PHYSICAL REVIEW C **72**, 015502 (2005)

Bound-state β decay of a neutron in a strong magnetic field

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(Received 9 December 2004; published 28 July 2005)

The β decay of a neutron into a bound (pe^-) state and an antineutrino in the presence of a strong uniform magnetic field ($B \gtrsim 10^{13}$ G) is considered. The β decay process is treated within the framework of the standard model of weak interactions. A Bethe-Salpeter formalism is employed for description of the bound (pe^-) system in a strong magnetic field. For the field strengths $10^{13} \lesssim B \lesssim 10^{18}$ G the estimate for the ratio of the bound-state decay rate w_b and the usual (continuum-state) decay rate w_c is derived. It is found that in such strong magnetic fields $w_b/w_c \sim 0.1$–0.4. This is in contrast to the field-free case, where $w_b/w_c \simeq 4.2 \times 10^{-6}$ [J. N. Bahcall, Phys. Rev. **124**, 495 (1961); L. L. Nemenov, Sov. J. Nucl. Phys. **15**, 582 (1972); X. Song, J. Phys. G: Nucl. Phys. **13**, 1023 (1987)]. The dependence of the ratio w_b/w_c on the magnetic field strength B exhibits a logarithmiclike behavior. The obtained results can be important for applications in astrophysics and cosmology.

DOI: 10.1103/PhysRevC.72.015502 PACS number(s): 13.30.Ce, 11.10.St, 32.60.+i, 97.20.Rp

I. INTRODUCTION

It is well known that besides the main decay mode of a free neutron in vacuum

$$n \to p + e^- + \bar{\nu}_e, \tag{1.1}$$

there is also the neutron decay into a bound (pe^-) state (hydrogen atom) and an antineutrino

$$n \to (pe^-) + \bar{\nu}_e. \tag{1.2}$$

Several theoretical estimates for the rate of the latter process have been performed (see Refs. [1–3] and references therein). Regardless of the framework employed for the description of the hydrogen atom (for example, such as the Schrödinger equation [1], Dirac theory [2], or relativistic Bethe-Salpeter formalism [3]) all the treatments yield the following estimate for the ratio of bound-state and continuum-state decay rates: $w_b/w_c \simeq 4.2 \times 10^{-6}$. Therefore, in general one might expect the effect of the neutron bound-state decay (1.2) to be subdominant if compared with the continuum-state decay (1.1). However, as shown below this conclusion does not hold true in the situation where a neutron decays in the presence of a strong magnetic field.

In this work we estimate the ratio w_b/w_c in the presence of a magnetic field with strength $B \gtrsim 10^{13}$ G. Our study is motivated by a widely accepted view that in diverse astrophysical and cosmological environments the physics of neutrinos in strong magnetic fields plays an important role. The existence of very strong magnetic fields in protoneutron stars and pulsars is well established. The surface magnetic fields of many supernovae and neutron stars are of the order of $B \sim 10^{12}$–10^{14} G [4,5]. The surface magnetic fields of magnetars are perhaps as large as $B \sim 10^{15}$–10^{16} G [6]. Very strong magnetic fields are also supposed to have existed in the early universe (see Ref. [7] for a recent review).

The effect of a constant magnetic field on the process (1.1) is well documented. In short, the constant magnetic field affects the motion of a charged particle in such a way that the energy associated with the transverse motion (with respect to the field direction) is quantized into Landau levels while the longitudinal motion remains free. The larger the field intensity B, the larger the energy separating the Landau levels. Thus, the decay rate w_c exhibits the following dependence on the field intensity B. For $0 < B \ll B_{\rm cr}$, where $B_{\rm cr} = (\Delta^2 - m_e^2)/2e = 1.2 \times 10^{14}$ G with $\Delta = m_n - m_p$ being the nucleon mass defect (hereafter we use the units $\hbar = c = 1$), the effect of a magnetic field on both an electron and a proton is small and w_c remains practically insensitive to B. As the field intensity approaches the value $B_{\rm cr}$ (note that $B_{\rm cr} > B_e$, where $B_e = m_e^2/e = 4.414 \times 10^{13}$ G is the so-called Schwinger field) the effect of Landau quantization on the transverse electron motion becomes considerable $\omega_c \sim B$ [8]. If $B_{\rm cr} < B \ll B'_{\rm cr}$, where $B'_{\rm cr} = [(m_n - m_e)^2 - m_p^2]/2e = 1.25 \times 10^{17}$ G, the electron (due to the energy conservation law) can occupy only the lowest Landau level, and it was shown in [8] that the decay rate grows as $w_c \sim B$. In the case of a superstrong magnetic field ($B \geqslant B'_{\rm cr}$) the proton can occupy only the lowest Landau level and one has again a monotonic dependence $w_c \sim B$ [9]. Finally, as the field strength exceeds the value $B \sim 1.5 \times 10^{18}$ G the modification of the strong forces, which bind the quarks in nucleons, is such that it can close the mass gap between the neutron and proton making the neutron stable [10] (see also [7]). In what follows we concentrate on field strengths in the range $B_e \lesssim B \ll B_p$, where $B_p = m_p^2/e = 1.5 \times 10^{20}$ G is the Schwinger field for the proton.

