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# Statistical mechanics of Yang-Mills classical mechanics

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Statistical mechanics (SM) of Yang-Mills classical mechanics is studied by using a toy model that resembles chaotic quartic oscillators. This nonlinear system attains the thermodynamic equilibrium *not* by collisions, which is generally assumed in SM, but by chaotic dynamics. This is a new mechanism of thermalization that may be relevent to the quark-gluon plasma (QGP) formation in relativistic heavy-ion collisions because the interactions governing QGP involve quantum chromodynamics (QCD), which is a Yang-Mills theory [SU(3)]. The thermalization time is estimated from the Lyapunov exponent. The Lyapunov exponent is evaluated using the recently developed monodromy matrix method. We also discuss the physical meaning of thermalization and SM in this system of few degrees in terms of chromo-electric and chromomagnetic fields. One of the consequence of thermalization, such as equipartition of energy and dynamical temperature, is also numerically verified.

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## I. INTRODUCTION

Yang-Mills (YM) theory is a non-Abelian gauge theory that is used to explain fundamental interactions such as the weak interaction and the strong interaction. It is a nonlinear theory and hence its classical equation of motion exhibits chaos. An extensive study of the chaos of a homogeneous SU(2) YM system, which is known as Yang-Mills classical mechanics (YMCM), was carried out by Matinyan and Savvidy [1,2]. Savvidy [2] speculated that this chaos in YM fields, which results from its non-Abelian nature, may be related to the confinement problem in the strong interaction, which is a SU(3) the YM theory called quantum chromodynamics (QCD). Recently, the study of chaos of the YM theory has again become important in the context of the deconfinement problem of QCD [3]. It is believed that at low energy or temperature QCD exhibits a property of confinement; that is, the fundamental particles and fields of QCD, quarks and gluons, are confined and can never be isolated. Quarks and gluons can only exist inside hadrons, mesons, glueballs, etc. However, at high energy or temperature, QCD predicts [4] that quarks and gluons become deconfined and can exist freely. Such a system is called a quark gluon plasma (QGP). There are numerous relativistic heavy-ion collisions (RHICs) experiments going on at BNL, CERN, etc. [5] to produce such matter in the laboratory. Heavy nuclei are accelerated to ultrarelativistic energies and are made to collide. During the collision sufficient energy is deposited at the collision region to produce a QGP. At such high energy, the fireball, immediately after collision, may be in a highly nonequilibrium state; it then equilibrates to QGP, expands, and freezes out in the form of mesons and baryons.

The interactions governing nonequilibrium QGP, formed immediately after the collision, are those of QCD. It is argued in Ref. [3] that chaos in YM theory leads to equilibration of the system. This is a new method of equilibration of a system by chaos rather than by collisions, which we generally study in statistical mechanics (SM), kinetic theory, etc. This is true even for a system with two degrees of freedom, but with chaos, as we demonstrate here using a toy model related to the YMCM system. Since chaos drives the system almost ergodic in phase space, the situation is very similar to that of SM where frequent collisions lead to ergodicity. Ergodicity is very essential for the success of SM. Of course, the statistics arose over the infinitely many possible quantum states and thermodynamics is obtained by averageing over these microstates. Here, in classical theory, we have finite phase space spanned by the system because of chaos, and by dividing it by $h^N$, where $h$ is the Planck's constant and $N$ is the degrees of freedom, we get the number of microstates. Since we have a conservative system here, energy $E$ is constant. Hence we have a microcanonical system.

