] and [18], here we use equal-time Green's functions to formulate a 3D field-theoretic description. We show that Faddeev's idea, which renders the three-body kernel compact, namely, to reexpress the three-body equations in terms of two-body *t* matrices rather than two-body potentials, also enables one to sum up *exactly* the infinite series of Eq. (48) for the pair-interaction kernel.

II. IN-MATTER FOUR-DIMENSIONAL THREE-BODY EQUATIONS

The interactions of three identical particles at finite density and temperature are described in quantum field theory by the

Green's function $\mathscr{G}$, defined by

$$(2\pi)^4\delta^4(p_1' + p_2' + p_3' - p_1 - p_2 - p_3)\mathscr{G}(p_1'p_2'p_3'; p_1p_2p_3)$$
$$= \int d^4y_1 d^4y_2 d^4y_3 d^4x_1 d^4x_2 d^4x_3$$
$$\times e^{i(p_1'\cdot y_1 + p_2'\cdot y_2 + p_3'\cdot y_3 - p_1\cdot x_1 - p_2\cdot x_2 - p_3\cdot x_3)}$$
$$\times \mathrm{Tr}\{\rho\,\mathscr{T}[\Psi(y_1)\Psi(y_2)\Psi(y_3)\Psi^\dagger(x_3)\Psi^\dagger(x_2)\Psi^\dagger(x_1)]\}, \quad (1)$$

where Ψ and $\Psi^\dagger$ are Heisenberg fields with respect to the Hamiltonian $K = H - \mu N$, $\mathscr{T}$ is the time-ordering operator, and

$$\rho = \frac{e^{-\beta K}}{\mathrm{Tr}\,e^{-\beta K}} \quad (2)$$

is the statistical operator of the grand canonical ensemble [16]. Besides being the central quantity for the description of three-body observables, this Green's function is also needed to calculate the vacuum properties of the system with the help of the dressed single-particle propagator; for example, in the four-point interaction model, the single-particle self-energy diagram is completely defined by the pph Green's function [19].

In the zero-temperature case ($\beta = 1/k_B T \to \infty$) only the ground state survives in the trace of Eq. (1), and the Green's function reduces to the usual QFT expectation value:

$$(2\pi)^4\delta^4(p_1' + p_2' + p_3' - p_1 - p_2 - p_3)\mathscr{G}(p_1'p_2'p_3'; p_1p_2p_3)$$
$$= \int d^4y_1 d^4y_2 d^4y_3 d^4x_1 d^4x_2 d^4x_3$$
$$\times e^{i(p_1'\cdot y_1 + p_2'\cdot y_2 + p_3'\cdot y_3 - p_1\cdot x_1 - p_2\cdot x_2 - p_3\cdot x_3)}\langle 0|\mathscr{T}\Psi(y_1)\Psi(y_2)$$
$$\times \Psi(y_3)\Psi^\dagger(x_3)\Psi^\dagger(x_2)\Psi^\dagger(x_1)|0\rangle, \quad (3)$$

where $|0\rangle$ is the physical ground state. For the latter, straightforward use of Wick's theorem gives a perturbation theory with Feynman rules. Analogously, two types of perturbation theory, so-called "imaginary-time" and "real-time," have been derived for Eq. (1) [20].

Here we use the real-time formulation of perturbation theory in which the number of degrees of freedom is doubled [21]; this complication, with respect to the zero-temperature case, comes from the sum over the complete set of states (trace) in Eq. (1).[1] For example, the free one-body propagator, in the nonrelativistic case, is given by [22]

$$d^f(p) = i\left[\frac{\bar{n}(\mathbf{p})}{p^0 - \omega + i\epsilon} + \frac{n(\mathbf{p})}{p^0 - \omega - i\epsilon}\right] \quad (4)$$

where $\omega = \omega_\mathbf{p} = \mathbf{p}^2/2m - \mu$ (we take $\hbar = 1$), μ is the chemical potential, and n, $\bar{n}$ are 2×2 matrices whose elements are simple functions of the distribution function

$$f(\omega) = \frac{1}{e^{\beta\omega} \pm 1}, \quad (5)$$

with the upper sign $(+)$ for fermions and the lower sign $(-)$ for bosons (see the appendix). Correspondingly, the elementary vertices have an extra double-valued index for each particle

leg; e.g., the four-point interaction $\bar{v}$ used to define the Hamiltonian of Eq. (17) enters the formalism with doubled degrees of freedom through the quantity $\bar{\mathbf{v}}$, whose matrix structure is given as $\bar{\mathbf{v}}_{ijkl} = (\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} - \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2})\bar{v}$. Note that the first diagonal element is just the potential itself, $\bar{\mathbf{v}}_{1111} = \bar{v}$. Similarly, the first diagonal element of Eq. (4), d_{11}^f, for which

$$n_{11}(\mathbf{p}) = 1 - \bar{n}_{11}(\mathbf{p}) = \pm\frac{1}{e^{\beta\omega} \pm 1} = \pm f(\omega), \quad (6)$$

corresponds to the usual one-body free Green's function at zero temperature [16]. In some cases one can use a simplified propagator consisting of just this d_{11}^f element of Eq. (4) [23].

For the identical particle case considered here, the field-theoretic expression of Eq. (1) automatically guarantees the appropriate symmetry of the three-particle Green's function $\mathscr{G}$. Moreover, in the doubled-degrees-of-freedom formalism, the matrix Green's function G, whose first diagonal element is $\mathscr{G}$, is likewise properly symmetric in the case of bosons and antisymmetric in the case of fermions. On the other hand, the disconnected Green's function G_0, defined by

$$G_0(p_1'p_2'p_3', p_1p_2p_3) = d(p_1)d(p_2)d(p_3)(2\pi)^4$$
$$\times \delta^4(p_2' - p_2)(2\pi)^4\delta^4(p_3' - p_3), \quad (7)$$

where $d(p_i)$ is the dressed propagator of particle i, does not possess identical particle symmetry; thus G_0 is not equal to the fully disconnected part of G (which we denote by G_d). Indeed, it can be easily shown that, to obtain G_d, one need only symmetrize (or antisymmetrize) G_0 according to

$$\sum_P G_0(1'2'3', 123) \equiv G_0^P(1'2'3', 123) = G_d(1'2'3', 123),$$
$$(8)$$

where the sum is over all permutations P of either the initial or final state particle labels, and for fermions is understood to include a factor of $(-1)^P = +1$ or -1 depending on whether the permutation is even or odd, respectively. In Eq. (8) we use symbolic notation in which integers represent the momenta plus all quantum numbers of the corresponding particles, with primes distinguishing the final states.

Defining the kernel K to be the set of all possible three-particle irreducible Feynman diagrams for the $3 \to 3$ process, we may write the equation for the Green's function G as [24]

$$G = G_0^P + \frac{1}{3!}G_0 K G, \quad (9)$$

where the $1/3!$ factor reflects the fact that both G and K are fully symmetric or antisymmetric in their particle labels. The disconnected part of K, indicated by subscript d, can be expressed in terms of the identical particle two-body potential v as

$$K_d(1'2'3', 123) = \sum_{L_c R_c} v(2'3', 23)d^{-1}(1)\delta(1', 1), \quad (10)$$

where $\delta(1', 1)$ represents the momentum conserving Dirac δ function $(2\pi)^4\delta^4(p_1' - p_1)$, and L_c and R_c indicate that sums are taken over cyclic permutations of the left-hand labels $(1'2'3')$ and right-hand labels (123), respectively (note that the sums are restricted to cyclic permutations because the potential v is already properly symmetric or antisymmetric in its labels).

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[1]A similar discussion on the basis of the imaginary-time formalism and its comparison with this note is presented elsewhere.

Defining

$$V_i(1'2'3, 123) = v(j'k', jk)d^{-1}(i)\delta(i', i), \qquad (11)$$

where (ijk) is a cyclic permutation of (123), we have that

$$K_d = \sum_{P_c}(V_1 + V_2 + V_3), \qquad (12)$$

where it makes no difference over which labels, left hand or right hand, the cyclic permutations are taken. Denoting the connected part of the kernel by K_c, we define the $3 \rightarrow 3$ potential v by

$$V = \tfrac{1}{2}(V_1 + V_2 + V_3) + \tfrac{1}{6}K_c. \qquad (13)$$

Although v is not fully symmetric or antisymmetric, it does have the useful symmetry property

$$P_{ij}V P_{ij} = V, \qquad (14)$$

where P_{ij} is the operator that exchanges the i'th and i'th momentum, spin, and isospin labels. Because

$$K = \sum_P V, \qquad (15)$$

Eq. (9) can be written as

$$G = G_0^P + G_0 V G. \qquad (16)$$

Formally, Eq. (16) differs from the equivalent relation for distinguishable particles, $G = G_0 + G_0 V G$, only in the explicit symmetrization of the inhomogeneous term.

