Number of states with given spin *J* of *n* fermions in a *j* orbit

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A recursion formula for the number of states with a given value of total spin J of n identical fermions in a j orbit, $N(J, j, n)$, is derived. That number is expressed in terms of the number of states with some values of J, of $n, n-1$, and $n-2$ fermions in a $(j-1)$ orbit. This formula may be used in calculating $N(J, j, n)$. In this paper the formula is used to prove some interesting results that were found empirically by Zhao and Arima [1].

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Zhao and Arima [1] found empirical formulas for the number of states with a given angular momentum obtained by antisymmetric coupling of individual spins of three and four identical fermions in a j orbit. There were attempts to find general expressions for the number of states of n j fermions with spin J. In their paper [1], Zhao and Arima list some of the references and point out that no explicit formulas were obtained before, even for $n = 3$ and $n = 4$. They tested the formulas that they obtained to very high j values. More recently Zhao and Arima [2] used group theoretical methods to prove their results for $n = 4$.

In the present paper, a recursion relation is derived relating the number of J states in the j^n configuration to the number of some states in the configuration of $(j-1)$ fermions whose numbers are $n, n-1$, and $n-2$. This relation may be used for calculating successively the number of states with a given value of J. Here, it will be used to prove the results of Zhao and Arima for the $n = 3$ case.

A special case of the formulas of Ref. [1] is the number of $J = j$ states of three fermions with spin j. This number $[(2j + 3)/6]$, the largest integer not exceeding $(2j + 3)/6$, is equal to the number of $J = 0$ states obtained by Ginocchio and Haxton [3] for the case of four fermions. This equality has physical significance, since it was shown [4] that a necessary and sufficient condition for a two-body interaction to be diagonal in the seniority scheme is to have vanishing matrix elements between the $v = 1$, $J = j$ state and all $v = 3$, $J = j$ states of the j^3 configuration. It was later shown [5] that an equivalent condition is to have vanishing matrix elements between the $v = 0$, $J = 0$ state and all $v = 4$, $J = 0$ states of the j^4 configuration. The number of independent conditions must be the same in both cases. Very recently, an alternate derivation of this number was published [6]. These numbers for $n = 3$ and $n = 4$ will simply follow from special cases of the recursion relation for any n, to be derived below.

The simplest way to find the number of states with given J is to use the m scheme. The number of states with given J is obtained by subtracting the number of states with projection $M = J + 1$ from the one with $M = J$ [4]. For identical j fermions, the m states are defined by the projections of the individual spins of the fermions, $m_1, m_2, \ldots, m_n$ onto the z axis. According to the Pauli principle, they should be different and it is convenient to choose the arrangement $m_1 > m_2 > \cdots > m_n$. The projection of the total spin J onto the z axis is equal to $M = m_1 + m_2 + \cdots + m_n$. The number of states with a given value of M in the j^n configuration will be denoted $D(M, j, n)$.

Starting with the highest possible value of J,

$$J_{\max} = M_{\max} = j + j - 1 + j - 2 + \cdots + j - n + 1 \\ = n(2j + 1 - n)/2,$$

there are states for which $m_1 = j$ and the absolute values of all other m values are not larger than $j - 1$. This is guaranteed by the condition $m_n > -j$. The number of such states with given M is the same as the number of states with $M - j$ because there are $n - 1$ fermions in the $(j - 1)$ orbit, $D(M - j, j - 1, n - 1)$. There may be states with $m_1 = j$ and $m_n = -j$; their number is equal to the number of states with the same value of M, with $n - 2$ fermions of spin $j - 1$, $D(M, j - 1, n - 2)$. There are also states where $m_1 < j, m_n > -j$; their number is simply $D(M, j - 1, n)$. The other type of state in which the j orbit is evident is when $m_1 < j$ but $m_n = -j$. The number of those is equal to $D(M + j, j - 1, n - 1)$. The sum of these four numbers is equal to the total number of states with given M of the j^n configuration,

$$D(M, j, n) = D(M, j - 1, n) + D(M + j, j - 1, n - 1) \\ + D(M, j - 1, n - 2) \\ + D(M - j, j - 1, n - 1). \tag{1}$$