To our knowledge, no theoretical analysis has been published for the process (1.2) in a (strong) magnetic field. It should be noted that such study is hampered by the fact that in calculating the decay rate w_b one must know the eigenenergies and eigenstates of the hydrogen atom in an

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external magnetic field. In the case of a strong field the latter problem has been studied in a number of papers, where numerical and semianalytical results have been derived within different approaches based on the Schrödinger [11–20], Dirac [21–23], and Breit [24] equations. A common result of all the treatments is that, compared to the field-free case, the ground state of the hydrogen atom is (1) very tightly bound and (2) well separated in energy from excited states. The shape of the atom is also affected by the field. The transverse atomic size is approximately determined by the magnetic length $a_B = 1/\sqrt{eB} \ll a_0$, where $a_0 = (e^2 m_e)^{-1}$ is the Bohr radius, and the longitudinal atomic size $a_\parallel$ depends on the binding atomic energy. In particular, in the case $B \gtrsim B_{\rm cr}$ one has for the ground state a cigarlike shape with $a_\parallel \ll a_0$, while for the excited states $a_\parallel \sim a_0$ [20]. It should be noted that these remarkable effects of a strong magnetic field on the hydrogen atom are obtained assuming the absence of transverse motion of the atom as a whole. However, even at a moderate transverse atomic velocity a rather strong electric field is induced in the rest frame of the atom pushing the electron and the proton apart. As a result, the binding energy decreases as the transverse velocity increases. Thus, in contrast to the field-free case, the center-of-mass atomic motion does not decouple from the relative internal motion, and therefore a correct description of the hydrogen atom in a strong magnetic field should involve the two-body approach [25,26].

By analogy with the field-free case [2], the ratio w_b/w_c can be roughly estimated using a phase-space argument that does not depend on a formal theory of weak interactions. The phase-space volume available for the final products in the process (1.1) can be presented by the function $f(B,\Delta)$ that depends on the field strength and nucleon mass defect. For the bound-state decay (1.2), the available phase-space volume is given by $\varepsilon_\nu^2|\psi(0)|^2$, where ε_ν is the neutrino's energy and $\psi(0)$ is the ground-state wave function of a hydrogen atom evaluated at zero distance between an electron and a proton. The ratio of the bound-state and continuum-state decay rates is approximately equal to the ratio of the corresponding phase-space volumes

$$\frac{w_b}{w_c} \sim \frac{\varepsilon_\nu^2|\psi(0)|^2}{f(B,\Delta)}. \tag{1.3}$$

For example, using the phase-space argument in the field-free case ($B = 0$) we get the result

$$\frac{w_b}{w_c} \sim \pi e^6 \left(\frac{\Delta}{m_e} - 1\right)^2 \approx 2.9 \times 10^{-6}, \tag{1.4}$$

which is of the same order of magnitude as the accurate calculations of Refs. [1–3]. In the case of a strong field ($B_{\rm cr} < B \ll B'_{\rm cr}$) we have

$$f(B,\Delta) \sim eBm_e^3/\pi^2$$

and

$$|\psi(0)|^2 \sim \left(\pi a_\parallel a_B^2\right)^{-1} = (\pi a_\parallel/eB)^{-1}.$$

Using the estimates for $a_\parallel$ reported in Ref. [20], we might expect [in accordance with Eq. (1.3)] that

$$\frac{w_b}{w_c} \sim \frac{\pi e^2 a_0}{a_\parallel} \left(\frac{\Delta}{m_e} - 1\right)^2 \gtrsim 0.1. \tag{1.5}$$

This result is almost 5 orders of magnitude larger than in the field-free case (1.4) and thus it calls for a more detailed and rigorous theoretical analysis, such as carried out below.