For the sake of simplicity, we consider the model given by the second quantized Hamiltonian

$$H = \sum_1 \omega_1 a_1^\dagger a_1 + \sum_{1234} \bar{v}(1234)a_1^\dagger a_2^\dagger a_4 a_3, \qquad (17)$$

where $\omega_1 = \mathbf{q}_1^2/2m$ is the single-particle nonrelativistic kinetic energy and $\bar{v}(1234) = \bar{v}(s_1\mathbf{q}_1, s_2\mathbf{q}_2, s_3\mathbf{q}_3, s_4\mathbf{q}_4)$ is a Gallilei invariant function of spins s_i and 3D momenta $\mathbf{q}_i$ of particles 1, 2, 3, and 4. Then the first diagonal element of the pair-interaction potential v of Eq. (11), v_{1111}, is not $\bar{v}$ in the general case of finite density and temperature, as the presence of the medium makes it possible for particle-hole pairs to be exchanged in the t- channel. As a result of these exchanges, $v = v(s_1q_1, s_2q_2, s_3q_3, s_4q_4)$ also depends on the energy variables $q_1^0, q_2^0, q_3^0, q_4^0$ (of which only three are independent because $q_1^0 + q_2^0 = q_3^0 + q_4^0$). In the case of zero density and temperature, $\mu = 0$, $T = 0$, we obtain pure quantum mechanics with particle-number conservation and $v_{1111} = \bar{v}$. In this case, single-particle propagators do not have an advanced part, $n = 0$, and this fact allows us to derive the standard 3D Faddeev equations for the three-particle system in which we obtain input two-body t matrices (in three-body space) from physical two-body t matrices (in two-body space) simply by subtracting the spectator energy from the total energy. In the next two sections we discuss how 3D three-body equations can also be derived for the case of finite temperature and density.

III. THREE-DIMENSIONAL EQUAL-TIME REDUCTION

Being a 4D integral equation, Eq. (16) is not very convenient for practical calculations: It involves integrations over relative times (or relative energies) that, because of the presence of cuts and singularities in all four quadrants of the integration plane, cannot be easily handled numerically. For this reason, we would like to implement a 3D reduction of this equation. To do this we follow the current literature and effect this reduction by equating times in initial states and separately in final states. A quantity A with such equated times is denoted by $i\langle A\rangle$ (an extra factor of i has been included for later convenience). Thus our central quantity is the two-time Green's function $\langle\mathscr{G}\rangle$, which we obtain from the 4D Green's function $\mathscr{G}$ by equating times, as just described. In momentum space, $\langle\mathscr{G}\rangle$ is therefore given by

$$(2\pi)^3\delta^3(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3' - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)i\langle\mathscr{G}\rangle$$
$$\times (E, \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = \int d^3y_1\, d^3y_2\, d^3y_3\, d^3x_1\, d^3x_2$$
$$\times d^3x_3 dt e^{i(Et - \mathbf{p}_1'\cdot\mathbf{y}_1 - \mathbf{p}_2'\cdot\mathbf{y}_2 - \mathbf{p}_3'\cdot\mathbf{y}_3 + \mathbf{p}_1\cdot\mathbf{x}_1 + \mathbf{p}_2\cdot\mathbf{x}_2 + \mathbf{p}_3\cdot\mathbf{x}_3)}$$
$$\times \text{Tr}\{\rho\mathscr{T}[\Psi(t, \mathbf{y}_1)\Psi(t, \mathbf{y}_2)\Psi(t, \mathbf{y}_3)\Psi^\dagger(0, \mathbf{x}_3)$$
$$\times \Psi^\dagger(0, \mathbf{x}_2)\Psi^\dagger(0, \mathbf{x}_1)]\}. \qquad (18)$$

In the doubled-degrees formalism, the two-time Green's function of Eq. (18) is the first diagonal element of the matrix Green's function $\langle G\rangle$. Our goal in this section is to develop the 3D integral equation, analogous to Eq. (16), for $\langle G\rangle$. Previously, such 3D equations for the two-time Green's function have been considered in the context of (zero-density) relativistic quantum field theory in Ref. [25] for the two-particle case and in Ref. [14] for the three-particle case. We base our derivation of the in-matter three-body equation on that of Ref. [14]. By contrast, other recent many-body formulations of three-body equations have been closely related to the work of Ref. [17].

It can be shown that one can obtain the momentum-space two-time Green's function of Eq. (18) directly from the 4D Green's function of Eq. (1) by integrating out all the relative energies:

$$i\langle\mathscr{G}\rangle(E, \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = \int \frac{dp_1'^0}{2\pi}\frac{dp_2'^0}{2\pi}\frac{dp_3'^0}{2\pi}\frac{dp_1^0}{2\pi}\frac{dp_2^0}{2\pi}\frac{dp_3^0}{2\pi}$$
$$\times \mathscr{G}(p_1'p_2'p_3', p_1p_2p_3)(2\pi)^2\delta(p_1'^0 + p_2'^0 + p_3'^0 - E)$$
$$\times \delta(p_1^0 + p_2^0 + p_3^0 - E). \qquad (19)$$

Correspondingly, in the doubled-degrees-of-freedom formalism, the equal-time matrix Green function $\langle G\rangle$ is related to its 4D counterpart G by

$$i\langle G\rangle(E, \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = \int \frac{dp_1'^0}{2\pi}\frac{dp_2'^0}{2\pi}\frac{dp_3'^0}{2\pi}\frac{dp_1^0}{2\pi}\frac{dp_2^0}{2\pi}\frac{dp_3^0}{2\pi}$$
$$\times G(p_1'p_2'p_3', p_1p_2p_3)(2\pi)^2\delta(p_1'^0 + p_2'^0 + p_3'^0 - E)$$
$$\times \delta(p_1^0 + p_2^0 + p_3^0 - E). \qquad (20)$$

Thus the 4D free three-body Green function, defined as

$$G_0^f(p_1'p_2'p_3', p_1p_2p_3) = d^f(p_1)d^f(p_2)d^f(p_3)(2\pi)^8$$
$$\times \delta^4(p_2' - p_2)\,\delta^4(p_3' - p_3), \qquad (21)$$

where d^f is given by Eq. (4), leads to the following two-time free three-body Green's function:

$$\langle G_0^f\rangle(E, \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$$
$$= \langle G_0^f\rangle(E, \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)(2\pi)^6\delta^3(\mathbf{p}_2' - \mathbf{p}_2)\,\delta^3(\mathbf{p}_3' - \mathbf{p}_3), \quad (22)$$

where[2]

$$\langle G_0^f\rangle(E, \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = -i\int\frac{dp_2^0}{2\pi}\frac{dp_3^0}{2\pi}d^f\big(E - p_2^0 - p_3^0, \mathbf{p}_1\big)$$
$$\times d^f\big(p_2^0, \mathbf{p}_2\big)d^f\big(p_3^0, \mathbf{p}_3\big)$$
$$= \frac{\bar{n}(\mathbf{p}_1)\bar{n}(\mathbf{p}_2)\bar{n}(\mathbf{p}_3)}{E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}_3} + i\epsilon}$$
$$+ \frac{n(\mathbf{p}_1)n(\mathbf{p}_2)n(\mathbf{p}_3)}{E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}_3} - i\epsilon}. \quad (23)$$

In a similar way, we can define the Green's function $\mathscr{D}$ of two identical particles at finite density and temperature as

$$(2\pi)^4\delta^4(p_1' + p_2' - p_1 - p_2)\mathscr{D}(p_1'p_2'; p_1p_2)$$
$$= \int d^4y_1 d^4y_2 d^4x_1 d^4x_2 e^{i(p_1'\cdot y_1 + p_2'\cdot y_2 - p_1\cdot x_1 - p_2\cdot x_2)}$$
$$\times \mathrm{Tr}\{\rho\mathscr{T}[\Psi(y_1)\Psi(y_2)\Psi^\dagger(x_2)\Psi^\dagger(x_1)]\}, \quad (24)$$

from which the two-time Green's function $\langle\mathscr{D}\rangle$ follows:

$$i\langle\mathscr{D}\rangle(E, \mathbf{p}_1'\mathbf{p}_2', \mathbf{p}_1\mathbf{p}_2) = \int\frac{dp_1'^0}{2\pi}\frac{dp_2'^0}{2\pi}\frac{dp_1^0}{2\pi}\frac{dp_2^0}{2\pi}$$
$$\times \mathscr{D}(p_1'p_2', p_1p_2)(2\pi)^2\delta\big(p_1'^0 + p_2'^0 - E\big)$$
$$\times \delta\big(p_1^0 + p_2^0 - E\big). \quad (25)$$

In the doubled-degrees-of-freedom formalism, a similar equation relates the two-body matrix Green's function D to its two-time version $\langle D\rangle$. Applying this relation to the free two-body Green's function

$$D_0^f(p_1'p_2', p_1p_2) = d^f(p_1)d^f(p_2)(2\pi)^4\delta^4(p_2' - p_2), \quad (26)$$

one obtains the two-time free two-body Green's function:

$$\langle D_0^f\rangle(E, \mathbf{p}_1'\mathbf{p}_2', \mathbf{p}_1\mathbf{p}_2) = \langle D_0^f\rangle(E, \mathbf{p}_1\mathbf{p}_2)(2\pi)^3\delta^3(\mathbf{p}_2' - \mathbf{p}_2),$$
$$(27)$$

where

$$\langle D_0^f\rangle(E, \mathbf{p}_1\mathbf{p}_2) = -i\int\frac{dp_2^0}{2\pi}d^f\big(E - p_2^0, \mathbf{p}_1\big)d^f\big(p_2^0, \mathbf{p}_2\big)$$
$$= \frac{\bar{n}(\mathbf{p}_1)\bar{n}(\mathbf{p}_2)}{E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} + i\epsilon} - \frac{n(\mathbf{p}_1)n(\mathbf{p}_2)}{E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - i\epsilon}.$$
$$(28)$$

It also convenient to express the one-body propagator as

$$\langle d^f\rangle(E, \mathbf{p}) = -i d^f(p)|_{p^0=E} = \frac{\bar{n}(\mathbf{p})}{E - \omega_{\mathbf{p}} + i\epsilon} + \frac{n(\mathbf{p})}{E - \omega_{\mathbf{p}} - i\epsilon}.$$
$$(29)$$

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[2]To save on notation, we use the same symbol with differing numbers of arguments to represent disconnected Green's functions with and without the momentum-conserving δ functions.