The total number of states with $M \geqslant 0$ is equal to the sum of numbers of all states with given $J \leqslant M$,

$$D(M, j, n) = \sum N(J, j, n), \qquad J \leqslant M.$$

On the other hand, for $M \leqslant 0$,

$$D(M, j, n) = \sum N(J, j, n), \qquad J \geqslant |M|.$$

Hence, if $M \geqslant 0$, we obtain

$$N(J, j, n) = D(M = J, j, n) - D(M = J + 1, j, n). \tag{2}$$

If, however, $M < 0$ and $M + 1 \leqslant 0$, the difference is equal to

$$D(M = -J, j, n) - D(M = -J + 1, j, n) = -N(J - 1, j, n). \tag{3}$$

A special case is when $M < 0$ while $M + 1 > 0$, which occurs If $M = -1/2$. In such a case, the number of $M = -1/2$ states

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is equal to the number of $M = 1/2$ states, and, hence,

$$D(M = -1/2, j, n) - D(M = 1/2, j, n) = 0. \quad (4)$$

Using these expressions for spin $j - 1$, we can express $N(J, j, n)$ by subtracting the expressions on the right-hand side (r.h.s.) of Eq. (1) for $M = J + 1$ from the one where we set $M = J$.

Taking M to be non-negative, the projections in the first three terms on the r.h.s. of Eq. (1), M and $M + j$, are also non-negative. This is true for $M = J$ and certainly for $M = J + 1$. When the subtraction is carried out, Eq. (2) may be safely used. In the last term, however, the situation depends on the value of J. If $J \geqslant j$, then the projection $J - j$ is also non-negative and Eq. (2) may be used. If $J < j$, then the projection $J - j$ is negative, and Eq. (3) or (4) should be applied.

These considerations lead to the following recursion relations.

For $J \geqslant j$

$$\begin{aligned} N(J, j, n) = {} & N(J, j-1, n) + N(J + j, j-1, n-1) \\ & + N(J, j-1, n-2) \\ & + N(J - j, j-1, n-1). \end{aligned} \quad (5)$$

For $J = j - 1/2$

$$\begin{aligned} N(J = j - 1/2, j, n) = {} & N(J = j-1/2, j-1, n) \\ & + N(J = 2j - 1/2, j-1, n-1) \\ & + N(J = j - 1/2, j-1, n-2). \end{aligned} \quad (6)$$

For $J \leqslant j - 1$

$$\begin{aligned} N(J, j, n) = {} & N(J, j-1, n) + N(J + j, j-1, n-1) \\ & + N(J, j-1, n-2) \\ & - N(j - J - 1, j-1, n-1) \end{aligned} \quad (7)$$

By its definition, $N(J, j, n)$ vanishes if there is no state with the given value J in the j^n fermion configuration. The recursion relations (5)–(7) also hold for $n = 2$ and $n = 1$ (and trivially for $n = 0$) provided we define $N(J, j, n) = 0$ for $n < 0$ and $N(J, j, n = 0) = \delta_{J0}$. To calculate the number of states, it is sufficient to limit the value of n to $(2j + 1)/2$. That number for n fermions is equal to the one for n holes, i.e., $2j + 1 - n$ fermions. The recursion relations derived above hold for any value of n, and they satisfy the condition $N(J, j, 2j + 1 - n) = N(J, j, n)$. This is easily verified by using the particle-hole symmetry of fermions in the $(j - 1)$ orbit.