The question of application of SM concepts to a chaotic system has been studied for the past few years by Berdichevsky and Alberti [6] and Bannur and coworkers [6,7]. Various properties, such as the equipartioning of energy, notion of temperature, and distribution functions, with few degrees of freedom, are discussed in the literature. Since we have a finite $N$ system in the study of the SM of chaotic systems, another question debated in the literature involves the SM from a dynamical point of view using these chaotic systems [8]. For example, the derivation of the Fourier heat law has been debated in the literature [9] using chaotic systems. Thus we see that chaos can lead to equilibration or thermalization of a nonequilibrium system like a QGP formed immediately after RHICs. Again there has been considerable work done in this direction by Müller and coworkers [3] by simulating classical YM theory such as SU(2) and SU(3) on the lattice in $(3+1)$ dimensions. They estimated Lyapunov exponents ($\lambda$) and their dependence on energy, thermalization time, etc. However, the existance of chaos in YM has already been shown by Matinyan and Savvidy and here by including the spatial part, making the system more complex and hence probably more chaotic. The various uncertainties in the numerical accuracy of these simulations are debated in Refs. [10,11] and there is as yet no consensus on the dependance of Lyapunov exponents on energy. Müller *et al.* [3] reported that the Lyapunov exponent scales linearly with energy/plaquette ($E$) in their numerical simulations. Nielsen *et al.* [11] reanalyzed the problem and argue that at low energy the Lyapunov exponent is propotional to $E^{1/4}$ and at high energy the results on $\lambda$ are not reliable owing to finite lattice size. Again Müller *et al.* [10] recalculated their numerical simulations with better accuracy and disproved the claim of Nielsen *et al.* In this context it is interesting to study the SM of homogeneous YM theory using the simplest toy model that retains the essential ingredients, namely chaos. In another scheme [12] the study of the of as quark-antiquark plasma in one dimensional as well as $(1+1)$-dimensional simulations revealed that non-Abelian features of the QGP lead to chaos and hence thermalization, termed collective thermalization. Again the essential ingredient of thermalization here is also the chaos present in YM theory.

## II. A TOY MODEL TO REPRESENT YMCM

The YMCM Hamiltonian is a SU(2) YM Hamiltonian density in temperal gauge, $A_0^a = 0$, constructed by assuming that the vector potentials $\vec{A}^a(t)$ are functions of time only. It is given by [2]

$$H_{\rm YM} = \frac{1}{2}\sum_a \dot{\vec{A}}_a^2 + \frac{1}{4}g^2\sum_{a,b}(\vec{A}_a \times \vec{A}_b)^2 , \qquad (1)$$

where $g$ is the gauge coupling constant and $a = 1, 2, 3$ are color quantum numbers. The Hamiltonian $H_{\rm YM}$, may be rewritten in the form

$$H_{\rm YM} = H_{\rm FS} + T_{\rm YM} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}g^2(x^2y^2 + y^2z^2 + z^2x^2) + T_{\rm YM}, \qquad (2)$$

where $H_{\rm FS}$ is called a fundamental subsystem (FS) of YMCM and $T_{\rm YM}$ describes quasi-rotational freedoms. Let us consider a simpler two-dimensional model [$a = 1, 2$ in Eq. (1)] without $T_{\rm YM}$. The corresponding Hamiltonian is

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}g^2x^2y^2 . \qquad (3)$$

On redefining variables $X_1 \equiv gx$ and $X_2 \equiv gy$, this reduces to

$$g^2H = \tfrac{1}{2}\left(\dot{X}_1^2 + \dot{X}_2^2\right) + \tfrac{1}{2}X_1^2X_2^2 . \qquad (4)$$

This is very similar to the quartic oscillator (QO) system with two degrees of freedom, which has been studied extensively both classicaly and quantum mechanically and is given by [13]

$$\tilde{H} \equiv g^2H = \frac{1}{2}\left(\dot{X}_1^2 + \dot{X}_2^2\right) + \frac{(1-\alpha)}{12}\left(X_1^4 + X_2^4\right) + \frac{1}{2}X_1^2X_2^2, \qquad (5)$$

where $\alpha$ is a parameter. For $\alpha = 1$ Eq. (5) reduces to Eq. (4). The QO system is highly chaotic for $\alpha = 1$ and becomes less chaotic as $\alpha$ decreases, as shown in Ref. [13].

To connect YMCM and QO, we note that the variables $X_i$ have dimensions of energy and that $g^2H$ is a constant of motion, say $g^2\varepsilon$, and has dimension of the fourth power of energy and hence energy density in a natural system of units. Hence we normalize all variables by the fourth root of $g^2\varepsilon$ (say, $E$) to make them dimensionless. Thus we get

$$H' = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{(1-\alpha)}{12}\left(q_1^4 + q_2^4\right) + \frac{1}{2}q_1^2q_2^2 , \qquad (6)$$

where $H' = 1$, $q_i \equiv X_i/E$, $\dot{q}_i \equiv dq_i/d\tau$, and $\tau \equiv Et$.