A. Three-body equal-time quasipotential

One can write a 3D equation for the two-time Green's function $\langle G\rangle$ of the same form as 4D Eq. (16):

$$\langle G\rangle = \langle G_0\rangle^P + \langle G_0\rangle\tilde{V}\langle G\rangle, \quad (30)$$

where $\tilde{V}$ is the 3D "quasi potential." As is well known [25], $\tilde{V}$ should be expressible in terms of a perturbation series in the 4D potential v; however, this series involves the inverse of the disconnected Green's function $\langle G_0\rangle$, which in the many-body case is a problem, as this inverse may not exist. To see this, it is sufficient to consider the free propagator of Eq. (23) for the case of fermions at $T = 0$ where real-time perturbation theory is expressed in single degrees of freedom. In this case the functions n and $\bar{n}$ reduce to

$$n(\mathbf{p}) = \theta(p_F - |\mathbf{p}|), \quad \bar{n}(\mathbf{p}) = 1 - n(\mathbf{p}) = \theta(|\mathbf{p}| - p_F),$$
$$(31)$$

where $p_F = \mu(0)$ is the Fermi momentum. It is clear that n and $\bar{n}$ are projection operators, and this in turn means that $\langle G_0^f\rangle$ is also a projection operator. Thus $\langle G_0^f\rangle$ does not project onto the full space of momenta, but only onto the subspace projected by the operator:

$$\mathscr{N} = n(\mathbf{p}_1)n(\mathbf{p}_2)n(\mathbf{p}_3) + \bar{n}(\mathbf{p}_1)\bar{n}(\mathbf{p}_2)\bar{n}(\mathbf{p}_3); \quad (32)$$

as a result, $\langle G_0^f\rangle$ cannot be inverted in the full space of momenta. Exactly the same conclusion is reached in the case of finite temperature in the formalism with doubled degrees of freedom, as n and $\bar{n}$ remain projection operators, but now as matrices. More specifically (see the appendix), n and $\bar{n}$ satisfy the following relations that define their projection properties: $n + \bar{n} = g$, $ng\bar{n} = \bar{n}gn = 0$, $ngn = n$, and $\bar{n}g\bar{n} = \bar{n}$, where g is defined as

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for fermions}, \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for bosons}.$$
$$(33)$$

To get around the noninvertability of $\langle G_0\rangle$, we introduce a modified Green's function,

$$\tilde{G}_0 = \langle G_0\rangle + (ggg - \mathscr{N})\Delta, \quad (34)$$

which for nonzero Δ is not singular and can be inverted. In Eq. (34), ggg is a direct product of g's in which one g is in the space of particle 1, another is in the space of particle 2, and the third is in the space of particle 3. It is important to note that $\tilde{G}_0$ is identical to $\langle G_0\rangle$ in the subspace projected by $\mathscr{N}$, i.e., $\mathscr{N}ggg\tilde{G}_0 = \mathscr{N}ggg\langle G_0\rangle = \langle G_0\rangle$. The operator Δ is required to be fully disconnected, so that

$$\tilde{G}_0(E, \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$$
$$= \tilde{G}_0(E, \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)(2\pi)^6\delta^3(\mathbf{p}_2' - \mathbf{p}_2)\,\delta^3(\mathbf{p}_3' - \mathbf{p}_3), \quad (35)$$

but can otherwise be chosen according to one's own convenience (later we will also be able to let Δ go safely to zero). For the free case, one can write the inverse of $\tilde{G}_0^f$ explicitly:

$$\big(\tilde{G}_0^f\big)^{-1}(E, \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = ggg\big[\mathscr{N}\big(E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}_3}\big)$$
$$+ (ggg - \mathscr{N})\Delta^{-1}\big]ggg. \quad (36)$$

As a consequence of introducing $\tilde{G}_0$, we likewise introduce a modified full 3D Green's function $\tilde{G}$ defined as

$$\tilde{G} = \langle G \rangle + (ggg - \mathscr{N})\Delta^P \qquad (37)$$

and redefine the 3D quasi potential $\tilde{V}$ to satisfy the equation

$$\tilde{G} = \tilde{G}_0^P + \tilde{G}_0 \tilde{V} \tilde{G}. \qquad (38)$$

As it stands, Eq. (16) is not convenient to work with as the inhomogeneous term contains an explicit sum over the permutations of G_0's labels. For this reason we define the unsymmetrized Green's function G^u as the solution of

$$G^u = G_0 + G_0 V G^u. \qquad (39)$$

There we obtain the full Green's function G by summing over the permutations of the right-hand labels of G^u, which we symbolically write as $G = G^u P$. Similarly, in the 3D case we define

$$\tilde{G}^u = \tilde{G}_0 + \tilde{G}_0 \tilde{V} \tilde{G}^u, \qquad (40)$$

where $\tilde{G} = \tilde{G}^u P$. It follows that

$$\tilde{G}^u = \langle G^u \rangle + (ggg - \mathscr{N})\Delta. \qquad (41)$$

Iterating Eq. (39), equating initial and final times, and using Eqs. (34) and (41), we obtain

$$\tilde{G}^u = \tilde{G}_0 + \langle G_0 V G_0 \rangle + \langle G_0 V G_0 V G_0 \rangle + \cdots +, \qquad (42)$$

where in momentum space the angle brackets indicate the integration of relative energies as in Eq. (20). Because the inverse $\tilde{G}_0^{-1}$ exists by construction,

$$\tilde{G}_0^{-1} \tilde{G}^u = 1 + \tilde{G}_0^{-1}\langle G_0 V G_0 \rangle + \tilde{G}_0^{-1}\langle G_0 V G_0 V G_0 \rangle + \cdots +, \qquad (43)$$

so that

$$(\tilde{G}^u)^{-1}\tilde{G}_0 = 1 - \tilde{G}_0^{-1}\langle G_0 V G_0 \rangle - \tilde{G}_0^{-1}\langle G_0 V G_0 V G_0 \rangle + \tilde{G}_0^{-1}\langle G_0 V G_0 \rangle \tilde{G}_0^{-1}\langle G_0 V G_0 \rangle - \cdots -. \qquad (44)$$

Using this in Eq. (40) gives an explicit perturbation series for the quasi potential:

$$\tilde{V} = \tilde{G}_0^{-1}[\langle G_0 V G_0 \rangle + \langle G_0 V G_0 V G_0 \rangle - \langle G_0 V G_0 \rangle \tilde{G}_0^{-1}\langle G_0 V G_0 \rangle + \cdots +]\tilde{G}_0^{-1}. \qquad (45)$$

In the present case of a three-particle system, v consists of a sum of pair interactions and three-body forces, as given in Eq. (13). The quasi potential $\tilde{V}$ must be expressible similarly as

$$\tilde{V} = \tfrac{1}{2}(\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3) + \tfrac{1}{6}\tilde{K}_c, \qquad (46)$$

where $\tilde{V}_i$ is a pair interaction with particle i as spectator, and $\tilde{K}_c$ is a three-particle irreducible connected term (the three-body force). We can thus write $\tilde{V}_i$ as

$$\tilde{V}_i(\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = (2\pi)^3\delta^3(\mathbf{p}_i' - \mathbf{p}_i)\tilde{V}_i(\mathbf{p}_j'\mathbf{p}_k', \mathbf{p}_j\mathbf{p}_k; \mathbf{p}_i), \qquad (47)$$

where, to save on notation, we use the same symbol $\tilde{V}_i$, but with different arguments, to denote the pair interaction with and without the spectator δ function. Note that, in general, the

$\tilde{V}_i$ without the δ function still depends on the momentum $\mathbf{p}_i$ of the spectator. It follows from Eq. (45) that

$$\frac{\tilde{V}_i}{2} = \tilde{G}_0^{-1}\left[\left\langle G_0 \frac{V_i}{2} G_0 \right\rangle + \left\langle G_0 \frac{V_i}{2} G_0 \frac{V_i}{2} G_0 \right\rangle - \left\langle G_0 \frac{V_i}{2} G_0 \right\rangle \tilde{G}_0^{-1}\left\langle G_0 \frac{V_i}{2} G_0 \right\rangle + \cdots +\right]\tilde{G}_0^{-1}. \qquad (48)$$

We are interested in the usual case in which the three-body force $\tilde{K}_c$ is neglected. Then $\tilde{V}_i$ provides the sole interaction that describes three-body observables. However, what enters the three-body Faddeev equations is the pair-interaction t matrix in three-body space, $\tilde{T}_i$, defined in terms of the quasi potential $\tilde{V}_i$ by the Lippmann-Schwinger equation

$$\tilde{T}_i = \tilde{V}_i + \tfrac{1}{2}\tilde{V}_i \tilde{G}_0 \tilde{T}_i. \qquad (49)$$

In correspondence with Eq. (47), we write the disconnectedness structure of $\tilde{T}_i$ as

$$\tilde{T}_i(\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3', \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = (2\pi)^3\delta^3(\mathbf{p}_i' - \mathbf{p}_i)\tilde{T}_i(\mathbf{p}_j'\mathbf{p}_k', \mathbf{p}_j\mathbf{p}_k; \mathbf{p}_i). \qquad (50)$$

The task of constructing $\tilde{T}_i$ appears to be formidable. Just to determine $\tilde{V}_i$ by summing the infinite series of Eq. (48) seems already a practical impossibility. That is why most, if not all, works on this subject keep only the linear term in the input two-body interaction [17]. In our case, this would mean keeping only the first term of the series in Eq. (48). However, as we show subsequently, there is another way of solving this problem that gives the exact t matrix $\tilde{T}_i$, namely the one that results from a complete summation of Eq. (48) followed by an exact solution of the Lippmann-Schwinger equation, Eq. (49).