A simple result of Eq. (7) is obtained for $J = 1/2$, $n = 3$. It is well known that there are no $J = 1/2$ states in the fermion j^3 configuration (see, e.g., Ref. [4] or [7]). Setting $J = 1/2$ and $n = 3$, we obtain for this number the expression

$$\begin{aligned} N(J = 1/2, j, 3) = {} & N(1/2, j-1, 3) \\ & + N(j + 1/2, j-1, 2) \\ & + N(1/2, j-1, 1) - N(j - 3/2, j-1, 2). \end{aligned} \quad (8)$$

The vanishing of this expression may be proved by induction with respect to j. For $j = 7/2$, $n = 3$, there is no $J = 1/2$ state. There is a $J = 1/2$ state neither in the $n = 3$ case nor in the $n = 1$ case of $j = 5/2$. The two other terms cancel each other, as will be shown below, for any value of j. If the nonexistence of $J = 1/2$ state in states with three spin $j - 1$ fermions is assumed, then this property follows also for the configuration of three j fermions. Since it is assumed that the first and third terms in Eq. (8) vanish, it is necessary to show that the sum of the other terms is also zero. In a two-fermion configuration, only states with even values of J exist. Thus, the sum considered is equal to

$$[1 + (-1)^{j+1/2}]/2 - [1 + (-1)^{j-3/2}]/2 = 0 \quad \text{for any } j.$$

Equation (8) is also consistent for $j = 3/2$, since the first term is not present for $j - 1 = 1/2$; the second term vanishes while the third term is equal to 1, but it is canceled by the last term, which is also equal to 1.

It is more interesting to use the recursion relations obtained above to prove, by induction with respect to j, some of the results of Zhao and Arima [1]. In their paper, they use the notation $D(3, J)_j$ for the number of states with spin J in the j^3 configuration. For states with $J \leqslant j$, they found the expression $N(J, j, 3) = [(2J + 3)/6]$, the largest integer not exceeding $(2J + 3)/6$. Their conclusion can be directly verified for $j = 5/2$, and from its validity for $j - 1$ we will show that it is also valid for spin j. Using the recursion relations, we obtain in this case

$$\begin{aligned} N(J, j, 3) = {} & N(J, j-1, 3) + N(J + j, j-1, 2) \\ & + N(J, j-1, 1) - N(j - J - 1, j-1, 2). \end{aligned} \quad (9)$$

The first term on the r.h.s. of Eq. (9) is equal, according to the assumption, to $[(2J + 3)/6]$. Since it is independent of j, it is necessary to show that the sum of the other terms vanishes. The third term vanishes unless J is equal to $j - 1$, $\delta(J, j - 1)$. The second term vanishes if $J + j$ is odd and is equal to 1 if it is even. Similarly, the last term vanishes if $j - J - 1$ or $j + J + 1-2j$ is odd, and it contributes -1 otherwise. Thus,

$$\begin{aligned} N(J, j, 3) = {} & [(2J + 3)/6] + [1 + (-1)^{J+j}]/2 \\ & + \delta(J, j-1) - [1 + (-1)^{j-J-1}]/2. \end{aligned}$$

If $J < j - 1$, the contributions of the second and fourth terms on the r.h.s. either vanish (if $J + j$ is odd) or cancel each other, which completes the proof. The vanishing of the last three terms on the r.h.s. of Eq. (9) also holds for $J = j - 1$. In that case, the second term does not contribute, since there is no state of two $(j - 1)$ fermions with total spin $j - 1 + j = 2j - 1$. The last two terms cancel each other.

In the other case, if $J \geqslant j$, Zhao and Arima found the number of states $N(J, j, 3)$ to be given by $[(J_{\max} - J)/6] = [(3j - 3 - J)/6]$ plus another term. The other term is equal to 0 if r, the remainder of dividing $3j - 3 - J$ by 6, is equal to 1 and is equal to 1 otherwise. The remainder may be defined by $6\{(J_{\max} - J)/6 - [(J_{\max} - J)/6]\}$. It is the smallest integer r defined by $r \equiv J_{\max} - J \pmod 6$. The formula of Zhao and Arima may thus be expressed by

$$\begin{aligned} & [(3j - 3 - J)/6] + 1 - \delta(6\{(J_{\max} - J)/6 \\ & \quad - [J_{\max} - J)/6]\}, 1). \end{aligned} \quad (10)$$

In the special case of $J = j$, $J_{\max} - J = 2j - 3$ is an even number for any value of j, and Eq. (10) reduces to

$$[(2j-3)/6] + 1 = [(2j-3)/6 + 6/6] = [(2j+3)/6],$$

as found by Ginocchio and Haxton [3].

