Nuclear incompressibility in asymmetric systems at finite temperature and entropy

A. Z. Mekjian,[1] S. J. Lee,[2] and L. Zamick[1]

[1]*Department of Physics and Astronomy, Rutgers University Piscataway, New Jersey 08854, USA*
[2]*Department of Physics and Institute of Natural Sciences, Kyung Hee University, Suwon, KyungGiDo, Korea*

The nuclear incompressibility κ is investigated in asymmetric systems in a mean field model. The calculations are done at zero and finite temperatures and include surface, Coulomb and symmetry energy terms for several equations of state. Also considered is the behavior of the incompressibility at constant entropy κ_Q which is shown to have a very different behavior than the isothermal κ. Namely, κ_Q decreases with increasing entropy while the isothermal κ increases with increasing T. A duality is found between the adiabatic κ_Q and the $T = 0$ isothermal κ. Analytic and also simple approximate expressions for κ are given which illustrate the role of various terms that enter into κ. Namely, properties of the interaction used, the value of the effective mass, and its density dependence all play an important role.

I. INTRODUCTION

Heavy ion collisions produce hot and dense hadronic matter [1,2]. Such matter is also encountered in nuclear astrophysics and in supernovae explosions. Future rare isotope accelerators will explore nuclei at the limit of isospin asymmetry. In neutron stars one also encounters a system with a large neutron to proton ratio. Thermodynamic quantities play an important role in characterizing properties of such systems. One such quantity is the nuclear incompressibility. While the nuclear incompressibility at zero temperature has been studied for an extended period [2–5], it is only relatively recently that its temperature dependence has been of concern [6,7]. In this paper we will focus on the temperature dependence of the incompressibility in asymmetric systems, including the role of surface, symmetry and Coulomb terms. We will also study the adiabatic incompressibility which at $T = 0$ would be the same as the isothermal incompressibility, but differs from it at finite T. The isothermal incompressibility will increase with T while the adiabatic incompressibility will decrease with entropy [7].

There is still some uncertainty in the value of incompressibility κ at $T = 0$ of the order of 10–20% [8–11]. Nonrelativistic calculations [12,13] have values of $\kappa \approx 220 \sim 235$ MeV while recent relativistic calculations [14–16] have higher values with $\kappa \approx 250 \sim 270$ MeV when contributions of the negative energy sea are included. A larger range of values from $211 \sim 350$ MeV is reported in Ref. [17]. Several recent papers [8,9,18] explain this difference between nonrelativistic and relativistic calculations from differences in the symmetry or isospin dependent part of the EOS and its associated κ. The incompressibility of asymmetric nuclei is somewhat uncertain as noted in Ref. [19]. The experimental determination of κ involves a study of the breathing mode or isoscalar giant monopole resonance. Since nuclei are finite, charged and in the case of heavy nuclei asymmetric, the measured incompressibility has contributions from surface, Coulomb, and symmetry energy terms besides the volume term or infinite nuclear matter term. Recent experimental data [20,21] on nuclei ^{90}Zr, ^{116}Sn, ^{144}Sm, and ^{208}Pb allow for

an extraction of κ with an uncertainty of ±12 MeV [10,11]. For light nuclei extraction of κ is complicated by the fragmentation of the strength of the isoscalar monopole. For heavy nuclei, uncertainties in the asymmetry term are present. A value of $\kappa \sim 230 \pm 12$ MeV is often quoted by several groups.

Determinations of the nuclear incompressibility and nuclear EOS are also important in heavy ion collisions. References [2,22,23] illustrate the importance of κ in both sideward and elliptic flow and in the transverse momentum response of the spectators arising from an explosion of the participant region. In Refs. [24,25] the incompressibility is related to isospin diffusion [1,26], a process in which large clusters are nearly symmetric in isospin and the nuclear gas is neutron rich. Isospin diffusion is sensitive to the density dependence of symmetry energy. Knowledge of the detailed features of the symmetry energy is also important in nuclear astrophysics, where neutron stars probe the isospin limits of this term. Besides the giant isoscalar monopole, the isobaric analog resonance has important information on the isovector part of the nuclear potential. This potential is referred to the Lane potential and the nuclear symmetry energy is responsible for the splitting of the analog resonance from its isospin lower partners, called the anti-analogue states [27]. The EOS of asymmetric systems appears also in the theory of phase transition in such systems [28–30]. In many of these cases information about the EOS and associated incompressibility is at $T \neq 0$. Thus we explore the $T \neq 0$ situation in this paper.

As a baseline, we will begin with a mean field discussion of its behavior with T to see the contributions to incompressibility of effective mass, excluded volume, symmetry energy, surface energy, and Coulomb energy. In Sec. II, we will summarize a general description of incompressibility and results of Ref. [7] which considered the temperature and entropy dependence of isothermal and adiabatic incompressibility of infinite symmetric nuclear matter with a simple Skyrme interaction. In Secs. III and IV we will consider the contributions of effective mass and excluded volume. In Sec.V we consider the incompressibility of asymmetric finite nuclear system which has surface energy, symmetric energy, and Coulomb energy contribution. The conclusion is given in Sec. VI.

II. GENERAL CONSIDERATIONS

First, we define a quantity κ, incompressibility coefficient, as

$$\kappa = k_F^2 \frac{d^2(E/A)}{dk_F^2} = 9\rho^2 \frac{d^2(E/A)}{d\rho^2}$$
$$= 9V^2 \frac{d^2(E/A)}{dV^2} = R^2 \frac{d^2(E/A)}{dR^2}. \quad (1)$$

This quantity is evaluated at the saturation density ρ_0 where E/A has a minimum. The giant monopole resonance energy is related to κ and this relation is $E_0 = \sqrt{\frac{\hbar^2 \kappa A}{m \langle r^2 \rangle}}$. If the temperature is kept constant in the above derivatives we have the isothermal incompressibility κ, and if the entropy is held fixed, the result is the adiabatic incompressibility κ_Q. This κ is related to the second order Taylor expansion coefficient of E/A with ρ or $E/A = a + b(\rho - \rho_0)^2/2$. Thus $\kappa = 9\rho_0^2 b$. The quantity κ defined above is not the isothermal compressibility defined in thermal physics as $K = -(\frac{1}{V})(dV/dP)_T$ with T held fixed and here P is the pressure. Since $P = -dF/dV|_T$, we have $K = (\frac{1}{V})(1/\frac{d^2F}{dV^2})_T$. At $T = 0$, $F = E - TS = E$, and thus $\kappa = 9/(\rho_0 K)$. This reciprocal connection between K and κ is no longer true at finite T. A reciprocal relation would exist between K and a κ defined by an equation similar to Eq. (1), but with energy E replaced with the free energy F. The K can be connected to the density fluctuations through the equation $\langle \rho^2 \rangle - \langle \rho \rangle^2 = (\langle \rho \rangle^2/V)TK$ [31]. Near a first order phase transition and at its critical point, the K can become very large since small changes in the pressure can produce large changes in the volume. Large density fluctuations at a critical point give rise to the phenomena of critical opalescence where the presence of droplets of all sizes scatter light. In the limit of an ideal gas $K = 1/P$. For a Van der Waals gas, whose equation of state is $(P + a(N/V)^2)(V - bN) = NT$, the isothermal compressibility K will diverge above and near the critical temperature T_c as $1/(T - T_c)^\gamma$, where the critical exponent $\gamma = 1$ which is the mean field value. Its reciprocal will go to zero. The energy per particle for the Van der Waals gas is $E/N = (3/2)T - a(N/V)$ for $V > bN$, while the entropy per particle is $S/N = \log[(V - bN)/(N\lambda^3)] + 5/2$. The λ is the De Broglie or quantum wavelength $\lambda = h/(2\pi mT)^{1/2}$. The energy per particle monotonically decreases with increasing density $\rho = N/V$ and is at its minimum value at the smallest density determined by the excluded volume $V = bN$. Besides the isothermal compressibility, an adiabatic compressibility can be obtained by keeping the entropy constant. The adiabatic compressibility is $K_Q = -(\frac{1}{V})(dV/dP)_S$ with the derivative at constant entropy. The reciprocal is related to κ_Q, the adiabatic incompressibility as $\kappa_Q = 9/(\rho_0 K_Q)$. The energy per particle of a Van der Waals gas can have a minimum with V at constant entropy. Since the natural variables for energy are entropy and volume from $dE = TdS - PdV$ variations of the energy with V at constant S bear a similar relation to variations of the Helmholtz free energy with V at constant T where $dF = -SdT - PdV$. Thus the adiabatic incompressibility of Eq. (1) can go to zero for a Skyrme interaction as we shall see. The minimum point (also maximum point) in the energy occurs at zero pressure since dE/dV at constant entropy is $-P$. Therefore E at constant S has the same maximum and minimum points with variations in V or R or density as F at constant T since both derivatives are $-P$ which is set to 0.

A. Review of infinite matter incompressibility

Our mean field discussion is based on a Skyrme interaction. The Skyrme interaction shares some common features with the above Van der Waals equation such as an intermediate density attraction [the $a(N/V)^2$ term above] and a higher density repulsion (the excluded volume term in Van der Waals). The excluded volume leads to a much stronger repulsive term than the Skyrme repulsive term since it involves an expansion to all orders in the density. To keep the discussion simple to begin with, we consider uncharged symmetric nuclear matter with no surface energy terms. The Skyrme interaction energy is then

$$U/A = -a_0 \rho + a_\alpha \rho^{1+\alpha}. \quad (2)$$

The a_0 term gives a medium range attraction while the a_α term is a short range repulsion.