We finish this subsection by pointing out that the pair-interaction part of the three-body quasi-potential $\tilde{V}_i$ (without the spectator δ function) is related in a most nontrivial way to the corresponding two-body quasi-potential $\tilde{v}_i$ arising from equating times in the two-body Green's function. Indeed, if we write the two-body equivalent of Eq. (16),

$$D = D_0^P + \tfrac{1}{2}D_0 v D, \qquad (51)$$

where D is the two-body Green's function originating from Eq. (24), D_0 is the two-body disconnected Green's function given by

$$D_0(p_1'p_2', p_1p_2) = d(p_1)d(p_2)(2\pi)^4\delta^4(p_2' - p_2), \qquad (52)$$

and v is the fully symmetric (or antisymmetric) two-body kernel, then we can determine the two-body quasi potential $\tilde{v}$ by following the same procedure as the preceding one to determine $\tilde{V}$. Clearly, we shall obtain the following perturbation series for $\tilde{v}$:

$$\frac{\tilde{v}}{2} = \tilde{D}_0^{-1}\left[\left\langle D_0 \frac{v}{2} D_0 \right\rangle + \left\langle D_0 \frac{v}{2} D_0 \frac{v}{2} D_0 \right\rangle - \left\langle D_0 \frac{v}{2} D_0 \right\rangle \tilde{D}_0^{-1}\left\langle D_0 \frac{v}{2} D_0 \right\rangle + \cdots +\right]\tilde{D}_0^{-1}, \qquad (53)$$

where

$$\tilde{D}_0 = \langle D_0 \rangle + (gg - \mathscr{N}^{(2)})\Delta^{(2)}, \qquad (54)$$

with

$$\mathscr{N}^{(2)} = n(\mathbf{p}_1)n(\mathbf{p}_2) + \bar{n}(\mathbf{p}_1)\bar{n}(\mathbf{p}_2), \qquad (55)$$

and $\Delta^{(2)}$ is any convenient disconnected function that makes $\tilde{D}_0$ invertible. It is easy to see that, in general, $\tilde{V}_i$ will be related to $\tilde{v}_i$ in a complicated and nonlinear way through an infinite series.

B. Exact three-body equal-time disconnected t matrix

As the determination of the exact quasi potential $\tilde{V}_i$ is difficult, as discussed in the preceding subsection it would seem that the exact determination of the corresponding t matrix $\tilde{T}_i$, needed as input to the 3D Faddeev equations, is not attainable, at least not through the iteration of the exact $\tilde{V}_i$. Yet it often happens that working with t matrices *directly* is much simpler than working with the underlying potential—one example being the problem of disconnectedness in the three-body problem, which Faddeev solved by formulating equations in terms of two-body t matrices rather than two-body potentials.

Thus, rather than expressing $\tilde{V}_i$ in terms of V_i, as we have done above in Eq. (48), we now attempt to express the exact $\tilde{T}_i$ directly in terms of the corresponding 4D disconnected t matrix T_i. We start by noting that even though $\tilde{V}_i$ may not be known, one can nevertheless formally write the exact $\tilde{T}_i$ as the solution of the Lippmann-Schwinger equation, Eq. (49). Thus, if $\tilde{G}_i^u$ is the disconnected part of $\tilde{G}^u$ with particle i as spectator, so that, by Eq. (40),

$$\tilde{G}_i^u = \tilde{G}_0 + \tfrac{1}{2}\tilde{G}_0\tilde{V}_i\tilde{G}_i^u, \tag{56}$$

it follows that

$$\tilde{G}_i^u = \tilde{G}_0 + \tfrac{1}{2}\tilde{G}_0\tilde{T}_i\tilde{G}_0. \tag{57}$$

Defining the equal-time three-body t matrix $\tilde{T}$ through

$$\tilde{G}^u = \tilde{G}_0 + \tilde{G}_0\tilde{T}\tilde{G}_0, \tag{58}$$

one can clearly write

$$\tilde{T} = \tfrac{1}{2}(\tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3) + \tilde{T}_c, \tag{59}$$

where $\tilde{T}_c$ is the part of $\tilde{T}$ that is fully connected.

In the 4D sector one can similarly express the unsymmetrized Green's function G^u in terms of the three-body t matrix T as

$$G^u = G_0 + G_0 T G_0, \tag{60}$$

where

$$T = V + V G_0 T. \tag{61}$$

By analogy with Eqs. (11) and (13), we write

$$T = \tfrac{1}{2}(T_1 + T_2 + T_3) + T_c \tag{62}$$

where

$$T_i(1'2'3', 123) = t(j'k', jk)d^{-1}(i)\delta(i', i) \tag{63}$$

is the disconnnected part of T with particle i as spectator and $t(j'k', jk) \equiv t_i$ is the 4D two-body t matrix, and T_c is the connected part. It follows that

$$T_i = V_i + \tfrac{1}{2}V_i G_0 T_i. \tag{64}$$

The disconnected part of G^u, with particle i as spectator, is given by

$$G_i^u = G_0 + \tfrac{1}{2}G_0 T_i G_0. \tag{65}$$

when the equal-time operation is performed on Eq. (65),

$$\langle G_i^u \rangle = \langle G_0 \rangle + \tfrac{1}{2}\langle G_0 T_i G_0 \rangle, \tag{66}$$

it follows from Eqs. (34) and (41) that

$$\tilde{G}_i^u = \tilde{G}_0 + \tfrac{1}{2}\langle G_0 T_i G_0 \rangle. \tag{67}$$

Comparing this equation with Eq. (57), we obtain

$$\tilde{G}_0\tilde{T}_i\tilde{G}_0 = \langle G_0 T_i G_0 \rangle. \tag{68}$$

It is seen that, in contrast to the quasi potential $\tilde{V}_i$, which is related to the 4D potential V_i in a very complicated way, the t matrix $\tilde{T}_i$ corresponding to the quasi potential, defined by the *exact* solution of Eq. (49), is connected to the 4D t matrix T_i in a very simple way.

Indeed, using Eqs. (20) and (63), one can write the right-hand side of Eq. (68) as

$$
\begin{aligned}
\langle G_0 T_i G_0 \rangle = &-i\int \frac{dp_1'^0}{2\pi}\frac{dp_2'^0}{2\pi}\frac{dp_3'^0}{2\pi}\frac{dp_1^0}{2\pi}\frac{dp_2^0}{2\pi}\frac{dp_3^0}{2\pi}[D_{0i}t_i D_{0i}]\\
&\times (p_j'p_k', p_j p_k)(2\pi)^4\delta^4(p_i' - p_i)d_i(p_i)(2\pi)^2\\
&\times \delta\big(p_1'^0 + p_2'^0 + p_3'^0 - E\big)\delta\big(p_1^0 + p_2^0 + p_3^0 - E\big)\\
= &-i(2\pi)^3\delta^3(\mathbf{p}_i' - \mathbf{p}_i)\int \frac{dp_j'^0}{2\pi}\frac{dp_k'^0}{2\pi}\frac{dp_i^0}{2\pi}\frac{dp_j^0}{2\pi}\\
&\times \frac{dp_k^0}{2\pi}[D_{0i}t_i D_{0i}](p_j'p_k', p_j p_k)d_i(p_i)(2\pi)^2\\
&\times \delta\big(p_i^0 + p_j'^0 + p_k'^0 - E\big)\delta\big(p_i^0 + p_j^0 + p_k^0 - E\big)\\
= &\,(2\pi)^3\delta^3(\mathbf{p}_i' - \mathbf{p}_i)\int \frac{dp_i^0}{2\pi}i\langle D_{0i}t_i D_{0i}\rangle\\
&\times \big(E - p_i^0, \mathbf{p}_j'\mathbf{p}_k', \mathbf{p}_j\mathbf{p}_k\big)\langle d_i\rangle\big(p_i^0, \mathbf{p}_i\big), \tag{69}
\end{aligned}
$$

where

$$
\begin{aligned}
&[D_{0i}t_i D_{0i}](p_j'p_k', p_j p_k)\\
&\qquad = d(p_j')d(p_k')t(p_j'p_k', p_j p_k)\,d(p_j)d(p_k). \tag{70}
\end{aligned}
$$

The result of Eq. (68) can thus be written as

$$\tilde{G}_0\tilde{T}_i\tilde{G}_0 = (2\pi)^3\delta^3(\mathbf{p}_i' - \mathbf{p}_i)\langle D_{0i}t_i D_{0i}\rangle \otimes \langle d_i \rangle, \tag{71}$$

or, without the spectator δ function, as

$$\tilde{G}_0\tilde{T}_i\tilde{G}_0 = \langle D_{0i}t_i D_{0i}\rangle \otimes \langle d_i \rangle \tag{72}$$

where the symbol $\otimes$ denotes the convolution integral:

$$a \otimes b(E) \equiv \frac{i}{2\pi}\int_{-\infty}^{\infty} a(E - z)b(z)\,dz. \tag{73}$$

The preceding analysis of three-body Green's functions can be repeated for two-body Green's functions, thereby yielding the two-body version of Eq. (68):

$$\tilde{D}_0\tilde{t}\tilde{D}_0 = \langle D_0 t D_0 \rangle, \tag{74}$$

where

$$\tilde{t} = \tilde{v} + \tfrac{1}{2}\tilde{v}\tilde{D}_0\tilde{t}. \tag{75}$$

Using this in Eq. (72), we obtain the essential result

$$\tilde{G}_0 \tilde{T}_i \tilde{G}_0 = \tilde{D}_{0i} \tilde{t}_i \tilde{D}_{0i} \otimes \langle d_i \rangle, \tag{76}$$

which expresses the exact t matrix $\tilde{T}_i$, forming the "spectator" plus interacting pair" input to the three-body Faddeev equations, in terms of a convolution of the spectator propagator and the subsystem equal-time two-body t matrix.