This simple looking Hamiltonian represents our toy model to study the statistical mechanics of the YMCM system. The actual model, Eq. (3), has the problem that its phase space is logarithmically diverging in classical theory, as discussed in detail by Matinyan and Ng [14]. They also showed that the finite $(1-\alpha)$, as in our toy model [Eq. (5)], however small it may be, regularizes the divergence. Further, it contains the essential ingredient, nonlinearity according to non-Abelian guage theory, that leads to chaos or approximate ergodicity [6,7] and hence to the equilibration of energy. As we have noted earlier as well as discussed in Ref. [6,7], even though this system has only two degrees of freedom, it exhibits chaos for $\alpha$ close to 1 and hence we may talk about thermalization and SM of the system. Note that these two degrees of freedom are the collective modes of YM theory. Let us first consider the thermalization problem.

## III. ESTIMATE OF THERMALIZATION TIME

As pointed out in Ref. [3] the thermalization time $t_{\rm th}$ may be estimated from knowledge of the Lyapunov exponent $\lambda$. To evaluate $\lambda$, define the distance between two trajectories, $D(\tau)$, in phase space, as

$$D^G(\tau) = \sqrt{\sum_{i=1}^{2}\left((q_i - q_i')^2 + (\dot{q}_i - \dot{q}_i')^2\right)}, \qquad (7)$$

where primed and unprimed variables describe two different trajectories. Superscript $G$ refers to the fact that it is the general definition used in textbooks [15] and Ref. [13]. Note that without normalized variables, $D(t)$ is not dimensionally correct. Now the maximum Lyapunov exponent is defined as

$$\lambda_1 \equiv \lim_{\tau\to\infty}\frac{1}{\tau}\ln\left|\frac{D(\tau)}{D(0)}\right| = \frac{\lambda_E}{E}, \qquad (8)$$

where

$$\lambda_E \equiv \lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{D(t,E)}{D(0,E)}\right|. \qquad (9)$$

Here $D(t, E)$ is given by

$$D(t,E) = \frac{1}{E}\sqrt{\sum_{i=1}^{2}\left((X_i - X_i')^2 + (\dot{X}_i - \dot{X}_i')^2/E^2\right)}, \qquad (10)$$

$\lambda_E$ is equal to $\lambda_1$ for $E = 1$, and, for arbitrary $E$ or $g^2\varepsilon$, $\lambda_E$ scales with $E$ or the fourth root of $\varepsilon$, which has the dimension of energy. This is similar to the results of Müller *et al.*, obtained numerically, whereas here it follows from dimensional arguments. We have also numerically verified Eq. (8) with our model; this is tabulated in Table I.

The maximum Lyapunov exponent may be evaluated following the procedure of Joy and Sabir [13] using the

TABLE I. Maximum LE ($\lambda_E$) for various energies ($g^2\varepsilon$).

| $g^2\varepsilon$ | 1 | 16 | 81 | 256 |
|---|---|---|---|---|
| $\lambda_E$ | 0.39 | 0.78 | 1.18 | 1.58 |

Hamiltonian, Eq. (5) [17]. Here we use the monodromy matrix method, which was recently used and studied in a much more complicated system by Fulop and Biro [16]. The Hamilton equations of motion, to be solved numerically, are

$$\begin{aligned}\dot{X}_1 &= X_3;\ \dot{X}_3 = (\alpha-1)X_1^3/3 - X_1X_2^2;\\ \dot{X}_2 &= X_4;\ \dot{X}_4 = (\alpha-1)X_2^3/3 - X_2X_1^2,\end{aligned} \tag{11}$$

which follow from Eq. (5). The numerical solutions of these equations are used to construct the monodromy matrix $M$. The monodromy matrix is the linear stability matrix of Eq. (11),

$$M = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ (\alpha-1)X_1^2 - X_2^2 & -2X_1X_2 & 0 & 0\\ -2X_1X_2 & (\alpha-1)X_2^2 - X_1^2 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $M$, $\Lambda_i$, are found at every instant and the Lyapunov exponent (LE) spectra ($\lambda_i$) is obtained from their time averages. That is,

$$\lambda_i = \frac{1}{t}\int_0^t d\tau\ \Lambda_i(\tau). \tag{12}$$

The real values of $\lambda_i$ give four LEs as $t \to \infty$. Since our system is conservative two of them are zero. The other two come in pairs with positive and negative values. The positive value is the largest LE ($\lambda$) of our model. $\lambda$s are plotted for various energies in Fig. 1 as a function of time. The result is in good agreement with the scaling relation Eq. (8).