At $T = 0$, the kinetic energy $E_K/A = (\frac{3}{5})E_F(\rho)$. The coefficients a_0 and a_α are fixed to give a binding energy per particle $E_B/A = 16$ MeV at saturation density $\rho = \rho_0 = 0.15$ fm^3. At this density $E_F = \frac{\hbar^2}{2m}(\frac{6\pi^2}{4}\rho)^{2/3} = 35$ MeV and $E_K/A = 21$ MeV. From $E = E_K + U$ and $d(E/A)/d\rho = 0$ at ρ_0 the coefficients are

$$a_0 \rho_0 = b_0 = (E_K + E_B)/A + b_\alpha$$
$$= (E_K + E_B)/A + (1/\alpha)(E_K/3 + E_B)/A$$
$$= 37 + 23/\alpha,$$
$$\alpha a_\alpha \rho_0^{1+\alpha} = \alpha b_\alpha = (E_K/3 + E_B)/A = 23,$$
$$(1 + \alpha)b_\alpha = b_0 - \tfrac{2}{3}E_K/A. \quad (3)$$

The incompressibility coefficient κ at $T = 0$ is then

$$\kappa = -2E_K/A + 9(1 + \alpha)\alpha a_\alpha \rho_0^{1+\alpha}$$
$$= E_K/A + 9E_B/A + \alpha(3E_K/A + 9E_B/A)$$
$$= 165 + 207\alpha. \quad (4)$$

The last result is for the case of $E_B/A = 16$ MeV and $\rho_0 = 0.15$ fm^{-3} with nucleon mass m. For an effective mass $m^* \neq m$, $E_F = 35(m/m^*)$ MeV and $E_K/A = 21(m/m^*)$ MeV and

$$\kappa = (144 + 21m/m^*) + \alpha(144 + 63m/m^*) \quad (5)$$

when the density dependence of m^* is neglected. Detailed discussions of the role of m^* are presented in Sec. III. For $m^*/m = 1, \alpha = 1/3, \kappa = 234$ MeV at ρ_0 while for $m^*/m = 0.7, \kappa = 234$ MeV at $\alpha = 60/234 \sim 1/4$. These values are in agreement with Refs. [10,11]. Smaller values of α lead to softer equations of state and lower κ. Similar results were obtained in Ref. [32] with a density dependent effective mass. In the limit $\alpha \to 0$, logarithmic terms appear in Eq. (2) coming from the presence of a factor $(x/\alpha)(1 - x^\alpha) = (x/\alpha)(1 - e^{\alpha \log x}) \to -x \log(x)$. The $x = \rho/\rho_0 = (R_0/R)^3$ with $R_0^3 = A/(\rho_0 4\pi/3)$. The $\alpha \to 0$ limit is the softest EOS allowed by Eq. (2), and this limit gives from Eq. (4) a value of $\kappa = 165$ MeV at $m^*/m = 1$. A stiff EOS has $\alpha = 1$ and $\kappa = 372$ MeV for $m^*/m = 1$. These extreme values for $\alpha \to 0$ and $\alpha = 1$ are unrealistic. The $T = 0$ results for κ are somewhat

uncertain, but recent calculations [18,33,34], put κ in the range $210 \sim 270$ MeV and suggest a value of $\alpha = 1/3$ for $\kappa = 234$ MeV in a Skyrme type approach with $m^* = m$. When $m^* < m$, a value of $\kappa = 234$ MeV is achieved with smaller values of α. A larger range of values of κ, from $211 \sim 350$ MeV, were reported in Ref. [17]. In this paper we will keep α as a variable to illustrate its role in conjunction with m^* and its density dependence. Because of these uncertainties in κ from more realistic forces, we will present results for various values of α, from $\alpha \approx 0$ to $\alpha = 1$.

B. Corrections to the infinite matter result

If the corrections to the infinite nuclear matter incompressibility from finite size terms or nonzero temperature or entropy terms are small, then these corrections can be obtained by using the following method. Let $E_0(R)$ be the nuclear matter energy per particle EOS which has a minimum at R_0 and an incompressibility κ_0. If we add to this a term $E_x(R)$, so that $E(R) = E_0(R) + E_x(R)$ then the minimum shifts to a new point $R_m = R_0 + \Delta R_x$. The new minimum and κ can be found by making a Taylor expansions around of R_0. The shift is given by

$$\Delta R_x/R_0 = -R_0 (dE_x(R)/dR)/\kappa_0 \tag{6}$$

with the derivative evaluated at R_0 since $R^2 \frac{d^2 E/A}{dR^2}\big|_{R_m} \approx R^2 \frac{d^2 E/A}{dR^2}\big|_{R_0}$ for small ΔR_x. The new κ is

$$\kappa = R^2 \frac{d^2 (E(R)/A)}{dR^2} = \kappa_0 + R^2 \left(\frac{E_x(R)}{A}\right)'' - 2R \left(\frac{E_x(R)}{A}\right)'$$
$$- R \left(\frac{E_x(R)}{A}\right)' \frac{Sk}{\kappa_0}, \tag{7}$$

$$Sk = R^3 \frac{d^3 (E_0(R)/A)}{dR^3}\bigg|_{R_0} = -k_F^3 \frac{d^3 (E_0(R)/A)}{dk_F^3}. \tag{8}$$

The various quantities are evaluated at R_0 and each $'$ represents one derivative with respect to R. Corrections to κ involving skewness Sk or third derivative of the energy were pointed out in Refs. [4,5]. However note here that the skewness Sk is defined with opposite sign from others. Ellis *et al.* [35] used the correlation between compression modulus and skewness coefficient to examine the implications in a relativistic Hartree approximation where the E_x is the Coulomb energy. The above expression is a modified version of their result. When two small correction terms are present then the above result is easily generalized and each adds its own separate contribution. If the corrections are large then the above approximate expression will not be very accurate and it is best to calculate the shift and κ exactly. We will use these results for the shift and κ to find simple expressions for κ for the general case of arbitrary α. Equation (4) gives an expression for the incompressibility at $T = 0$ in terms of E_K and E_B. This will be $\kappa_0 = \kappa_0(\alpha)$ in Eqs. (6) and (7). The skewness $Sk = R^3 (E_0(R)/A)'''$, in terms of E_K, E_B, and α, is

$$Sk(\alpha) = -27(E_B/A)(3 + 4\alpha + \alpha^2)$$
$$- (E_K/A)(11 + 36\alpha + 9\alpha^2) \tag{9}$$
$$= -3(509 + 828\alpha + 207\alpha^2).$$

The last equality in Eq. (9) follows when $E_K = 21$ MeV and $E_B = 16$ MeV and $m^* = m$. If $m^* \neq m$, then

$$Sk(\alpha, m^*) = -3[(432 + 77m/m^*) + (576 + 252m/m^*)\alpha$$
$$+ (144 + 63m/m^*)\alpha^2]. \tag{10}$$

Comparing this last equation with the κ of Eqs. (4) or (5) we see that the ratio of Sk/κ_0 is of the order of 10 and somewhat insensitive to α and m^*/m. In particular for $\alpha = 0$, 1/3, and 1 and $m^*/m = 1$ this ratio is -9.25, -10.36, and -12.45. For these α's and $m^*/m = 0.75$ this ratio is -9.33, -10.45, -12.56. We will apply this procedure or exact solutions to evaluate the contribution of surface, Coulomb, symmetry energy, temperature, and entropy to the nuclear incompressibility. For the particular case $\alpha = 1/3$, exact solutions are also easy to obtain in several situations which lead to a quadratic equation for the equilibrium point R when E is written in terms of R.

C. Nonzero temperature and entropy considerations

At nonzero T and with $T \ll E_F$, the kinetic energy is $E_K/A = (\frac{3}{5})E_F + (\frac{\pi^2}{4})T^2/E_F$ [31,36]. The energy per particle is

$$E/A = 21 \left(\frac{m}{m^*}\right) x^{2/3} + \left(\frac{\pi^2}{140}\right) \left(\frac{m^*}{m}\right) \frac{T^2}{x^{2/3}} - b_0 x + b_\alpha x^{1+\alpha}. \tag{11}$$

This equation cannot be used for very small x since the condition $T \ll E_F$ will fail. We can use the procedure of Sec. II B [7]. Taking $E_x(R, T) = 0.0517(m^*/m)T^2 R^2/A^{2/3}$ in Eqs. (7) and (6), we obtain the following expression for κ and x_m:

$$\kappa(\alpha) = R^2 \frac{d^2(E(R)/A)}{dR^2}\bigg|_T$$
$$= \kappa_0(\alpha, m^*) - 0.1034(m^*/m)T^2 R_0^2/A^{2/3}$$
$$\times \{Sk(\alpha)/\kappa_0(\alpha) + 1\}, \tag{12}$$
$$x_m(\alpha) = \rho_m/\rho_0 = (R_0/R_m)^3 = (1 + \Delta R_x/R_0)^{-3}$$
$$= (1 - 0.1034(m^*/m)T^2/A^{2/3})^{-3}. \tag{13}$$