IV. IN-MATTER THREE-DIMENSIONAL THREE-BODY EQUATIONS

A. General description

The unsymmetrized three-body equal-time Green's function $\tilde{G}^u$ is given in terms of the three-body t matrix $\tilde{T}$ by Eq. (58). If the three-body force $\tilde{K}_c$ of Eq. (46) is neglected, one can express $\tilde{T}$ in the Faddeev form,

$$\tilde{T} = \sum_{i=1}^{3} X_i, \tag{77}$$

where

$$X_i = \frac{1}{2} \tilde{T}_i + \frac{1}{2} \tilde{T}_i \sum_{k \neq i} \tilde{G}_0 X_k. \tag{78}$$

Alternatively, one can express $\tilde{G}^u$ in terms of Alt-Grassberger-Sandhas (AGS) amplitudes U_{ij} [26],

$$\tilde{G}^u = \tilde{G}_i \delta_{ij} + \tilde{G}_i U_{ij} \tilde{G}_j, \tag{79}$$

where the U_{ij} satisfy the AGS equations:

$$U_{ij} = \tilde{G}_0^{-1} \delta_{ij} + \frac{1}{2} \sum_k \delta_{ik} \tilde{T}_k \tilde{G}_0 U_{kj}. \tag{80}$$

In either case, the input consists of the disconnected amplitudes $\tilde{T}_i$ that are specified in terms of equal-time two-body t matrices $\tilde{t}_i$ according to Eq. (76).

The preceding equations constitute our general formulation of the finite-temperature equal-time in-matter three-body problem. What is noteworthy is that the neglect of the three-body force $\tilde{K}_c$ is the only approximation made; in particular, the 4D two-body potential V_i of Eq. (11), which is specified by the underlying Hamiltonian, is included exactly and to all orders within the equal-time approach. Although the preceding equations can be used directly for calculations, in the next subsection we show that, for the case of instantaneous potentials and effective single-particle dressings, they can be greatly simplified.

B. Description for instantaneous potentials and freelike dressed propagators

Here we consider the commonly used approximations in which the 4D two-body potential v is assumed to be instantaneous and in which single-particle dressings are taken into account only through effective masses and effective chemical potentials.

Thus we assume that the dressed propagator d has exactly the same structure as that of the free propagator d^f, given in

Eq. (4), but with a modified (effective) mass m^* and a modified chemical potential μ^*. From Eq. (29) this means we can write for particle i,

$$\langle d_i \rangle = \bar{n}_i d_i^r + n_i d_i^a \tag{81}$$

$$\equiv \langle d_i^r \rangle + \langle d_i^a \rangle, \tag{82}$$

where

$$d_i^{r,a} = \frac{1}{E - \omega_i \pm i\epsilon}, \tag{83}$$

d_i^r being specified with $+i\epsilon$ and d_i^a with $-i\epsilon$, $\bar{n}_i = \bar{n}(\mathbf{p}_i)$, $n_i = n(\mathbf{p}_i)$, and $\omega_i = \mathbf{p}_i^2/2m^* - \mu^*$. The disconnected equal-time two-body propagator $\langle D_0 \rangle$ will then take the same form as that of $\langle D_0^f \rangle$ given in Eq. (28), and we similarly write

$$\langle D_{0i} \rangle = \bar{n}_j \bar{n}_k D_i^r - n_j n_k D_i^a \tag{84}$$

$$\equiv \langle D_{0i}^r \rangle + \langle D_{0i}^a \rangle, \tag{85}$$

where

$$D_i^{r,a} = \frac{1}{E - \omega_j - \omega_k \pm i\epsilon}. \tag{86}$$

The disconnected equal-time three-body propagator $\langle G_0 \rangle$ can likewise be written as

$$\langle G_0 \rangle = \bar{n}_i \bar{n}_j \bar{n}_k G^r + n_i n_j n_k G^a \tag{87}$$

$$\equiv \langle G_0^r \rangle + \langle G_0^a \rangle, \tag{88}$$

where

$$G^{r,a} = \frac{1}{E - \omega_i - \omega_j - \omega_k \pm i\epsilon}. \tag{89}$$

At the same time, the assumption of an instantaneous two-body potential means, in momentum space, that potential v, and therefore the corresponding t matrix t, do not depend of zero components of relative momenta. Thus $\langle D_0 t D_0 \rangle = \langle D_0 \rangle it \langle D_0 \rangle$, so that Eq. (74) becomes

$$\tilde{D}_0 \tilde{t} \tilde{D}_0 = \langle D_0 \rangle it \langle D_0 \rangle. \tag{90}$$

Now, because the disconnected equal-time two-body propagator $\langle D_0 \rangle$ has the form specified by Eq. (85), it follows that $\langle D_0 \rangle = \langle D_0 \rangle gg \mathcal{N}^{(2)}$, and therefore $\langle D_0 \rangle = \tilde{D}_0 gg \mathcal{N}^{(2)}$. Applying this to Eq. (90), one obtains the useful identity

$$\langle D_0 \rangle \tilde{t} \langle D_0 \rangle = \langle D_0 \rangle it \langle D_0 \rangle = \tilde{D}_0 \tilde{t} \tilde{D}_0. \tag{91}$$

Interestingly, the quasi potential $\tilde{v}$ and the instantaneous potential v obey a similar equation,

$$\langle D_0 \rangle \tilde{v} \langle D_0 \rangle = \langle D_0 \rangle iv \langle D_0 \rangle = \tilde{D}_0 \tilde{v} \tilde{D}_0. \tag{92}$$

Indeed, for the case in which v is instantaneous and $\langle D_0 \rangle$ is specified by Eqs. (85) and (74) holds also for potentials, i.e.,

$$\tilde{D}_0 \tilde{v} \tilde{D}_0 = \langle D_0 v D_0 \rangle. \tag{93}$$

To see this, consider the second-order term in v of Eq. (53) under the assumption of instantaneous v. Up to a constant

factor, we have that

$$\langle D_0 v D_0 v D_0 \rangle - \langle D_0 v D_0 \rangle \tilde{D}_0^{-1} \langle D_0 v D_0 \rangle$$

$$= \langle D_0 \rangle i v \langle D_0 \rangle i v \langle D_0 \rangle - \langle D_0 \rangle i v \langle D_0 \rangle \tilde{D}_0^{-1} \langle D_0 \rangle i v \langle D_0 \rangle$$

$$= \langle D_0 \rangle i v \langle D_0 \rangle \left(1 - \tilde{D}_0^{-1} \langle D_0 \rangle \right) i v \langle D_0 \rangle. \qquad (94)$$

Then, taking $\langle D_0 \rangle$ to be given as in Eq. (85), so that $\langle D_0 \rangle = \langle D_0 \rangle g g \mathcal{N}^{(2)} = \tilde{D}_0 g g \mathcal{N}^{(2)}$,

$$\langle D_0 v D_0 v D_0 \rangle - \langle D_0 v D_0 \rangle \tilde{D}_0^{-1} \langle D_0 v D_0 \rangle$$

$$= \langle D_0 \rangle i v \langle D_0 \rangle \left(g g \mathcal{N}^{(2)} - \tilde{D}_0^{-1} \langle D_0 \rangle \right) i v \langle D_0 \rangle \qquad (95)$$

$$= \langle D_0 \rangle i v \langle D_0 \rangle \tilde{D}_0^{-1} \left(\tilde{D}_0 g g \mathcal{N}^{(2)} - \langle D_0 \rangle \right) i v \langle D_0 \rangle = 0. \quad (96)$$

In a similar way, all higher-order contributions in v are zero in Eq. (53), and the result of Eq. (93) follows.

Applying Eq. (91) to Eq. (76), one obtains

$$\tilde{G}_0 \tilde{T}_i \tilde{G}_0 = \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i \rangle, \qquad (97)$$

which is to be used as the input to the equal-time Faddeev equations previously discussed.

1. Split form of equations

Under the assumptions of this subsection, the disconnected two- and three-particle dressed propagators, $\langle D_0 \rangle$ and $\langle G_0 \rangle$, have the same projection properties as the corresponding free propagators. These projection properties lead to a substantial simplification of the in-matter three-body equations. We show that, for instantaneous potentials and the freelike dressed propagators of Eqs. (86) and (89), the in-matter Faddeev equation, Eq. (78), can be split into two equations, one involving only retarded parts of propagators, the other involving only the advanced parts.