From the maximum LE and its dependence on $E$, we estimate the thermalization time using the relation, given by Müller [3], $t_{\rm th} = 1/(\lambda\ E)$. In our model, we get $t_{\rm th} \approx 1$ fm/$c$ for $g = 1$ and an energy density $\varepsilon = 1$ GeV/fm$^3$, parameters relevent to RHICs.

## IV. STATISTICAL MECHANICS AND THERMODYNAMICS OF OUR MODEL

So far we have discussed the chaotic property of the YMCM toy model. Let us now look at the SM properties

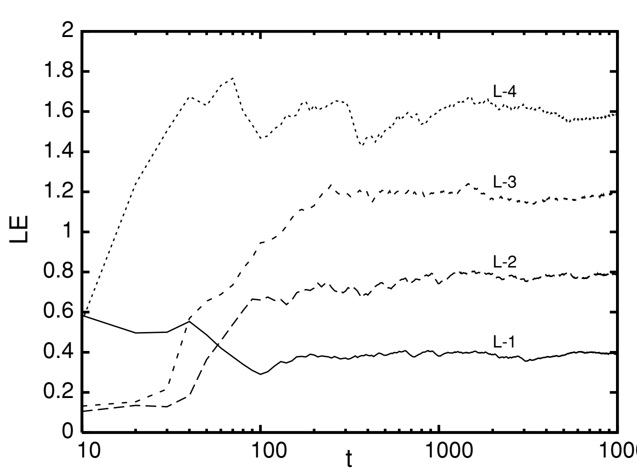


FIG. 1. Plots of LE as a function of $t$ for $E = 1$ (L-1), 16 (L-2), 81 (L-3), 256 (L-4).

of the system. Even with only two degrees of freedom, the system is almost ergodic and hence we can look for statistical and thermodynamic properties in line with Refs. [6,7]. Note that in the case of normal SM, ergodicity arises because of frequent collisions and hence we need $N$ to be large. The dependence of $N$ becomes important in the calculations of the distribution function [6], correlation function, etc. However, many properties of SM such as entropy, temperature, and equipartition of energy are found to be obeyed by our system and also give an estimate of thermalization, from the Lyapunov exponent, of the same order as that discussed in RHICs. Following the standard procedure of calculating the thermodynamic functions from SM, we have the phase space volume

$$\Gamma(E) = \int_{H\leqslant\varepsilon} \frac{dX_1 dX_2 dP_1 dP_2}{h^2} = CE^{3/2}, \tag{13}$$

where $P_1$, $P_2$ are canonical conjugate momenta of $X_1$ and $X_2$, respectively, and $C$ is a constant independent of $\varepsilon$. However, for $\alpha = 1$, $C$ is infinite because of the divergence of phase space and the divergence may be regularized by not taking $\alpha$ exactly as 1, as noted by Matinyan and Ng [14]. For numerical stability, we also used $\alpha = 0.99$ and hence this ambiguity is removed. Furthermore, for the problems addressed here involving equipartitioning of energy, temperature, etc., $C$ gets canceled on taking averages.

In the literature there are two definition of temperature, which are discussed and compared in Ref. [7]. One follows from the equipartition theorem,

$$\begin{aligned}\left\langle \dot{X}_1 \frac{\partial \tilde{H}}{\partial \dot{X}_1}\right\rangle &= \left\langle \dot{X}_2 \frac{\partial \tilde{H}}{\partial \dot{X}_2}\right\rangle = \left\langle X_1 \frac{\partial \tilde{H}}{\partial X_1}\right\rangle = \left\langle X_2 \frac{\partial \tilde{H}}{\partial X_2}\right\rangle\\ &= T_B \equiv \left[\frac{\partial \ln \Gamma}{\partial E}\right]^{-1} = \frac{2E}{3},\end{aligned} \tag{14}$$

where the first two averages are twice the average kinetic energies associated with each degree of freedom, which we call $T_{B1}$ and $T_{B2}$, respectively. The remaining two averages are associated with potential energies. These results are from SM calculations. If the system is ergodic we can calculate the temperature dynamically; the dynamical temperature is