Note here that $x_m > 1$, i.e., the saturation density is larger than ρ_0 in an isothermal process. Since $A/(4\pi R_0^3/3) = \rho_0 = 0.15$ fm^3 and thus $R_0^2/A^{2/3} = 1.363$ fm^2 we have the following final simple results: For $m^* = m$,

$$\kappa(\alpha = 0) = 165 + 1.16T^2,$$
$$\kappa(\alpha = 1/3) = 234 + 1.32T^2, \tag{14}$$
$$\kappa(\alpha = 1) = 373 + 1.61T^2.$$

From Eq. (14) we see that the first term is very sensitive to α but the finite temperature correction is somewhat insensitive to α [7]. Since Sk/κ_0 is insensitive to α with a value of about -10, the second term does not scale with the first term in Eq. (14) as α changes. At $T = 7.5$ MeV the T dependent corrections in Eq. (14) are 65, 74, and 91 MeV while the first terms are 165, 234, and 373 MeV for $\alpha = 0$, $\alpha = 1/3$, and $\alpha = 1$. At low T, κ and ρ_m increase with T and thus the GMR energy

will be increased as T increases. However the T dependence of GMR energy would depend on a mean field used and also on its extrapolation to finite T. These results [Eqs. (12)–(14)] are very close to the exact solution result of Ref. [7]. The exact result is given by

$$\kappa(T) = -42x_m^{2/3} + 0.705T^2/x_m^{2/3} + 9\alpha(1+\alpha)b_\alpha x_m^{1+\alpha}. \quad (15)$$

At $T = 2.5$, 5, and 7.5 MeV, and for $\alpha = 1/3$, the values of κ are 242, 265, and 302 MeV, respectively. The value of $\alpha = 1/3$ and $T = 0$ is also useful. It leads to an energy whose minimum can be obtained from a quadratic equation in the radius using $\rho = A/(4\pi R^3/3)$. Namely, $E(R)/A$ is

$$E(R)/A = 21\left(\frac{R_0}{R}\right)^2 - 106\left(\frac{R_0}{R}\right)^3 + 69\left(\frac{R_0}{R}\right)^4$$
$$= 28.626\frac{A^{2/3}}{R^2} - 168.7\frac{A}{R^3} + 128.2\frac{A^{4/3}}{R^4}. \quad (16)$$

At finite T a term $(\pi^2/140)T^2(R/R0)^2 = 0.0517T^2R^2/A^{2/3}$ is added to the right side of Eq. (16). When T is replaced with entropy per particle S/A then this T dependent term becomes $4.836(S/A)^2A^{2/3}/R^2$ since $S = (\pi^2/2)TA/E_F$ at low T. This S/A term can simply be added to the first term on the right side of Eq. (16) since both have the same $A^{2/3}/R^2$ dependence. In the limit $\alpha \to 0$, x_m satisfies the equation $14(x^{2/3} - x) + 23x\log x = (\pi^2/210)T^2/x^{2/3}$. At $T = 0, 2.5$, 5, 7.5, the $x_m = 1, 1.0157, 1.0598$, and 1.1251, while $\kappa = -42x^{2/3} + 0.705T^2/x^{2/3} + 207x$ has values 165, 172, 193, 224 MeV, respectively.

At fixed entropy, the second derivative of $E(R)/A$ has a very different behavior than at fixed T. Namely, it decreases with R for a given S/A. This can easily be seen by noting that $E_x(R, S) = 4.836(S/A)^2A^{2/3}/R^2$ compared to $E_x(R, T) = 0.0517T^2R^2/A^{2/3}$. Here we used $S = \frac{\pi^2}{2}TA/E_F$ at low T. Because R now is in the denominator, the first derivative is negative and κ_Q will decrease

$$\kappa_Q = R^2\frac{d^2(E(R)/A)}{dR^2}\bigg|_S$$
$$= \kappa_0(\alpha) + 9.672(S/A)^2A^{2/3}/R_0^2\{Sk(\alpha)/\kappa_0(\alpha) + 5\}, \quad (17)$$
$$x_m = \left(1 + 9.672(S/A)^2A^{2/3}/R_0^2\right)^{-3}. \quad (18)$$

Note here that $x_m < 1$, i.e., the saturation density is lower than ρ_0 in an adiabatic process in contrast to an isothermal process. We have the following final results at $m^* = m$:

$$\kappa_Q(\alpha = 0) = 165 - 30(S/A)^2,$$
$$\kappa_Q(\alpha = 1/3) = 234 - 38(S/A)^2, \quad (19)$$
$$\kappa_Q(\alpha = 1) = 373 - 53(S/A)^2$$

in MeV.

At higher T, the nearly degenerate Fermi gas kinetic energy term is replaced by a virial expansion in $\rho\lambda^3$, where λ is the quantum wavelength. Namely, $E_K/A = (3/2)T(1 + \sum c_n(\rho\lambda^3/4)^n)$ where the sum over $n = 1, 2, 3, \cdots$ has coefficients: $c_1 = 1/2^{5/2} = 0.177$, $c_2 = (1/8 - 2/3^{5/2}) = -3.3 \times 10^{-3}$, $c_3 = 1.11 \times 10^{-4}$, $\cdots$ [31,36]. These coefficients come

from antisymmetrization effects only. The role of clusters in the virial expansion is discussed in Ref. [7]. Since the c_n's become small rapidly, we will keep terms up to c_2. Then κ is given by

$$\kappa(T) = -(2196/T^2)x_m^2 + 9\alpha(1+\alpha)b_\alpha x_m^{1+\alpha}. \quad (20)$$

The x_m is again the minimum of E/A, but now evaluated with the new kinetic energy. The x_m is affected by both the c_1 and c_2 terms. A limiting value of κ can be obtained by taking T very large where an ideal gas $E_K/A = (3/2)T$ leads to very simple results [7].

We also investigate the case of constant entropy in the ideal gas limit using the Sackur-Tetrode law [37]: $S/A = 5/2 - \ln(\lambda^3\rho/4)$. This law connects T to ρ or V as $T = C_S\rho^{2/3}$. Here $C_S = [2\pi(\hbar c)^2/(mc^2)]\exp[(2/3)(S/A) - 5/3]$. Then $E_K/A = (3/2)C_S\rho^{2/3}(1 + \rho^{2/3}\sum c_n([2\pi\hbar^2/mC_S]^{3/2})^n)$. The resulting $E(R)/A$ has a structure similar to the result for a degenerate Fermi gas since both have a $\rho^{2/3}$ dependence for the kinetic energy term but with different coefficients. This feature and a similar result at lower T suggests a duality in the energy per particle EOS at constant entropy and its associated κ_Q and the $T = 0$ EOS and its associated constant T κ [7].

It should be noted that the x_m's for the constant T case are usually close to 1 except at high T, and greater then 1, so that the minimum of the energy occurs at a density that is greater than ρ_0. This feature is not the case for the Helmholtz free energy $F = E - TS$. Since $P = -dF/dV$, F is minimum when $P = 0$. At low T, the entropy $S = (\pi^2/2)TA/E_F$, and thus $F_K = E_K - TS$, the part of the Helmholtz free energy coming from the kinetic energy part of E, is $F_K/A = (3E_F/5)(1 - (5\pi^2/12)(T/E_F)^2)$. The complete $F/A = F_K/A + U/A$, where U/A is given by Eq. (2). The free energy per particle at low T can be obtained from Eq. (11) by just changing the sign of the T^2 term: $F/A = 21x^{2/3} - (\pi^2/140)T^2/x^{2/3} - b_0x + b_\alpha x^{1+\alpha}$. As with the E/A equation, this F/A equation is only valid when $T \ll E_f$. For higher T, we use the virial expansion for E_K involving the coefficients c_n mentioned above. Similarly, the $TS/A = (5/2)T - T\ln(\lambda^3\rho_0x/4) + c_1(\lambda^3\rho_0x/4)/2 + \cdots$ [31,36]. We note that the minimum of the free energy $F(T, V)$ and energy $E(T, V)$ at constant T don't coincide except at $T = 0$. However, the minimum of free energy $F(T, V)$ at constant T and the minimum of energy $E(S, V)$ at constant S are at $P = 0$. The minimum of F is at subnuclear density, or $x_m < 1$, at nonzero T. For $\alpha = 1/3$ and at $T = 0, 2.5, 5$, 7.5 MeV the $x_m = 1, 0.988, 0.951, 0.878$ and the curvature function $\kappa = 9x_m^2(d^2(F/A)/dx_m^2) = 234, 225, 199, 150$ MeV, respectively. The F also has a maximum at even smaller x than its minimum point. The pressure $P = \rho xd(F/A)/dx = \rho_0x^2d(F/A)/dx$ and is $P = 2.1x^{5/3} + 0.00705T^2x^{1/3} - 0.15b_0x^2 + 0.15(1+\alpha)b_\alpha x^{2+\alpha}$ for $T \ll E_F$. At high T and/or small x, the $P = \rho_0xT\{1 + \sum c_n(\lambda^3\rho_0/4)^nx^n\} - 0.15b_0x^2 + 0.15(1+\alpha)b_\alpha x^{2+\alpha}$. The spinodal line is obtained by setting $dP/dV = 0$ or $d^2(F/A)/dV^2 = 0$. The critical point occurs at the top of the spinodal line where $d^2P/dV^2 = 0$ also. Similar remarks also apply to $E(S, V)$ with S held constant and for different values of S. The specific heat at constant volume is also easily obtained from the above results. At low T, C_V is

linear in T and is $(\pi^2/2)T/E_F$. The behavior of C_V at higher T can be obtained from the virial expansion of E_K/A given above. At very high T the $C_V \to 3/2$. The $T = 0$ limit where $C_V = 0$ is connected to this very high T limit of $3/2$ in a monotonic way with increasing T with no peak or cusp in C_V. This is to be compared with the strong peak found in C_V when inhomogeneities are included [38]. In this latter case, cost functions arising from the surface energy associated with the formation of clusters are responsible for the very large peak as mentioned. These calculation [38] were done at a freeze-out density of about $1/3$ nuclear matter density. At high density, i.e., above nuclear matter, heated matter may first form bubbles in a uniform background of liquid. The formation of a bubble has associated with it a surface energy. This aspect will be considered in a future paper.