In view of Eq. (97), we begin by defining the quantities $\tilde{T}_i^R$ and $\tilde{T}_i^A$ by[3]

$$\tilde{G}_0 \tilde{T}_i^R \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^r \rangle, \quad (98a)$$

$$\tilde{G}_0 \tilde{T}_i^A \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^a \rangle, \quad (98b)$$

so that

$$\tilde{T}_i = \tilde{T}_i^R + \tilde{T}_i^A. \qquad (99)$$

We can now check that $\tilde{T}_i^R \tilde{G}_0 \tilde{T}_k^A = 0$, when $i \neq k$:

$$\tilde{G}_0 \tilde{T}_i^R \tilde{G}_0 \tilde{T}_k^A \tilde{G}_0 = \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^r \rangle \tilde{G}_0^{-1} \langle D_{0k} \rangle \tilde{t}_k \langle D_{0k} \rangle \otimes \langle d_k^a \rangle$$

$$= \langle D_{0i} \rangle \tilde{t}_i \left(\bar{n}_j \bar{n}_k D_i^r - n_j n_k D_i^a \right) \otimes \bar{n}_i d_i^r \tilde{G}_0^{-1}$$

$$\times \left(\bar{n}_i \bar{n}_j D_k^r - n_i n_j D_k^a \right) \tilde{t}_k \langle D_{0k} \rangle \otimes n_k d_k^a = 0, \qquad (100)$$

where we used Eq. (36) for $\tilde{G}_0^{-1}$ and the fact that

$$\left(\bar{n}_i \bar{n}_j \bar{n}_k D_i^r - \bar{n}_i n_j n_k D_i^a \right) g g g \left(\bar{n}_i \bar{n}_j n_k D_k^r - n_i n_j n_k D_k^a \right) = 0. \qquad (101)$$

It is therefore clear that the solution to the Faddeev equation, Eq. (78), has the form $X_i = X_i^R + X_i^A$, where X_i^R and X_i^A satisfy the independent equations

$$X_i^R = \frac{1}{2} \tilde{T}_i^R + \frac{1}{2} \tilde{T}_i^R \sum_{k \neq i} \tilde{G}_0 X_k^R, \qquad (102a)$$

$$X_i^A = \frac{1}{2} \tilde{T}_i^A + \frac{1}{2} \tilde{T}_i^A \sum_{k \neq i} \tilde{G}_0 X_k^A. \qquad (102b)$$

In a similar way we have that

$$\tilde{G}_0 \tilde{T}_i^R \tilde{G}_0 \tilde{T}_k^R \tilde{G}_0 = \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^r \rangle \tilde{G}_0^{-1} \langle D_{0k} \rangle \tilde{t}_k \langle D_{0k} \rangle \otimes \langle d_k^r \rangle$$

$$= \langle D_{0i} \rangle \tilde{t}_i \left(\bar{n}_j \bar{n}_k D_i^r - n_j n_k D_i^a \right) \otimes \bar{n}_i d_i^r \tilde{G}_0^{-1}$$

$$\times \left(\bar{n}_i \bar{n}_j D_k^r - n_i n_j D_k^a \right) \tilde{t}_k \langle D_{0k} \rangle \otimes \bar{n}_k d_k^r$$

$$= \langle D_{0i} \rangle \tilde{t}_i \left(\bar{n}_j \bar{n}_k D_i^r \right) \otimes \bar{n}_i d_i^r \tilde{G}_0^{-1} \left(\bar{n}_i \bar{n}_j D_k^r \right) \tilde{t}_k$$

$$\times \langle D_{0k} \rangle \otimes \bar{n}_k d_k^r$$

$$= \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i}^r \rangle \otimes \langle d_i^r \rangle \tilde{G}_0^{-1}$$

$$\times \langle D_{0k}^r \rangle \tilde{t}_k \langle D_{0k} \rangle \otimes \langle d_k^r \rangle, \qquad (103)$$

and

$$\tilde{G}_0 \tilde{T}_i^A \tilde{G}_0 \tilde{T}_k^A \tilde{G}_0 = \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^a \rangle \tilde{G}_0^{-1} \langle D_{0k} \rangle \tilde{t}_k \langle D_{0k} \rangle \otimes \langle d_k^a \rangle$$

$$= \langle D_{0i} \rangle \tilde{t}_i \left(\bar{n}_j \bar{n}_k D_i^r - n_j n_k D_i^a \right) \otimes n_i d_i^a \tilde{G}_0^{-1}$$

$$\times \left(\bar{n}_i \bar{n}_j D_k^r - n_i n_j D_k^a \right) \tilde{t}_k \langle D_{0k} \rangle \otimes n_k d_k^a$$

$$= \langle D_{0i} \rangle \tilde{t}_i \left(-n_j n_k D_i^a \right) \otimes \bar{n}_i d_i^r \tilde{G}_0^{-1}$$

$$\times \left(-n_i n_j D_k^a \right) \tilde{t}_k \langle D_{0k} \rangle \otimes \bar{n}_k d_k^a$$

$$= \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i}^a \rangle \otimes \langle d_i^a \rangle \tilde{G}_0^{-1}$$

$$\times \langle D_{0k}^a \rangle \tilde{t}_k \langle D_{0k} \rangle \otimes \langle d_k^a \rangle. \qquad (104)$$

Thus, apart from external two-body propagators $\langle D_{0i} \rangle$ and $\langle D_{0k} \rangle$, all other propagators explicitly shown on the right-hand side of Eqs. (103) and (104) are truncated to their retarded or advanced parts. To take advantage of this simplification, we iterate Eq. (102a) once to obtain

$$X_i^R = \frac{1}{2} \tilde{T}_i^R + \frac{1}{4} \sum_{k \neq i} \tilde{T}_i^{Rr} \left(\tilde{G}_0 + \sum_{j \neq k} \tilde{G}_0 X_j^r \tilde{G}_0 \right) \tilde{T}_k^{rR}, \qquad (105a)$$

$$X_i^A = \frac{1}{2} \tilde{T}_i^A + \frac{1}{4} \sum_{k \neq i} \tilde{T}_i^{Aa} \left(\tilde{G}_0 + \sum_{j \neq k} \tilde{G}_0 X_j^a \tilde{G}_0 \right) \tilde{T}_k^{aA}, \qquad (105b)$$

where

$$\tilde{G}_0 \tilde{T}_i^{Rr} \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i}^r \rangle \otimes \langle d_i^r \rangle, \quad (106a)$$

$$\tilde{G}_0 \tilde{T}_i^{rR} \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i}^r \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^r \rangle, \quad (106b)$$

$$\tilde{G}_0 \tilde{T}_i^{Aa} \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i} \rangle \tilde{t}_i \langle D_{0i}^a \rangle \otimes \langle d_i^a \rangle, \quad (106c)$$

$$\tilde{G}_0 \tilde{T}_i^{aA} \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i}^a \rangle \tilde{t}_i \langle D_{0i} \rangle \otimes \langle d_i^a \rangle, \quad (106d)$$

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[3]The superscripts of $\tilde{T}_i^{R,A}$ are simply convenient labels and do not correspond exactly to the retarded or advanced θ functions of time as, for example, in the case of the single particle propagator of Eq. (81).

and where the amplitudes X_i^r and X_i^a satisfy the Faddeev equations

$$X_i^r = \frac{1}{2}\tilde{T}_i^r + \frac{1}{2}\tilde{T}_i^r \sum_{k \neq i} \tilde{G}_0 X_k^r, \qquad (107a)$$

$$X_i^a = \frac{1}{2}\tilde{T}_i^a + \frac{1}{2}\tilde{T}_i^a \sum_{k \neq i} \tilde{G}_0 X_k^a, \qquad (107b)$$

which are simpler than Eq. (102a) in that they utilize input t matrices whose adjoining propagators are either all retarded or all advanced:

$$\tilde{G}_0 \tilde{T}_i^r \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i}^r \rangle \tilde{t}_i \langle D_{0i}^r \rangle \otimes \langle d_i^r \rangle, \qquad (108a)$$

$$\tilde{G}_0 \tilde{T}_i^a \tilde{G}_0 = (2\pi)^3 \delta^3(\mathbf{p}_i' - \mathbf{p}_i) \langle D_{0i}^a \rangle \tilde{t}_i \langle D_{0i}^a \rangle \otimes \langle d_i^a \rangle. \qquad (108b)$$