$$T_{Bi}^{\rm dy} = \frac{1}{t}\int_0^t d\tau \dot{X}_i \frac{\partial \tilde{H}}{\partial \dot{X}_i}, \tag{15}$$

where $i = 1, 2$. It is just a time average. In fact, all four averaged quantities may be evaluated by time averaging. At thermodynamic equilibrium all of them approach the same value $T_B$ as $t \to \infty$ according to the ergodic theorem. Another definition of temperature, which we call $T_s$, is

$$T_s^{-1} = \frac{\partial \ln \frac{\partial \Gamma}{\partial \varepsilon}}{\partial E} = \frac{1}{2E}, \tag{16}$$

and the corresponding dynamical temperature is

$$\frac{1}{T_s^{\rm dy}} = \frac{1}{t}\int_0^t d\tau \Phi(\tau), \tag{17}$$

where $\Phi(\tau) \equiv \nabla \cdot \frac{\nabla \tilde{H}}{|\nabla \tilde{H}|^2}$. Here gradients and divergence are in phase space and hence $\Phi(\tau)$ may be a very complicated

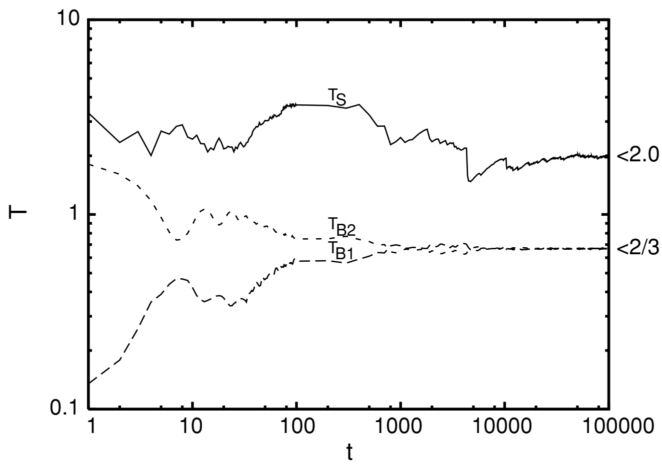


FIG. 2. Plots of $T_{B1}^{\rm dy}$, $T_{B2}^{\rm dy}$, and $T_S^{\rm dy}$ as a function of $t$ for $E = 1$.

expression. Again at thermodynamic equilibrium $T_s^{\rm dy}$ approaches $T_s$. Note that $\Phi(\tau)$ depends on all phase space variables and hence only on $T_s^{\rm dy}$.

When the system is almost ergodic (i.e, when the initial points are chosen in the chaotic region), our numerical calculations, as shown in Fig. 2, show that $T_{B1}^{\rm dy}$ and $T_{B2}^{\rm dy}$ both approach the value $T_B$ and $T_s^{\rm dy}$ approaches $T_s$ asymptotically with time. Still it is not clear which temperature definition is that of thermodynamic temperature for a finite $N$ system. Of course, in SM, when $N \to \infty$, $T_B = T_B^{\rm dy} \to T_s = T_s^{\rm dy}$. It is interesting that the dynamical temperature $T_s^{\rm dy}$ was discovered recently by Rugh [18] and reformulated in a simpler language and verified numerically by Bannur [7] using chaotic systems such as QOs quartic oscillators and Henon-Heiles oscillators. Let us discuss the physical meaning of equipartitioning of energy in this model.

## V. MEANING OF THERMALIZATION

The $(3+1)$-dimensional YM system, discussed in Ref. [3,8], is a system with large degrees of freedom resulting form spatial dependence, and the thermalization there occurs, because of equal energy sharing among various modes by nonlinear interactions. In our case, our YM fields without spatial dependence and energy sharing are due to the nonlinear interactions between only two degrees of freedom. As we noted earlier even this system with two degrees of freedom exhibits chaos and hence is almost ergodic because of its nonlinearity. To understand it further, we may rewrite the Hamiltonian, Eq. (3), as

$$H = \tfrac{1}{2}\left(E_1^2 + E_2^2\right) + \tfrac{1}{2}B_3^2, \qquad (18)$$

by using the defintion of electric and magnetic fields in terms of the field tensor