III. EFFECTIVE MASS EFFECTS

An effective mass m^* changes the kinetic energy term. In Sec. II, we presented results with m^* independent of density to keep analytic equations simple. In this section we include a density dependence in m^*. This will introduce another dependence into our expressions which has to do with the choice of the density behavior of m^*/m with ρ/ρ_0. If we parametrize it as $m^*/m = 1/(1 + r(\rho/\rho_0)^\gamma)$, then at zero temperature the E/A will simply read $E/A = (E_K/A)x^{2/3} + r(E_K/A)x^{2/3+\gamma} - b_0 x + b_\alpha x^{1+\alpha}$. The results of Sec. II correspond to the choice $\gamma = 0$ with $m/m^* = 1 + r$ or $r = m^*/m - 1$. The $E_K/A = (3/5)E_F = 21$ MeV, the value with the unrenormalized mass, and the effective mass correction appears in the additional second term with a $\gamma + 2/3$ power dependence. The r is a constant and has a value of $1/2$ when $m^*/m = 2/3$ at ρ_0. If $\alpha = \gamma - 1/3$, then this additional effective mass term could be simply added to the b_α term. Since the coefficients are determined to give the correct binding at $\rho = \rho_0$ or $x = 1$ where E/A has a minimum, the b_α coefficient must be adjusted also. The final incompressibility is still given by Eq. (4) with $\alpha = \gamma - 1/3$.

If we parametrize m^*/m with $\gamma = 1$ then for $\alpha = 2/3$ the effective mass term can be added to the b_α term and the final incompressibility is now given by Eq. (4) with $\alpha = 2/3$. Similarly if we parametrize m^*/m with $\gamma = 2/3$ then for $\alpha = 1/3$ the effective mass term can be added to the b_α term and the final incompressibility is now given by Eq. (4) with $\alpha = 1/3$. If $\gamma = 1/3$, the r dependent effective mass term can be added to b_0 term and there is no effect on κ and Sk. For general α and m^*/m: $\alpha b_\alpha = (E_B/A) + (1 - [3\gamma - 1]r)(E_K/A)/3$ and $b_0 = (1 + r)(E_K/A) + (E_B/A) + b_\alpha$ and

$$
\begin{aligned}
\kappa_0(\alpha, r) =\ & 9E_B/A + (1 + [3\gamma - 1]^2 r)E_K/A \\
& + \alpha(9E_B/A + 3(1 - [3\gamma - 1]r)E_K/A) \\
& \to 3(55 + 69\alpha + 7r[3\gamma - 1]([3\gamma - 1] - 3\alpha)). \quad (21)
\end{aligned}
$$

For $\gamma = 0$ and $r = m/m^* - 1$, the result of Eq. (21) reduces to the result of Eq. (5). For $\gamma = 1/3$ or for $\alpha = \gamma - 1/3$ this κ is r independent for reasons already given. When we parametrize m^*/m with $\gamma = 1$, which is the same case as considered in Ref. [32], for $\alpha = 1/3$, $\kappa = 234 + 42r$ and for $r = 1/2$, $\kappa = 255$ MeV. The skewness is also affected by the effective mass:

$$
\begin{aligned}
Sk =\ & -27(E_B/A)(3 + 4\alpha + \alpha^2) - (E_K/A)(11 + 36\alpha + 9\alpha^2 \\
& + r[3\gamma - 1]([3\gamma - 1] - 3\alpha)([3\gamma + 11] + 3\alpha)) \\
& \to -3(509 + 828\alpha + 207\alpha^2) - 21r[3\gamma - 1] \\
& \times ([3\gamma - 1] - 3\alpha)([3\gamma + 11] + 3\alpha). \quad (22)
\end{aligned}
$$

For $\gamma = 0$ and $r = m/m^* - 1$, the result of Eq. (22) reduces to the result of Eq. (10). For $\gamma = 1/3$ or for $\alpha = \gamma - 1/3$ this is independent of r and for $\alpha < \gamma - 1/3$, Sk/κ_0 decreases with r; for $\alpha > \gamma - 1/3$, Sk/κ_0 increases with r.

The effective mass also appears in the temperature dependent part of the energy per particle, modifying the additional term $E_x(R, T)$ to now be given by $E_x(R, T) = 0.0517(T^2 R^2/A^{2/3})(1/(1 + r(R_0/R)^{3\gamma}))$. The last factor in this expression is the change the effective mass produces. The result of Eq. (7) can be used to calculate the new κ. This leads to the following equation:

$$
\kappa = \kappa_0(\alpha, r) - 0.0705T^2 \left[\frac{(2 + ([3\gamma - 3][3\gamma + 2] + 10)r - [3\gamma - 1][3\gamma + 2]r^2)}{(1 + r)^3} + \frac{(2 + [3\gamma + 2]r)}{(1 + r)^2} \left(\frac{Sk(\alpha, r)}{\kappa_0(\alpha, r)} \right) \right]. \quad (23)
$$

For the case of $\gamma = 1$, at $T = 7.5$ MeV and $r = 1/2$: $\kappa = 283$ MeV for $\alpha = 0$, 335 MeV for $\alpha = 1/3$, 388 MeV for $\alpha = 2/3$ and 442 MeV for $\alpha = 1$. The $T = 0, r = 1/2$ values of κ are 207, 255, 303, and 351 MeV and

values of Sk are 2115, 2739, 3459, and 4275 MeV for each of these respective α values. For a fixed entropy, $E_x(R, T) = 4.836(S/A)^2(A^{2/3}/R^2)(1/(1 + r(R_0/R)^{3\gamma}))$ and thus

$$
\begin{aligned}
\kappa_Q =\ & \kappa_0(\alpha, r) \\
& + 3.548(S/A)^2 \left[\frac{(10 - ([3\gamma - 2][3\gamma + 9] - 2)r + [3\gamma - 2][3\gamma - 5]r^2)}{(1 + r)^3} + \frac{(2 - [3\gamma - 2]r)}{(1 + r)^2} \left(\frac{Sk(\alpha, r)}{\kappa_0(\alpha, r)} \right) \right]. \quad (24)
\end{aligned}
$$

At high T with effective mass, $E_K/A = (3/2)T(1 + \sum_n c_n(\rho(1 + r(\rho/\rho_0)^\gamma)^{3/2}\lambda^{3/4})^n)$ due to the mass dependence in the De Broglie wavelength λ. Due to the density dependence of the effective mass the c_1 term contributes directly to incompressibility now. Then κ is, up to c_1 term, given by

$$\kappa(T) = \frac{1505}{T^{1/2}} \frac{x_m}{(1 + rx_m^\gamma)^{1/2}} r x_m^\gamma \frac{3}{2} \gamma \left(1 + \gamma + \left(1 + \frac{3}{2}\gamma\right) r x_m^\gamma\right)$$
$$+ 9\alpha(1 + \alpha)b_\alpha x_m^{1+\alpha}. \tag{25}$$

The x_m is again the minimum of E/A, but now evaluated with the new kinetic energy. At fixed entropy $E_K/A = (3/2)C_S\rho^{2/3}(1 + \sum c_n([2\pi\hbar^2/mC_S]^{3/2}(1 + r(\rho/\rho_0)^\gamma)^{3/2})^n)$. The lowest order term having effective mass dependence is the c_1 term. Up to this term,

$$\kappa_Q(S) = -3C_S\rho_0^{2/3}x_m^{2/3} + \frac{3}{2}C_S\rho_0^{2/3}\left(\frac{2\pi\hbar^2}{mC_S}\right)^{3/2}$$
$$\times \frac{c_1 x_m^{2/3}}{(1 + rx_m^\gamma)^{1/2}}\left(-2 + \frac{(27\gamma^2 + 9\gamma - 8)}{2}rx_m^\gamma\right.$$
$$\left. + \frac{(9\gamma + 4)(9\gamma - 2)}{4}r^2 x_m^{2\gamma}\right) + 9\alpha(1 + \alpha)b_\alpha x_m^{1+\alpha}. \tag{26}$$

IV. EXCLUDED VOLUME EFFECTS

Excluded volume effects appear in the Van der Waals approach and modify the adiabatic incompressibility, leading to an equation of state of the form, for a given S,

$$E/A = (3/2)C_S[\rho/(1 - b\rho)]^{2/3} - a_0\rho, \tag{27}$$

where $C_S = [2\pi(\hbar c)^2/(mc^2)]\exp[(2/3)(S/A) - 5/3]$ which connects T to ρ or V as $T = C_S\rho^{2/3}$ for ideal gas. An expansion of $1/(1 - b\rho)^{2/3} = 1 + (2/3)(b\rho) + (5/9)(b\rho)^2 + \cdots$ leads to an infinite series of repulsive terms with various powers of the density. The numerator has $\rho^{2/3}$ so that additional terms such as $\rho^{5/3}, \rho^{8/3}, \rho^{11/3}, \ldots$ are generated in this expansion. These terms collectively give rise to a much high incompressibility. If we had truncated the series at $\rho^{5/3}$, then the resulting EOS would be similar to a Skyrme with $\alpha = 2/3$. A Skyrme with $\alpha = 2/3$ has an incompressibility of 303 MeV which is rather high compared to the often quoted range of $210 \sim 270$ MeV.