Note, however, that

$$\tilde{t}_i = \tilde{v}_i + \frac{1}{2}\tilde{v}_i \langle D_{0i} \rangle \tilde{t}_i, \qquad (109)$$

so that the internal propagators $\langle D_{0i} \rangle$, used in constructing the physical two-body t matrices $\tilde{t}_i$, retain both retarded and advanced parts. With the spectator δ function removed, one can invert the $\tilde{G}_0$'s in Eq. (108b) with the help of Eq. (36) to obtain

$$\tilde{T}_i^r(E) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dz$$
$$\times \frac{ggg\bar{n}_j'\bar{n}_k'(E - \omega_{ijk}')\tilde{t}_i(E - z)(E - \omega_{ijk})\bar{n}_i\bar{n}_j\bar{n}_k\, ggg}{(E - z - \omega_{jk}' + i\epsilon)(E - z - \omega_{jk} + i\epsilon)(z - \omega_i + i\epsilon)}, \qquad (110a)$$

$$\tilde{T}_i^a(E) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dz$$
$$\times \frac{gggn_j'n_k'(E - \omega_{ijk}')\tilde{t}_i(E - z)(E - \omega_{ijk})n_in_jn_k\, ggg}{(E - z - \omega_{jk}' - i\epsilon)(E - z - \omega_{jk} - i\epsilon)(z - \omega_i - i\epsilon)} \qquad (110b)$$

where $\quad \omega_{ijk}' = \omega_i + \omega_j' + \omega_k', \omega_{ijk} = \omega_i + \omega_j + \omega_k, \omega_{jk}' = \omega_j' + \omega_k'$, and $\omega_{jk} = \omega_j + \omega_k$. In turn, this result shows that the $\tilde{G}_0$ in Eq. (107a) can be replaced with $\langle G_0^r \rangle$ and the $\tilde{G}_0$ in Eq. (107b) can be replaced with $\langle G_0^a \rangle$. Similarly, replacements can be made in appropriate $\tilde{G}_0$'s in Eqs. (105a) and (106a), so that our final equations for the amplitude X_i are

$$X_i = X_i^R + X_i^A, \qquad (111)$$

$$X_i^R = \frac{1}{2}\tilde{T}_i^R + \frac{1}{4}\sum_{k \neq i}\tilde{T}_i^{Rr}\left(\langle G_0^r \rangle + \sum_{j \neq k}\langle G_0^r \rangle X_j^r \langle G_0^r \rangle\right)\tilde{T}_k^{rR}, \qquad (112a)$$

$$X_i^A = \frac{1}{2}\tilde{T}_i^A + \frac{1}{4}\sum_{k \neq i}\tilde{T}_i^{Aa}\left(\langle G_0^a \rangle + \sum_{j \neq k}\langle G_0^a \rangle X_j^a \langle G_0^a \rangle\right)\tilde{T}_k^{aA}, \qquad (112b)$$

where the amplitudes X_i^r and X_i^a satisfy the Faddeev equations

$$X_i^r = \frac{1}{2}\tilde{T}_i^r + \frac{1}{2}\tilde{T}_i^r \sum_{k \neq i}\langle G_0^r \rangle X_k^r, \qquad (113a)$$

$$X_i^a = \frac{1}{2}\tilde{T}_i^a + \frac{1}{2}\tilde{T}_i^a \sum_{k \neq i}\langle G_0^a \rangle X_k^a. \qquad (113b)$$

It is interesting to note that our equations, like the ones used in Ref. [27], contain a blocking factor ($\bar{n}$ or n) for each fermion line, in contrast to the approach of Refs. [5–10] where such factors appear only on non-spectator fermions.

2. Single-degree-of-freedom equations

The projection properties of the input two-body t matrices of Eq. (110a) enable one to eliminate the doubled degrees of freedom from the in-matter Faddeev equations, Eq. (113). To see this, we first note that the amplitude X_j^R of Eq. (112a) is determined by the quantity

$$\langle G_0^r \rangle X_j^r \langle G_0^r \rangle = G^r\bar{n}_1\bar{n}_2\bar{n}_3 X_j^r\bar{n}_1\bar{n}_2\bar{n}_3 G^r, \qquad (114)$$

which has the doubled-degrees-of-freedom Faddeev amplitude X_j^r completely surrounded by projection operators $\bar{n}$. The matrix structure of $\bar{n}$ is given in Eq. (A29) as

$$\bar{n} = \mathscr{U}(\omega)\mathscr{U}^\dagger(\omega), \qquad (115)$$

where $\mathscr{U}(\omega)$ is a column vector defined in Eq. (A10) for bosons and in Eq. (A20) for fermions, and $\mathscr{U}^\dagger(\omega)$ is the corresponding row vector. Thus amplitude X_j^R is expressible directly in terms of the single-degree-of-freedom amplitude:

$$\hat{X}_j^r = \mathscr{U}^\dagger(\omega_1')\mathscr{U}^\dagger(\omega_2')\mathscr{U}^\dagger(\omega_3')X_j^r\mathscr{U}(\omega_1)\mathscr{U}(\omega_2)\mathscr{U}(\omega_3). \qquad (116)$$

Similarly, the matrix structure of projection operator n, given in Eq. (A30) as

$$n = \pm\mathscr{V}(\omega)\mathscr{V}^\dagger(\omega), \qquad (117)$$

allows one to express amplitude X_j^A of Eq. (112b) directly in terms of the single-degree-of-freedom amplitude:

$$\hat{X}_j^a = \mathscr{V}^\dagger(\omega_1')\mathscr{V}^\dagger(\omega_2')\mathscr{V}^\dagger(\omega_3')X_j^a\mathscr{V}(\omega_1)\mathscr{V}(\omega_2)\mathscr{V}(\omega_3). \qquad (118)$$

Moreover, it follows from Eqs. (113) that $\hat{X}_i^r$ and $\hat{X}_i^a$ themselves satisfy Faddeev equations:

$$\hat{X}_i^r = \frac{1}{2}\hat{T}_i^r + \frac{1}{2}\hat{T}_i^r \sum_{k \neq i} G^r \hat{X}_k^r, \qquad (119a)$$

$$\hat{X}_i^a = \frac{1}{2}\hat{T}_i^a + \frac{1}{2}\hat{T}_i^a \sum_{k \neq i} G^a \hat{X}_k^a, \qquad (119b)$$

where, with the spectator δ function removed,

$$\hat{T}_i^r(E) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dz$$
$$\times \frac{(E - \omega_{ijk}')\hat{t}_i^R(E - z)(E - \omega_{ijk})}{(E - z - \omega_{jk}' + i\epsilon)(E - z - \omega_{jk} + i\epsilon)(z - \omega_i + i\epsilon)}, \qquad (120a)$$

$$\hat{T}_i^a(E) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dz$$

$$\times \frac{(E - \omega'_{ijk}) \, \hat{t}_i^A(E - z) \, (E - \omega_{ijk})}{(E - z - \omega'_{jk} - i\epsilon)(E - z - \omega_{jk} - i\epsilon)(z - \omega_i - i\epsilon)}, \tag{120b}$$

and

$$\hat{t}_i^R = \mathcal{U}^\dagger(\omega'_j) \mathcal{U}^\dagger(\omega'_k) \, \tilde{t}_i \, \mathcal{U}(\omega_j) \mathcal{U}(\omega_k), \tag{121a}$$

$$\hat{t}_i^A = \mathcal{V}^\dagger(\omega'_j) \mathcal{V}^\dagger(\omega'_k) \, \tilde{t}_i \, \mathcal{V}(\omega_j) \mathcal{V}(\omega_k). \tag{121b}$$

We note that in Eq. (119a) all doubled degrees of freedom have been eliminated. In particular, the two-body input to these equations is found from the convolution integrals, Eq. (120a), involving the single-degree-of-freedom t matrices $\hat{t}_i^R$ and $\hat{t}_i^A$ of Eq. (121a).

Although both $\hat{t}_i^R$ and $\hat{t}_i^A$ are single-degree-of-freedom quantities, they themselves are constructed from a 16-component doubled-degree-of-freedom t matrix $\tilde{t}_i$. Although it may not be difficult to solve the 16-component equation, Eq. (109), to obtain $\tilde{t}_i$, it is useful to note that this equation can be recast into a four-component (2×2) equation as follows. Writing Eq. (109) as

$$\tilde{t}_i = \tilde{v}_i + \tfrac{1}{2}\tilde{v}_i \left(\bar{n}_j \bar{n}_k D_i^r - n_j n_k D_i^a \right) \tilde{t}_i, \tag{122}$$

straightforward use of Eqs. (A29) and (A30) allows one to write

$$\hat{t}_i = \hat{v}_i + \tfrac{1}{2}\hat{v}_i \hat{D}_i \hat{t}_i, \tag{123}$$

where

$$\hat{t}_i = \begin{pmatrix} \mathcal{U}_j^\dagger \mathcal{U}_k^\dagger \tilde{t}_i \mathcal{U}_j \mathcal{U}_k & \mathcal{U}_j^\dagger \mathcal{U}_k^\dagger \tilde{t}_i \mathcal{V}_j \mathcal{V}_k \\ \mathcal{V}_j^\dagger \mathcal{V}_k^\dagger \tilde{t}_i \mathcal{U}_j \mathcal{U}_k & \mathcal{V}_j^\dagger \mathcal{V}_k^\dagger \tilde{t}_i \mathcal{V}_j \mathcal{V}_k \end{pmatrix}, \tag{124}$$

$$\hat{v}_i = \begin{pmatrix} \mathcal{U}_j^\dagger \mathcal{U}_k^\dagger \tilde{v}_i \mathcal{U}_j \mathcal{U}_k & \mathcal{U}_j^\dagger \mathcal{U}_k^\dagger \tilde{v}_i \mathcal{V}_j \mathcal{V}_k \\ \mathcal{V}_j^\dagger \mathcal{V}_k^\dagger \tilde{v}_i \mathcal{U}_j \mathcal{U}_k & \mathcal{V}_j^\dagger \mathcal{V}_k^\dagger \tilde{v}_i \mathcal{V}_j \mathcal{V}_k \end{pmatrix}, \tag{125}$$

$$\hat{D}_i = \begin{pmatrix} D_i^r & 0 \\ 0 & -D_i^a \end{pmatrix}. \tag{126}$$

V. SUMMARY

Using the real-time formalism, we have formulated equal-time three-body equations that describe three identical particles interacting by means of pairwise interactions at finite temperature and density. Starting with the 4D field-theoretic description of the $3 \to 3$ Green's function, we derived equal-time three-body equations without resorting to any approximations beyond that of the assumption of pairwise interactions.