Formula (10) will be now proved by induction with respect to j. It holds for $j = 3/2$ as well as for several higher j values. We assume that it holds for fermions with spin $j-1$ and prove that it holds also for fermions with spin j. This will prove it for all values of j. In the case of $n = 3$, $J > j$, the recursion relation (5) has only two terms with nonvanishing contributions. The second term on the r.h.s. of Eq. (5) vanishes, since there is no state with spin $J + j$ of two $(j-1)$ fermions. This is the case also for the third term, since J is certainly higher than a single fermion spin of $j-1$. Hence,

$$N(J, j, 3) = N(J, j-1, 3) + N(J-j, j-1, 2).$$

The first term is equal, according to the assumption, to

$$\begin{aligned}&[(3j-6-J)/6]\\ &\quad + 1 - \delta(6\{(3j-6-J)/6 - [(3j-6-J)/6]\}, 1)\\ &\quad = [(3j-J)/6] - \delta(6\{(3j-J)/6 - [(3j-J)/6]\}, 1).\end{aligned}$$

The second term is equal to 1 if $J - j$ is even and to 0 if it is odd, $(1 + (-1)^{J-j})/2$. The r.h.s. of Eq. (5) is then equal to

$$\begin{aligned}&[(3j-J)/6] - \delta(6\{(3j-J)/6\\ &\quad - [(3j-J)/6]\}, 1) + (1 + (-1)^{J-j})/2.\end{aligned} \tag{11}$$

To prove the equality of Eq. (11) and the Zhao and Arima result (10), we define $J_{\max} - J = 3j - 3 - J = 6k + r$ with $r < 6$. The number of states (10) is then given by $k + 1 - \delta(r, 1)$, which is equal to k for $r = 1$ and to $k + 1$ for all other values of r. Since $J - j$ may be expressed as $2j - 3 - 6k - r$, the term $(-1)^{J-j}$ in Eq. (11) may be written as $(-1)^r$, and expression (11) can be written as

$$\begin{aligned}&[1 + (-1)^r]/2 + k + [(r+3)/6]\\ &\quad - \delta(6\{(r+3)/6 - [(r+3)/6]\}, 1).\end{aligned}$$

It is now possible to calculate Eq. (11) for the values $r = 0$ to $r = 5$ and to compare them with those obtained from Eq. (10). This rather inelegant procedure yields the following results. For $r = 0$ and $r = 2$ we obtain $1 + k$. For $r = 1$ the result is k. For $r = 3$ and $r = 5$ we get $k + 1$, and for $k = 4$, $1 + k + 1 - 1 = k + 1$. These numbers are equal to those that Eq. (10) yields, and, hence, expressions (10) and (11) are equal in spite of their different appearance.

In their paper [1], Zhao and Arima also presented formulas obtained empirically for states of four j fermions. In a recent paper [2], they proved them. Using the recursion relations for $n = 4$ and the formulas of Zhao and Arima for $n = 3$, the explicit formulas for $n = 4$ may be obtained. Here, only the case of $J = 0$ states will be discussed, and the value of $N(0, j, 4)$ calculated. This is presented just as a simple application of the recursion relation derived above. In view of Ref. [6], there is no need for another derivation of this number, which was first obtained by Ginocchio and Haxton [3].