To explore higher density repulsive terms in a $T = 0$ Skyrme parametrization, we can use the duality between the EOS at constant entropy and the EOS at zero T [7]. Thus we take the specific form of Eq. (27) with $(3/2)C_S$ replaced by a constant parameter a_K at zero T as a very simple model for studying excluded volume like effects in our E/A. We can fix the coefficient a_0 and a_K to give a binding energy per particle of 16 MeV at $\rho = 0.15$ fm^3: $b_K = a_K(\frac{\rho}{1-b\rho})^{2/3} = 48(\frac{1-b\rho}{3b\rho-1})$ and $b_0 = a_0\rho = \frac{32}{(3b\rho-1)}$. The resulting κ is

$$\kappa = 96(6b\rho - 1)/((3b\rho - 1)(1 - b\rho)). \tag{28}$$

This κ has a minimum value of 754 MeV at $b = 3.60$. Alternatively if we fix a_K to be the $T = 0$ Fermi kinetic energy coefficient, $a_K\rho_0^{2/3} = 21$ MeV, and determine a_0 and b by having correct binding energy at saturation we get $b = 3.61$ and an incompressibility of 754 MeV. The two methods result

in very similar values for b and κ. An unrealistically high incompressibility of 754 MeV is equivalent to having a single Skyrme repulsive term with $\alpha = 2.85$ from Eq. (4). This high incompressibility is unreasonable but it can be moderated by having higher order attractive terms in the density which, for example, can come from higher order attractive correlations of nucleons [7]. The presence of both attractive and repulsive terms in various orders of the expansion of the equation of state leads to a very complicated density dependence which is beyond the scope of the present development presented here. The skewness is given by

$$Sk = 192\left(\frac{b\rho - 1}{3b\rho - 1}\right)\left[6 + 31\left(\frac{b\rho}{1 - b\rho}\right) + 45\left(\frac{b\rho}{1 - b\rho}\right)^2\right.$$
$$\left. + 20\left(\frac{b\rho}{1 - b\rho}\right)^3\right] + \frac{1920}{3b\rho - 1}. \tag{29}$$

The effects of temperature and entropy dependences coming from a kinetic energy can be obtained from Eqs. (12) and (17) with the κ_0 and Sk given above.

V. ASYMMETRIC AND FINITE SYSTEMS

For N, Z asymmetric systems with Coulomb and finite size effects included, the $T = 0$ κ is reduced [3,4]. This was qualitatively explained in Ref. [3] as a reduction in the binding energy term in Eq. (11). Detailed calculation done in Ref. [4] confirmed this feature. In this part of the paper we will present some very simple equations for the contribution of surface, symmetry and surface symmetry, and Coulomb and surface Coulomb. The surface, symmetry energy and Coulomb terms each reduce the binding energy per particle from 16 MeV to 8 MeV. For light nuclei around $A = 27$, the surface energy/$A \sim 6$ MeV, the symmetry energy/$A \sim 0$ MeV and the Coulomb energy/$A \sim 1.5$ MeV. For heavy nuclei around $A = 216$, the above numbers are changed to ~ 3 MeV, ~ 1 MeV and ~ 4 MeV for the surface, symmetry, and Coulomb terms, respectively. We will find a somewhat similar pattern of relative importance in κ, but the surface term will be dominant even for large nuclei. The Coulomb is of next importance followed by the symmetry term for nuclei near the valley of beta stability. Surface symmetry and surface Coulomb terms will also be included.

A. Surface contribution

First, let us investigate the role of finite size or surface correction, before considering the full complexity of the problem with Coulomb and symmetry energy terms included in the energy. The surface energy can be studied by using $\rho(r) = \rho_c/(\exp[(r - R)/a] + 1)$ as a density profile. Here ρ_c is the central density and a is a parameter that determines the surface thickness. This density can then be used in a Skyrme energy density functional: $E(\rho) = a_K\rho^{5/3} - a_0\rho^2 + a_\alpha\rho^{2+\alpha}$. The a_K is the kinetic energy constant and is taken as $a_K\rho_0^{2/3} = (3/5)E_F = 21$ MeV. The a_0 and a_α are given above. The energy per particle is the integral of $E(\rho)$ over all space divided by A. The resulting integrals can be simplified by an integration by parts and very accurately approximated by noting that the

derivative of the Woods-Saxon profile gives a narrow peak of width a in the vicinity of R. A substitution $y = (r - R)/a$ and a change in the lower limit of integration from $-R/a$ to $-\infty$ gives a very accurate approximation to these integrals when $R \gg a$, as is usually the case. As an example, the energy per particle from the a_0 attractive interaction term is $(a_0/A)(4\pi R^3/3)\rho_c^2[1 - 3(a/R) + \pi^2(a/R)^2 - \pi^2(a/R)^3]$. The central density ρ_c, R, and a are also connected since the integral of ρ over all space must give A. Using the integration procedure just mentioned, the condition $(4\pi R^3/3)\rho_c(1 + \pi^2(a/R)^2) = A$ is obtained. Note that in this last result the linear and cubic terms in a/R are absent in the constraint equation. This is not the case for the integrals involving the kinetic energy density, the a_0 and the a_α terms. Each of these has a form involving surface thickness corrections that is $(1 - c_1(a/R) + c_2(a/R)^2 - c_3(a/R)^3)$. For the kinetic energy term $c_1 = 2.2769$, $c_2 = 9.1046$, $c_3 = 7.805$. For the a_0 term $c_1 = 3$, $c_2 = c_3 = \pi^2$, and for the a_α term, for $\alpha = 1/3$, $c_1 = 3.585$, $c_2 = 10.819$, and $c_3 = 11.642$, for $\alpha = 1$, $c_1 = 9/2$, $c_2 = \pi^2 + 3$, and $c_3 = 3\pi^2/2$. The energy per particle can be rewritten as

$$E(R)/A = 28.626 C_K(A^{2/3}/R^2) - (58.89 + 36.61/\alpha) \times C_0(A/R^3) + (23/\alpha)C_\alpha(A/(0.8565R)^3)^{\alpha+1}. \tag{30}$$

The C_K, C_0, and, C_α are the surface correction factors, each involving their corresponding c_1, c_2, c_3 terms as $[1 - c_1(a/R) + c_2(a/R)^2 - c_3(a/R)^3]/[1 + \pi^2(a/R)^2]^j$, where the denominator arises from the constraint condition between R, ρ_c, and A. The $j = 5/3$ for the kinetic term, $j = 2$ for the a_0 term and $j = \alpha + 2$ for the a_α term. For $a/R \ll 1$, the surface energy can be approximated by keeping linear terms in a/R to give

$$\begin{aligned} E_{\mathrm{diff}}(R)/A = &-(a/R)(21 \times 2.2769(R_0/R)^2 - (37 + 23/\alpha) \\ &\times 3(R_0/R)^3 + (23/\alpha)(3.585)(R_0/R)^{3\alpha+3}) \\ = &-(a/R)(65.18A^{2/3}/R^2 - 506.11A/R^3 \\ &+ 459.65A^{4/3}/R^4) \end{aligned} \tag{31}$$

for $\alpha = 1/3$. This $E_{\mathrm{diff}} \to 0$ as $a \to 0$ and $E_{\mathrm{diff}} = 10.36A^{2/3}$ MeV at $a = 0.53$ fm and $\rho_0 = 0.15$ fm^{-3}. To have correct surface energy of $\sim 20A^{2/3}$ MeV, we also consider a gradient dependent energy density of $B(\nabla\rho)^2 = -B\rho\nabla^2\rho$. With the Woods-Saxon density profile this becomes

$$\begin{aligned} E_{\mathrm{grad}}/A &= B\frac{\rho_c}{2Ra}\frac{[1 + (a/R)^2(\pi^2/3 - 2)]}{[1 + \pi^2(a/R)^2]^2} \\ &\approx B\frac{\rho}{2Ra} = B\frac{3}{8\pi a}\frac{A}{R^4} = \frac{3B}{8\pi a^2}\frac{a}{R}\frac{A}{R^3} \\ &= \frac{3B}{8\pi a^2}\frac{a}{A^{1/3}}\left(\frac{4\pi\rho_0}{3}\right)^{4/3}\left(\frac{R_0}{R}\right)^4. \end{aligned} \tag{32}$$