Our resulting in-matter three-body equations, Eq. (78) for the general case and Eqs. (111)–(113) for the case of instantaneous potentials and freelike dressed propagators have the familiar Faddeev form, although they differ from the usual zero-density Faddeev equations in that they involve doubled degrees of freedom (inherent in the real-time formalism) and they utilize one-body thermal Green's functions (which have

retarded and advanced parts and depend on both temperature and chemical potential). At the same time, the form of our equations is similar to that of other formulations of the in-matter three-body problem, even though all other formulations have apparently been done either at zero temperature or for nonzero temperatures by use of the imaginary-time formalism. However, what distinguishes our approach in an essential way from all other derivations of the equal-time in-matter three-body problem is that we have managed to avoid any approximations in the equal-time 3D reduction of the original 4D field-theoretic formulation. Moreover, our resulting 3D equations remain practical in that the equal-time two-body t matrix in three-body space, $\tilde{T}_i$, which determines the integral equation kernel, is given in terms of the 4D two-body t matrix by a simple convolution integral: Eq. (72).

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APPENDIX: PROPAGATOR MATRIX STRUCTURE

In the doubled-degrees-of-freedom formalism, the matrix structure of propagators is discussed in detail, for example, in Ref. [22]. Here we give only a brief summary.

1. Bosons

In the doubled-degrees-of-freedom formalism, the boson propagator is given by [22]

$$d^f(p) = i \begin{pmatrix} \cosh \omega & \sinh \omega \\ \sinh \omega & \cosh \omega \end{pmatrix} \begin{pmatrix} d^r & 0 \\ 0 & -d^a \end{pmatrix} \begin{pmatrix} \cosh \omega & \sinh \omega \\ \sinh \omega & \cosh \omega \end{pmatrix}, \tag{A1}$$

where

$$d^r = \frac{1}{p^0 - \omega + i\epsilon}, \quad d^a = \frac{1}{p^0 - \omega - i\epsilon}, \quad \omega = \frac{\mathbf{p}^2}{2m} - \mu, \tag{A2}$$

and

$$\sinh \omega = \sqrt{f_B(\omega)}, \quad \cosh \omega = \sqrt{1 + f_B(\omega)},$$
$$f_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{A3}$$

Denoting

$$U_B(\omega) = \begin{pmatrix} \cosh \omega & \sinh \omega \\ \sinh \omega & \cosh \omega \end{pmatrix}, \quad g_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A4}$$

one finds that U_B is not unitary but satisfies

$$U_B(\omega) \, g_B U_B^\dagger(\omega) = g_B. \tag{A5}$$

It follows that

$$[d^f(p)]^{-1} = -i(p^0 - \omega)g_B. \tag{A6}$$

Comparison with Eq. (4) provides explicit expressions for $\bar{n}$ and n:

$$\bar{n} = U_B(\omega)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}U_B^\dagger(\omega), \qquad n = U_B(\omega)\begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}U_B^\dagger(\omega).$$
(A7)

One can check the following important properties of n and $\bar{n}$:

$$n + \bar{n} = g_B,$$
(A8a)

$$n g_B \bar{n} = \bar{n} g_B n = 0,$$
(A8b)

$$n\, g_B\, n = n, \quad \bar{n} g_B\, \bar{n} = \bar{n}.$$
(A8c)

For the case of bosons, these relations will define what is meant by the projection properties of the operators n and $\bar{n}$. Equation (A7) imply another convenient way of expressing $\bar{n}$ and n, namely, as products of column and row vectors:

$$\bar{n} = \mathscr{U}_B(\omega)\mathscr{U}_B^\dagger(\omega), \qquad n = -\mathscr{V}_B(\omega)\mathscr{V}_B^\dagger(\omega),$$
(A9)

where

$$\mathscr{U}_B(\omega) = \begin{pmatrix}U_{11}^B\\U_{21}^B\end{pmatrix} = \begin{pmatrix}\cosh\omega\\\sinh\omega\end{pmatrix},$$
$$\mathscr{V}_B(\omega) = \begin{pmatrix}U_{12}^B\\U_{22}^B\end{pmatrix} = \begin{pmatrix}\sinh\omega\\\cosh\omega\end{pmatrix},$$
(A10)

are the column vectors and $\mathscr{U}_B^\dagger, \mathscr{V}_B^\dagger$ are the corresponding row vectors. It follows that

$$\mathscr{U}_B^\dagger g_B \mathscr{U}_B = 1, \qquad \mathscr{V}_B^\dagger g_B \mathscr{V}_B = -1.$$
(A11)

2. Fermions

In the doubled-degrees-of-freedom formalism, the fermion propagator is given by [22]

$$d^f(p) = i\begin{pmatrix}\cos\omega & \sin\omega\\-\sin\omega & \cos\omega\end{pmatrix}\begin{pmatrix}d^r & 0\\0 & d^a\end{pmatrix}\begin{pmatrix}\cos\omega & -\sin\omega\\\sin\omega & \cos\omega\end{pmatrix},$$
(A12)

where d^r, d^a, ω are as in Eqs. (A2), and

$$\sin\omega = \sqrt{f_F(\omega)}, \qquad \cos\omega = \sqrt{1 - f_F(\omega)},$$
(A13)
$$f_F(\omega) = \frac{1}{e^{\beta\omega} + 1}.$$

Denoting

$$U_F(\omega) = \begin{pmatrix}\cos\omega & \sin\omega\\-\sin\omega & \cos\omega\end{pmatrix},$$
(A14)

it is clear that U_F is unitary:

$$U_F(\omega)U_F^\dagger(\omega) = U_F^\dagger(\omega)U_F(\omega) = 1.$$
(A15)

It follows that

$$[d^f(p)]^{-1} = -i(p^0 - \omega)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$
(A16)

Comparison with Eq. (4) provides explicit expressions for $\bar{n}$ and n:

$$\bar{n} = U_F(\omega)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}U_F^\dagger(\omega), \qquad n = U_F(\omega)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}U_F^\dagger(\omega).$$
(A17)

One can check the following projection properties of n and $\bar{n}$:

$$n + \bar{n} = 1,$$
(A18a)

$$n\bar{n} = \bar{n}n = 0,$$
(A18b)

$$n\, n = n, \quad \bar{n}\, \bar{n} = \bar{n}.$$
(A18c)

As in the boson case, we can write $\bar{n}$ and n in terms of column and row vectors:

$$\bar{n} = \mathscr{U}_F(\omega)\mathscr{U}_F^\dagger(\omega), \qquad n = \mathscr{V}_F(\omega)\mathscr{V}_F^\dagger(\omega)$$
(A19)

where

$$\mathscr{U}_F(\omega) = \begin{pmatrix}U_{11}^F\\U_{21}^F\end{pmatrix} = \begin{pmatrix}\cos\omega\\-\sin\omega\end{pmatrix},$$
$$\mathscr{V}_F(\omega) = \begin{pmatrix}U_{12}^F\\U_{22}^F\end{pmatrix} = \begin{pmatrix}\sin\omega\\\cos\omega\end{pmatrix},$$
(A20)

are the column vectors and $\mathscr{U}_F^\dagger, \mathscr{V}_F^\dagger$, are the corresponding row vectors. It follows that

$$\mathscr{U}_F^\dagger\mathscr{U}_F = 1, \qquad \mathscr{V}_F^\dagger\mathscr{V}_F = 1.$$
(A21)

3. General

By defining

$$g_F = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \qquad g_B = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},$$
(A22)

and

$$g = \begin{cases}g_B\\g_F\end{cases}, \qquad U = \begin{cases}U_B\\U_F\end{cases}, \qquad \mathscr{U} = \begin{cases}\mathscr{U}_B\\\mathscr{U}_F\end{cases},$$
$$\mathscr{V} = \begin{cases}\mathscr{V}_B & \text{(boson case)}\\\mathscr{V}_F & \text{(fermion case)},\end{cases}$$
(A23)

one can write, for both bosons and fermions,

$$d^f(p) = iUg\begin{pmatrix}d^r & 0\\0 & d^a\end{pmatrix}U^\dagger,$$
(A24)

$$[d^f(p)]^{-1} = -i(p^0 - \omega)g,$$
(A25)

$$UgU^\dagger = U^\dagger gU = g,$$
(A26)

$$\bar{n} = U\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}U^\dagger, \qquad n = Ug\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}U^\dagger,$$
(A27)

$$n + \bar{n} = g,$$
(A28a)

$$ng\bar{n} = \bar{n}gn = 0, \tag{A28b}$$

$$n\,g\,n = n, \quad \bar{n}g\,\bar{n} = \bar{n}. \tag{A28c}$$

Also, for both bosons and fermions,

$$\bar{n} = \mathscr{U}\mathscr{U}^{\dagger}, \qquad \mathscr{U}^{\dagger}g\mathscr{U} = 1, \tag{A29}$$

and

$$n = \pm\mathscr{V}\mathscr{V}^{\dagger}, \qquad \mathscr{V}^{\dagger}g\mathscr{V} = \pm 1, \tag{A30}$$

with the $+$ sign being for fermions and the $-$ sign being for bosons.

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