Also here, we will use induction to show that, as mentioned above,

$$N(0, j, 4) = N(j, j, 3) = [(2j+3)/6]. \tag{12}$$

This relation holds for $j = 3/2$ and, assuming that it holds for $j - 1$, it will be shown to hold for j also. This will prove it for any value of j. The recursion relation (7) becomes in this case

$$\begin{aligned}N(0, j, 4) &= N(0, j-1, 4) + N(j, j-1, 3)\\ &\quad + N(0, j-1, 2) - N(j-1, j-1, 3).\end{aligned}$$

According to the assumption, the first term is canceled by the last one. The third term is equal to 1, while the second term, as found in Ref. [1] and proved above, is equal to $[(2j-6)/6] + 1 - \delta(6\{2j/6 - [2j/6]\}, 1)$. Thus,

$$N(0, j, 4) = [2j/6] + 1 - \delta(6\{2j/6 - [2j/6]\}, 1).$$

To show that this expression is indeed equal to the r.h.s. of Eq. (12), we define, as above, $2j = 6k + r$, $r < 6$. With this definition we obtain

$$N(0, j, 4) = k + 1 + [r/6] - \delta(r, 1) = k + 1 - \delta(r, 1). \tag{13}$$

The value of Eq. (13) for $r = 1$ is k, and for the other possible values of r, which are 3 and 5, it is $k + 1$. The r.h.s. of Eq. (12) is $k + [(r+3)/6]$, which is equal to k if $r = 1$ and to $k + 1$ if $r = 3$ or $r = 5$. This proves equality (12) for all j values.

In any j^n configuration there are higher J states that cannot be present in the $(j-1)^n$ configuration. These are states with $J_{\max} = n(2j + 1 - n)/2$ and other states down to the maximum J of n $(j-1)$-fermions, which is equal to $n(2j - 1 - n)/2$. The number of such states may be obtained from Eq. (5), where only the last term need not vanish. The rather trivial result that there is only one state with $J_{\max}$ is a direct consequence of it. The value of $J_{\max} - j$ is equal to the maximum value of J in the $(j-1)^{n-1}$ configuration, namely, $(j-1)^n$

$$\begin{aligned}n(2j + 1 - n)/2 - j &= (n-1)(2j-n)/2\\ &= (n-1)\{2(j-1) + 1 - (n-1)\}/2.\end{aligned}$$

This relation holds for any $n < 2j$. Since there is only one state with $J_{\max}$ in the case $j = 1/2$, this follows for all values of j and n. Another trivial fact, the nonexistence of a state with $J_{\max} - 1$ follows in the same way from Eq. (5) due to the states nonexistence in the case $j = 3/2$.

More interesting is the result that two other states lower than $J_{\max} - 1$ and higher than $n(2j - 1 - n)/2$ are unique in j^n configurations. These are the states with spins $J_{\max} - 2$ and $J_{\max} - 3$. To prove this fact by induction, we limit n to be no larger than $(2j+1)/2$. The results hold for any value of n, but the proof is simpler with this limitation [the middle of the j orbit is different from the middle of the $(j-1)$ orbit]. Also, for these states only the last term in Eq. (5) may not vanish, and if it is equal to 1 for the $J - j$ state in the $(j-1)$ orbit, it is also equal to 1 for the state with J in the j orbit. By inspection we see that these states are unique in configurations with $n = 4$ for any j (it is true also for $n = 3$, but for the state with $J_{\max} - 3$ the first term in Eq. (5) equals 1, while the last term vanishes). From Eq. (5) this fact follows for any value of j and for any n value up to $n = 2j + 1 - 3$.

In their paper [1], Zhao and Arima proved a very interesting general result. The number of states with a high J with given value of $J_{\max} - J$ of $n\, j$ fermions is equal to that of n bosons with spin l, of states with total spin L with the value of $L_{\max} - L$ ($L_{\max} = nl$) equal to $J_{\max} - J$. In both cases this number is independent of j or l. In the present paper it turned out to be simpler to carry out the derivations by using fermions. The results can also be applied to bosons for values of L that satisfy the conditions determined in Ref. [1].

In the present paper a recursion relation is derived for the number of antisymmetric states with a given value of J due to coupling of states of n identical fermions in the j orbit. Some applications of this relation were presented above. This relation may be used to calculate these numbers in terms of the numbers of states of fermions with spin $j - 1$. Starting from these numbers of states with various J values in all $(j - 1)^n$ configurations, the recursion relations (5), (6), and (7) determine these numbers for all j^n configurations.

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