The approximate form of this term is for $a/R \ll 1$ and is the same form as the a_0 term in E_{diff}. The total surface energy then becomes

$$E_{\mathrm{surf}} = 4\pi R^2 \sigma(\rho) = 4\pi r_0^2 A^{2/3}\sigma(\rho) = E_{\mathrm{diff}} + E_{\mathrm{grad}}. \tag{33}$$

The surface energy has a contribution from the gradient term that varies with a as $1/a$ while the diffuseness term

varies directly with a. Note that as the surface gets sharper or $a \to 0$, the gradient term becomes very large and the decrease in binding from the surface diffuseness becomes small. Alternatively, as the surface thickness a becomes large, the gradient term is reduced while the other term increases. These behavior implies an extremum condition for E_{surf} versus a. We take this extremum to be at $a = 0.53$ fm and use this to determine B. This extremum condition on E_{surf} leads to the result that the gradient term and the the reduction in binding energy from diffuseness make equal contributions to E_{surf} at its saturation. For $a = 0.53$ fm, with the approximated form of E_{grad}, the $B = 85.47$ MeV fm^5 and $E_{\mathrm{diff}} = E_{\mathrm{grad}} = 10.36A^{2/3}$ MeV. The surface energy E_{surf} is the same form as Eq. (31) with modified coefficient of a_0 term to be 542.43, i.e.,

$$\begin{aligned} E_{\mathrm{surf}}(R)/A = &-(a/R)(65.18A^{2/3}/R^2 - 542.43A/R^3 \\ &+ 459.65A^{4/3}/R^4). \end{aligned} \tag{34}$$

The incompressibility will also have two contributions each from E_{diff} and E_{grad}. To have the same surface energy $E_{\mathrm{surf}} = E_{\mathrm{diff}} + E_{\mathrm{grad}}$ for other values of α, a can be determined to have $E_{\mathrm{diff}} = -(4\pi\rho_0/3)^{1/3}(a/A^{1/3})(21 \times 2.2769 - (37 + 23/\alpha)3 + (23/\alpha)C_{\alpha 1}) = 10.36/A^{1/3}$ with $C_{\alpha 1}$ the coefficient of a/R term in C_α. Then the parameter B can be set by $E_{\mathrm{grad}} = \frac{3B}{8\pi a^2}(\frac{4\pi\rho_0}{3})^{4/3}\frac{a}{A^{1/3}} = 10.36/A^{1/3}$.

The values of κ at the minimum E/A can be obtained from Eq. (30) with E_{grad} included. The energy $E(R)/A$ with E_{grad} is

$$\begin{aligned} E(R)/A = &\, 21C_K(a)(R_0/R)^2/D^{5/3} - (37 + 23/\alpha)C_0(a) \\ &\times (R_0/R)^3/D^2 + (23/\alpha)C_\alpha(a)(R_0/R)^{3\alpha+3}/D^{\alpha+2} \\ &+ 10.36C_G(a)(R_0/R)^4/D^2, \end{aligned} \tag{35}$$

where

$$\begin{aligned} C_K(a) &= 1 - 2.2769(a/R) + 9.1046(a/R)^2 - 7.805(a/R)^3, \\ C_0(a) &= 1 - 3(a/R) + \pi^2(a/R)^2 - \pi^2(a/R)^3, \\ C_{1/3}(a) &= 1 - 3.585(a/R) + 10.819(a/R)^2 - 11.642(a/R)^3, \\ C_1(a) &= 1 - (9/2)(a/R) + (\pi^2+3)(a/R)^2 - (3\pi^2/2)(a/R)^3, \\ C_G(a) &= 1 + (\pi^2/3 - 2)(a/R)^2, \\ D(a) &= 1 + \pi^2(a/R)^2. \end{aligned} \tag{36}$$

Since E_{surf} is not a small quantity we need exact calculation here. For $A = 216$, 125, and 64 with $a = 0.53$ fm, the respective values of κ are 145, 128, and 103 in MeV, at the minimum values R_m of E/A $R_m = 6.888$, 5.709, and 4.527 or $x_m = 1.0518$, 1.0691, and 1.0981, respectively. These values are well below the value of $\kappa = 234$ without finite size corrections. An approximate expression for these κ is given by

$$\kappa = \kappa(0) - 2.26\kappa(0)/A^{1/3}, \tag{37}$$

while approximate calculation up to first order of $a/R = (4\pi\rho_0/3)^{1/3}(R_0/R)(a/A^{1/3})$ gives $\kappa = \kappa(0)(1 - 2.01/A^{1/3})$. The form of this result not only applies for $\alpha = 1/3$, but also for softer EOS or smaller α, and harder EOS or larger α. The $\kappa(0)$ is κ at $R \to \infty$ which is the infinite A κ. Thus $\kappa(0) = 165$, 234, 372 for $\alpha = 0$, 1/3, 1, respectively. The

simple result given by the equation comes from the observation that the surface correction coefficient scales with the $\kappa(0)$ coefficient for different α. This observation came from the detailed calculation that were done for various A from $A = 64$ to $A = 216$ and various values of α. For example, when $\alpha = 1$ $\kappa(0) = 372$ and detailed calculations gave $\kappa = 249$, 224, 188 for $A = 216, 125, 64$ with $a = 0.478$ fm which can be approximated as $\kappa = \kappa(0)(1 - 1.98/A^{1/3})$.

Surface corrections to the $T = 0$ nuclear matter compressibility have been also considered in several recent papers. Reference [39] gives a wide range of values of surface to volume contributions. Specifically, their ratio K_{sf}/K_∞ runs from ~ 0.5 to ~ 1.8, depending on the force, where $\kappa = K_\infty - K_{sf}/A^{1/3}$. Reference [40] has the ratio of surface to volume coefficients with a Seyler-Blanchard interaction at ~ 1.2, while Ref. [41] finds values ~ 1.6 and 1.8 for two Skyrme forces. Reference [42] has this ratio at 1.8 for the parameter set NL1 in a relativistic Hartree approach. Our results for this ratio is ~ 2. Reference [43] gives a very large value of the surface incompressibility coefficient which is -800 MeV, which was questioned in Ref. [41].

The temperature and entropy dependence of the incompressibility can be evaluated using Eq. (7) for the T dependent kinetic energy with the κ_0 and Sk determined above for the surface energy included. Beside the kinetic energy, the surface energy itself can have a temperature dependence which needs further investigation.

B. Symmetry energy contribution

We next investigate the role of symmetry energy terms. Symmetry energy terms appear in both the kinetic energy and potential energy parts of the total energy. The symmetry energy appears as $(N - Z)^2/A^2$ in the binding energy per particle in the well-known mass formulae. At low temperatures, $T \ll E_F$, and for proton fractions $y = Z/A$, such that $|2y - 1| \ll 1$, the kinetic energy per particle is

$$E_K/A = 21x^{2/3}(1 + (5/9)(2y - 1)^2) + (\pi^2/140)(T^2/x^{2/3})(1 - (1/9)(2y - 1)^2). \quad (38)$$

At higher temperatures, an expansion in $\rho\lambda^3$ is more appropriate and the kinetic energy per particle now becomes [36]

$$E_K/A = (3T/2)\{1 + c_1(\lambda^3\rho_0/4)x(1 + (2y - 1)^2) + c_2(\lambda^3\rho_0/4)^2x^2(1 + (2y - 1)^2) + \cdots\}. \quad (39)$$

The y dependence in Eq. (39), to the expansion order c_2, is not restricted to $|2y - 1| \ll 1$, i.e., it is valid for any y. The potential energy per particle, U/A, is [36]

$$U/A = -b_0 x + b_\alpha x^{1+\alpha} + (2/3)(1/2 + x_0)b_0(2y - 1)^2 x - (2/3)(1/2 + x_3)b_\alpha(2y - 1)^2 x^{\alpha+1}. \quad (40)$$

The x_0 is a parameter which we fix to give the empirical symmetry energy coefficient in the binding energy mass formula. This symmetry energy coefficient is taken to be 25 MeV. Note that the kinetic energy contributes to this value and from Eq. (38) is $(35/3)$ MeV at $x = 1$ and $T = 0$. The remaining $(40/3)$ MeV comes from the third and fourth term on the right hand side (rhs) of Eq. (40). Thus $(40/3) = (2/3)$

$(1/2 + x_0)b_0 - (2/3)(1/2 + x_3)b_\alpha$ then x_0 and x_3 satisfy $(37 + 23/\alpha)x_0 - (23/\alpha)x_3 = 3/2$. For $\alpha = 1/3$, $106x_0 - 69x_3 = 3/2$. If $x_3 = 1$, which is the value taken for Skyrme forces ZR1, ZR2, ZR3 [44], then $x_0 = 0.6651$. If $x_3 = -1/2$, so that the fourth term in Eq. (40) is absent [36], then $x_0 = -0.3113$. If $x_0 = -1/2$, so that the third term is now absent, then $x_3 = -0.7899$. The nuclear incompressibility involves the second derivative of E/A with respect to x or ρ so that the fourth term in Eq. (40), which involves $x^{\alpha+1}$, is very important in the asymmetric part of the nuclear incompressibility. Terms involving linear powers in x do not make explicit contributions, but they do affect the minimum point in the evaluation of E/A. However because of the uncertainty in the value of x_3, and thus in the coefficient $1/2 + x_3$ that appears in the fourth term, the contribution of the asymmetry term to the nuclear incompressibility would seem to be somewhat uncertain. This turns out not to be the case because of cancellations so that the coefficient multiplying the $(1/2 + x_3)$ part in κ is small as will be found. We can evaluate the relative importance of the contribution of the symmetry energy to κ by isolating those terms associated with it and turning off the Coulomb and surface contributions. Let us consider the choice of a general x_3, which has an associated x_0 to give correct symmetry energy. Our $E(R)$ with symmetry terms included and is

$$E(R)/A = C_K(y)(R_0/R)^2 - C_0(y)(R_0/R)^3 + C_\alpha(y)(R_0/R)^{3(\alpha+1)} \quad (41)$$

with

$$C_K(y) = 21 + (35/3)(2y - 1)^2,$$
$$C_0(y) = (37 + 23/\alpha) - (40/3 + (46/3\alpha)(1/2 + x_3))(2y - 1)^2,$$
$$C_\alpha(y) = (23/\alpha) - (46/3\alpha)(1/2 + x_3)(2y - 1)^2. \quad (42)$$