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc}A_b^\mu A_c^\nu,$$

where $a$, $b$, and $c$ are color indices that take values 1, 2, and 3 and Lorentz indices are $\mu, \nu = 0, 1, 2, 3$ with metric $(1, -1, -1, -1)$. $\epsilon_{abc}$ is an antisymmetric Levi-Civita tensor. In our case $a = 1, 2$ and we use the hedgehog ansatz and hence we have electric fields $E_1$ and $E_2$ and magnetic field $B_3$. Now thermalization means sharing energy between electric and magnetic fields. If we start with whole energy, say, in electric field $E_1$, after thermalization the energy will be equally distributed among $E_1$, $E_2$, and $B_3$. As we discussed in [7], whenever a system is almost chaotic equipartition of energy takes place and

$$\left\langle \dot{x}_1 \frac{\partial H}{\partial \dot{x}_1}\right\rangle = \left\langle \dot{x}_2 \frac{\partial H}{\partial \dot{x}_2}\right\rangle = \left\langle x_1 \frac{\partial H}{\partial x_1}\right\rangle = \left\langle x_2 \frac{\partial H}{\partial x_2}\right\rangle, \qquad (19)$$

which is the same as

$$\left\langle E_1^2\right\rangle = \left\langle E_2^2\right\rangle = \left\langle B_3^2\right\rangle = \tfrac{2}{3}\varepsilon. \qquad (20)$$

Here the angular brackets indicate the time average, which is also equal to the phase space average by the ergodic theorem. Hence $t_{\rm th}$ gives the time required to equilibrate energy among electric and magnetic fields and their components. This is a very important property of YM fields, which, for example in RHICs, will thermalize the coherent fields formed immediately after the collision. This is due to the relaxation phenomena associated with color degrees of freedom rather than momentum relaxation. Furthermore, if we extrapolate the results of Ref. [7] on the QOs, with large degrees of freedom, to our system with large color degrees of freedom or large number of YMCM systems with few color degrees of freedom, the energy distribution in any one component of the fields may decay exponentially with decay constant inversely proportional to temperature. In other words, the corresponding field distribution is Gaussian with its width proportional to temperature. For example, from Ref. [7], one particle momentum distribution is given by

$$f(p_1) = \frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{p_1^2}{2T}\right) = \frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{E_1^2}{2T}\right), \qquad (21)$$

since $p_1 = \dot{x}_1 = E_1$ in our model; similar distributions for other field components are found. This resembles the case of momentum relaxation, where one gets a Gaussian momentum distribution of particles in statistical mechanics.

## VI. NUMERICAL RESULTS AND CONCLUSIONS

In our numerical calculations, we evaluated the Lyapunov exponent by using the monodromy matrix method. The results are plotted in Fig. 1 for different values of $E = 1$ (curve L-1), 16 (curve L-2), 81 (curve L-3), and 256 (curve L-4) units. Asymptotically each LE approaches almost a constant value, from which the theoretical scaling behavior of $\lambda$ with $E$ is confirmed as shown in Table I. The thermalization time is of the order of $1/(\lambda_1 E) \approx 1$ fm/$c$ for typical parameters of RHICs and inversely depends on the fourth root of energy density. In the case of RHICs, it may give an estimate of the order of magnitude of the time required to redistribute the energy, stored initially in coherent fields, among all degrees of freedom associated with spatial, color, electric and magnetic fields, etc.

Another interesting result we present deals with the consequence of thermalization, namely, equipartition of energy and dynamical temperatures in this model. We numerically evaluated $T_B^{\rm dy}$, or twice the average kinetic energies, associated with both the degrees of freedom and $T_s^{\rm dy}$ using Eqs. (15) and (17), respectively. As shown in Fig. 2, the results from our model with just two degrees of freedom agree with the results of statistical mechanics [i.e, Eqs. (14) and (16)]. We took $E = 1$ unit so that $T_B = 2/3$ and $T_s = 2$. Here the thermalization is *not* by collisions, which is generally assumed in SM, but by chaotic dynamics. Even though our model has only two degrees of freedom, these are the collective (classical) degrees of freedom of YM fields. Note that by definition from the ergodic theorem, the phase space or ensemble average is equal to the time average as $t \to \infty$. This is what we obtain from Fig. 2 and at present we are not clear how to obtain the thermalization time quantitatively from this plot.


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