Thus

$$0 = -2C_K(y)(R_0/R)^2 + 3C_0(y)(R_0/R)^3 - (3\alpha + 3)C_\alpha(y)(R_0/R)^{3\alpha+3}, \quad (43)$$
$$\kappa = 6C_K(y)(R_0/R)^2 - 12C_0(y)(R_0/R)^3 + (3\alpha + 3)(3\alpha + 4)C_\alpha(y)(R_0/R)^{3\alpha+3}, \quad (44)$$
$$Sk = -24C_K(y)(R_0/R)^2 + 60C_0(y)(R_0/R)^3 - (3\alpha + 3)(3\alpha + 4)(3\alpha + 5)C_\alpha(y)(R_0/R)^{3\alpha+3}. \quad (45)$$

For $\alpha = 1/3$, the minimum point of $E(R)/A$ is determined by a simple quadratic equation. The value of κ at this minimum is accurately given by, up to $(2y - 1)^2$ order,

$$\kappa = 234 - (300 - 16(1/2 + x_3))(2y - 1)^2 \quad (46)$$

with $x_m = (R_0/R_m)^3 = (1 - (0.271 - 0.197(1/2 + x_3))(2y - 1)^2)^3$. The small coefficient of 16 compared to 300 in front of the term involving x_3, means that uncertainties in x_3 do not have a pronounced effect on the value of κ. Similar results are also true for other values of α, with the second $1/2 + x_3$ term about 5% of the first term in the $(2y - 1)^2$ dependent part of κ. For general α, using either Eqs. (44) or (7),

$$\kappa = \kappa_0(\alpha) - \frac{182.9 + 403.33\alpha + 190\alpha^2}{0.7971 + \alpha}(2y - 1)^2 + \frac{(1/2 + x_3)(9.333 + 28\alpha)}{0.7971 + \alpha}(2y - 1)^2 \quad (47)$$

with $x_m = (1 - (0.3060 - 0.2222(1/2 + x_3))(2y - 1)^2/(0.7971 + \alpha))^3$. Surface symmetry corrections can also be easily included following the procedure given above. Since the volume and volume symmetry terms have the same density dependence, the surface corrections to each will scale in exactly the same way with their corresponding volume term. Thus we can rewrite the symmetry term in the above equation as $(300 - 16(1/2 + x_3))(2y - 1)^2(1 - 2.26)/A^{1/3}$ for $\alpha = 1/3$, where the $2.26/A^{1/3}$ part of this result is now the surface symmetry correction to the symmetry energy [c.f., Eq. (37)].

Symmetry corrections to the $T = 0$ nuclear matter compressibility can also be found in Refs. [40,42]. For example in Ref. [40], the symmetry energy coefficient is -393 MeV and its surface correction is 333 MeV. Reference [42] has the volume symmetry term at -677 MeV.

At nonzero but low T, Eq. (38) gives a small T dependent kinetic energy term $E_x = (\pi^2/140)T^2(R/R_0)^2(1 - (2y - 1)^2/9)$. Using Eq. (7),

$$\kappa(T) = \kappa(0) - (\pi^2/70)T^2(1 - (2y - 1)^2/9)(Sk/\kappa(0) + 1), \tag{48}$$

where $\kappa(0)$ is the incompressibility at zero T for an asymmetric system which is given by Eq. (47). On the other hand $E_x = (35/\pi^2)(S/A)^2(R_0/R)^2(1 - (2y - 1)^2/9)$ for a given entropy since $S/A = (\pi^2/2)T/E_F$. This gives the adiabatic incompressibility of

$$\kappa_Q(T) = \kappa(0) + (70/\pi^2)(S/A)^2(1 - (2y - 1)^2/9) \\ \times (Sk/\kappa(0) + 5). \tag{49}$$

At higher T, the energy is the sum of Eqs. (39) and (40) and we can neglect higher order terms in virial expansion keeping terms only up to c_2. Then κ is given by

$$\kappa(T) = 27\frac{c_2}{T^2}\left(\frac{2\pi\hbar^2}{m}\right)^3\left(\frac{\rho_0}{4}\right)^2 x_m^2(1 + (2y - 1)^2) \\ + 9\alpha(1 + \alpha)b_\alpha\left[1 - \frac{2}{3}\left(\frac{1}{2} + x_3\right)(2y - 1)^2\right]x_m^{1+\alpha}. \tag{50}$$

The x_m is again the minimum of E/A, but now evaluated with the new kinetic energy. The x_m is affected by both the c_1 and c_2 terms. A limiting value of κ can be obtained by taking T very large where an ideal gas $E_K/A = (3/2)T$ leads to very simple results. Adding this E_K/A to U/A given by Eq. (40) leads to a minimum x_m given by $x_m^\alpha = b_0[1 - (2/3)(1/2 + x_0)(2y - 1)^2]/(b_\alpha(1 + \alpha)[1 - (2/3)(1/2 + x_3)(2y - 1)^2])$. In this high T limit κ is given by the second term on the right side of Eq. (50) and is

$$\kappa(T \gtrsim 10\,\text{MeV}) \\ = 9\alpha b_0\left[1 - \frac{2}{3}\left(\frac{1}{2} + x_0\right)(2y - 1)^2\right] \\ \times \left(\frac{b_0\left[1 - \frac{2}{3}(\frac{1}{2} - x_0)(2y - 1)^2\right]}{(1 + \alpha)b_\alpha\left[1 - \frac{2}{3}(\frac{1}{2} + x_3)(2y - 1)^2\right]}\right)^{1/\alpha}. \tag{51}$$

At fixed entropy $E_K/A = (3/2)C_S\rho^{2/3}(1 + \sum c_n([2\pi\hbar^2/mC_S]^{3/2}(1 + (2y - 1)^2))$. The resulting $E(R)/A$ has a struc-

ture similar to the result for a degenerate Fermi gas since both have a $\rho^{2/3}$ dependence for the kinetic energy term but with different coefficients. This feature and a similar result at lower T suggests a duality in the energy per particle EOS at constant entropy and its associated κ_Q and the $T = 0$ EOS and its associated constant T κ even for asymmetric system.

C. Coulomb energy contribution

Next, let us consider the Coulomb energy contribution. For a uniformly charged sphere we have $E_c/A = (3/5)e^2(Z^2/A)/R = (3/5)(e^2/\hbar c)(\hbar c)(Z^2/A)/R$. As before, we turn off the other two contributions, now the surface and symmetry, to first study the isolated importance of the Coulomb interaction. In this case our energy per particle at $T = 0$ is

$$E(R)/A = 28.627A^{2/3}/R^2 - (58.89 + 36.61/\alpha)A/R^3 \\ + (23/\alpha)(A/(0.8565R)^3)^{\alpha+1} + 0.8628(Z^2/A)/R. \tag{52}$$

The Coulomb contribution to κ can be expressed in terms of a simple result, using Eq. (7),

$$\kappa_C = 4E_C/A + (E_C/A)(Sk/\kappa_0). \tag{53}$$

The skewness Sk, third derivative of $E_0(R)$, is negative. Here $E_0(R)$ is the energy without the Coulomb correction and κ_0 its associated incompressibility. The rhs is evaluated at the minimum without Coulomb corrections. For a uniform sphere of radius R, for $\alpha = 1/3$, the

$$\kappa = 234 - 4.75Z^2/A^{4/3}. \tag{54}$$

For $A = 216$, $Z = 86$, $R_m = 7.14$, and $\kappa = 207$ MeV; for $A = 125$, $Z = 50$, $R_m = 5.914$, and $\kappa = 215$ MeV; for $A = 64$, $Z = 28$, $R_m = 4.717$, and $\kappa = 220$ MeV. The Coulomb coefficient of -4.75 MeV can also be compared to the work of others. Reference [40] has this coefficient at -5.25 MeV while Ref. [42] gives a value of -5.83 MeV for it. If the density profile is taken to that of a Woods-Saxon form, the Coulomb energy can be approximated by $(3/5)(Z^2/A)(1 - (7\pi^2/6)(a/R)^2)/R$ [45]. Thus surface contributions appear both directly and indirectly, modifying the other terms. Adding surface contributions gives, using Eq. (7) for $\alpha = 1/3$,

$$\kappa = 234 - 4.75Z^2/A^{4/3} + 22.9Z^2/A^2. \tag{55}$$

Other values of α can also be easily evaluated. Neglecting surface-Coulomb we have for $\alpha = 1$, $\kappa = 372 - 6.25Z^2/A^{4/3}$ and for $\alpha = 0$, $\kappa = 165 - 3.875Z^2/A^{4/3}$. The ratio of the Coulomb coefficient of $Z^2/A^{4/3}$ to volume coefficient is not the same for each α so that a simple scaling result is not possible.

Since both the Coulomb energy and the T dependent part of kinetic energy are small the effects of these energies to the incompressibility are additive.

VI. CONCLUSIONS

In this paper we investigated the behavior of the asymmetric nuclear matter incompressibility at finite temperature and entropy using a mean field theory. The T dependence of incompressibility is important for a thermalized hot system produced in heavy ion collision or in supernovae explosions. The incompressibility is related to the curvature of the energy as a function of density, with the isothermal incompressibility having temperature held constant and the adiabatic incompressibility has entropy held constant. These two incompressibilities are the same at $T = 0$ or $S = 0$ but have very different behaviors with temperature T and entropy S. At all T or S these incompressibilities are very sensitive to the choice of α, the density dependent power of the repulsive interaction. Also important in the determination of κ are properties of the effective mass, such as its density dependence and its value at normal density. These properties are determined by two quantities called r and γ in Sec. III. For example, at $T = 0$, $\kappa = 234$ MeV for $\alpha = 1/3$ and for an effective mass $m^*/m = 1$ which is density independent. For $m^*/m = 2/3$ and m^* density independent, or $\gamma = 0$, $\kappa = 234$ MeV for a softer equation of state given by $\alpha = 13/53 \sim 1/4$. For density dependent m^* with $\gamma = 1$ and $r = 1/2$, $\kappa = 207, 255, 303$, and 351 MeV for $\alpha = 0, 1/3, 2/3$, and 1, respectively. However, for $\gamma = 1/3$, the $T = 0$ κ is independent of m^*/m or r. We look at extreme cases of a very soft equation of state where α goes to zero to a very hard equation of state with $\alpha = 1$. The range of κ is from 165 MeV to 372 MeV. For a moderate equation of state that is frequently used, $\alpha = 1/3$ and $\kappa = 234$ MeV. When a temperature dependence is included, the value of the isothermal incompressibility κ increases sharply with T. At low T or T much less than the Fermi energy, κ increases as T^2. For example at $T = 7.5$ MeV and $\alpha = 1/3$ with $m^*/m = 1$ and density independent or for $\gamma = 1/3$, κ is 302 MeV. By contrast, the adiabatic incompressibility was shown to decrease with increasing entropy and it eventually goes to zero. The adiabatic incompressibility is shown to arise from an equation of state (EOS) or energy per particle that has a structure that is similar to a $T = 0$ Fermi gas.

The incompressibility is also studied in finite systems where we investigated the role of surface, symmetry energy, and Coulomb energy effects. First we studied each of these terms separately to illustrate the relative importance of them. Of these three terms, the surface had the largest effect, followed by the Coulomb and then the symmetry term for nuclei in the valley of β stability. Uncertainties in the x_3 parameter of the Skyrme interaction were shown not to be so important. In our study of the surface term we used a Woods-Saxon density distribution. We found an approximation scheme to do various integrals (similar to the Sommerfeld method for Fermi gas integrals) that appear in our analysis and obtained very accurate analytic results for the surface contribution to the energy per nucleon.

[1] S. Das Gupta, A. Z. Mekjian, and M. B. Tsang, Adv. Nucl. Phys. **26**, 81 (2001).

[2] P. Danielewicz, R. Lacey, and W. G. Lynch, Science **298**, 1592 (2002).

[3] L. Zamick, Phys. Lett. **B45**, 313 (1973).

[4] J. P. Blaizot, D. Gogny, and B. Grammaticos, Nucl. Phys. **A265**, 315 (1976); J. P. Blaizot, Phys. Rep. **64**, 171 (1980).

[5] J. M. Pearson, Phys. Lett. **B271**, 12 (1991).

[6] H.-M. Müller, S. E. Koonin, R. Seki, and U. van Kolck, Phys. Rev. C **61**, 044320 (2000).

[7] A. Z. Mekjian, S. J. Lee, and L. Zamick, Phys. Lett. **B621**, 239 (2005).

[8] B. K. Agrawal, S. Shlomo, and V. KimAu, Phys. Rev. C **68**, 031304(R) (2003).

[9] B. K. Agrawal and S. Shlomo, Phys. Rev. C **70**, 014308 (2004).

[10] G. Colo, N. V. Giai, J. Meyer, K. Bennaceur, and P. Bonche, Phys. Rev. C **70**, 024307 (2004).

[11] G. Colo and N. VanGai, Nucl. Phys. **A731**, 15 (2004).

[12] G. Colo, P. F. Bortignon, N. VanGai, A. Bracco, and R. A. Broglia, Phys. Lett. **B276**, 279 (1992).

[13] J. P. Blazot, J. F. Berger, J. Descharge, and N. Girod, Nucl. Phys. **A591**, 435 (1995).

[14] Z. Ma, N. VanGai, A. Wandelt, and P. Vretenar, Nucl. Phys. **A686**, 173 (2001).

[15] D. Vretenar, A. Wandelt, and P. Ring, Phys. Lett. **B487**, 334 (2000).

[16] D. Vretenar, T. Niksic, and P. Ring, Phys. Rev. C **68**, 024310 (2003).

[17] G. A. Lalazissis, J. König, and P. Ring, Phys. Rev. C **55**, 540 (1997).

[18] J. Piekarewicz, Phys. Rev. C **69**, 041301(R) (2004); **66**, 034305 (2003).

[19] S. Shlomo and D. H. Youngblood, Phys. Rev. C **47**, 529 (1993).

[20] D. H. Youngblood, H. L. Clark, and Y. W. Lui, Phys. Rev. Lett. **82**, 691 (1994).

[21] D. H. Youngblood, Nucl. Phys. **A687**, 1c (2001).

[22] L. Shi, P. Danielewicz, and R. Lacey, Phys. Rev. C **64**, 034601 (2001).

[23] P. Danielewicz, nucl-th/0203002; nucl-th/0201032 (2002).

[24] M. B. Tsang, W. A. Friedman, C. K. Gelbke, W. G. Lynch, G. Verde, H. Xu, Phys. Rev. Lett. **86**, 5023 (2001); **92**, 062701 (2004).

[25] B. A. Li, Phys. Rev. C **69**, 034614 (2004).

[26] C. B. Das, S. Das Gupta, W. G. Lynch, A. Z. Mekjian, and M. B. Tsang, Phys. Rep. **406**, 1 (2005).

[27] A. Z. Mekjian, Adv. Nucl. Phys. **7**, 1 (1973).

[28] H. Muller and B. D. Serot, Phys. Rev. C **52**, 2072 (1995).

[29] S. J. Lee and A. Z. Mekjian, Phys. Rev. C **68**, 014608 (2003).

[30] J. B. Silva, A. Delfino, J. S. Sá Martins, S. Moss de Oliveira, and C. E. Cordeiro, Phys. Rev. C **69**, 024606 (2004).

[31] K. Huang, *Statistical Mechanics*, 2nd ed. (John Wiley & Sons, New York, 1987), p. 153.

[32] B. Cochet, K. Bennaceur, P. Bonche, T. Duguet, and J. Meyer, Nucl. Phys. **A731**, 34 (2004).

[33] V. B. Soubbotin, V. I. Tselyaev, and X. Vinas, Phys. Rev. C **69**, 064312 (2004).

[34] D. Vretenar, T. Niksic, and P. Ring, Phys. Rev. C **68**, 024310 (2003).

[35] P. J. Ellis, E. K. Heide, S. Rudaz, and M. Prakash, Phys. Rep. **242**, 379 (1994).

[36] S. J. Lee and A. Z. Mekjian, Phys. Rev. C **63**, 044605 (2001).

[37] F. W. Sears and G. L. Salinger, *Thermodynamics, Kinetic Teory, and Statistical Thermodynamics*, 3rd ed. (Addison-Wesley, Reading, 1975), p. 354.

[38] S. Das Gupta and A. Z. Mekjian, Phys. Rev. C **57**, 1361 (1998).

[39] S. K. Patra, M. Centelles, X. Vinas, and M. Del Estal, Phys. Rev. C **65**, 044304 (2002).

[40] C. S. Wang, K. C. Chung, and A. J. Santiago, Phys. Rev. C **55**, 2844 (1997).

[41] V. S. Uma Maheswari, V. S. Ramamurthy, and L. Satpathy, Phys. Rev. C **46**, 2305 (1992).

[42] D. Von-Eiff, J. M. Pearson, W. Stocker, and M. K. Weigel, Phys. Rev. C **50**, 831 (1994).

[43] M. M. Sharma, W. Stocker, P. Greissel, and M. Brack, Nucl. Phys. **A504**, 337 (1989).

[44] H. R. Jaqaman, A. Z. Mekjian, and L. Zamick, Phys. Rev. C **29**, 2067 (1984).

[45] A. Bohr and B. R. Motelson, *Nuclear Structure*, Vol. 1 (W. A. Benjamin, New York, 1969), p. 160.