The paper is organized as follows. In Sec. II a general formulation of the problem is given in the context of the standard quantum field theory employing a Bethe-Salpeter formalism. Then, in Sec. III, we discuss in detail a structure of the wave function of a bound (pe^-) state in a strong magnetic field. Further, specific approximations to the Bethe-Salpeter equation are developed in the cases $B_{\rm cr} < B \ll B'_{\rm cr}$ and $B'_{\rm cr} < B \ll B_p$. Section IV is devoted to the derivation of an estimate for the bound-state decay rate. In addition, the asymptotic formula for the ratio of bound-state and continuum-state decay rates is obtained. The conclusions are drawn in Sec. V.

II. GENERAL THEORY

In the case of field strengths relevant to this work ($B_e \lesssim B \ll B_p$) we can neglect the effect of a magnetic field on both the propagator of W boson and the weak form factors. Therefore, the transition matrix element for the decay process (1.2) can be written as follows:

$$T_{fi} = \frac{G}{\sqrt{2}} \int \bar{\chi}^E(x,x)\gamma_\mu(1+\alpha\gamma_5)\psi_n\gamma^\mu(1+\gamma_5)\psi_\nu d^4x, \tag{2.1}$$

where ψ_n and ψ_ν are the Dirac wave functions of the neutron and neutrino, respectively, $G = G_F\cos\theta_c$, θ_c is the Cabibbo angle, and $\alpha = 1.26$ is the ratio of the axial and vector constants. The conjugate $\bar{\chi}^E$ of the wave function χ^E describing the bound (pe^-) system with energy E is determined as $\bar{\chi}^E = \gamma^0\chi^{E\dagger}\gamma^0$. In the ladder approximation the wave function χ^E satisfies the following Bethe-Salpeter equation:

$$\chi^E(x_e,x_p) = ie^2 \int G^e(x_e,x'_e)\gamma^\mu G^p(x_p,x'_p)\gamma^\nu \times D_{\mu\nu}(x'_e,x'_p)\chi^E(x'_e,x'_p)d^4x'_e d^4x'_p. \tag{2.2}$$

Here $D_{\mu\nu}(x,x')$ is the photon propagator. The electron and proton propagators satisfy the Dirac equation in an external magnetic field

$$[\gamma^\mu(-i\partial_\mu \mp eA_\mu) + m_{e/p}]G^{e/p}(x,x') = \delta^{(4)}(x-x'). \tag{2.3}$$

In what follows we use the symmetric axial gauge of the vector potential

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2}[\mathbf{B}\times\mathbf{r}]. \tag{2.4}$$

The photon propagator is supposed to be invariant under the time and space translations

$$D_{\mu\nu}(x,x') = D_{\mu\nu}(\mathbf{r}-\mathbf{r}',t-t'), \tag{2.5}$$

whereas the electron and proton propagators are invariant under the time translation

$$G^{e/p}(x,x') = G^{e/p}(\mathbf{r},\mathbf{r}',t-t'). \tag{2.6}$$

Their properties with respect to the space translation are determined according to Eqs. (2.3) and (2.4) as follows:

$$G^{e/p}(\mathbf{r}-\mathbf{r}_C,\mathbf{r}'-\mathbf{r}_C,t-t') = \exp\left\{\pm i\frac{e}{2}[\mathbf{B}\times\mathbf{r}_C](\mathbf{r}-\mathbf{r}')\right\}G^{e/p}(\mathbf{r},\mathbf{r}',t-t'). \quad (2.7)$$

From Eqs. (2.5), (2.7), and (2.2) we obtain

$$\chi^E(x_e,x_p) = \exp(-iET)\exp\left\{i\left(\mathbf{P}+\frac{e}{2}[\mathbf{B}\times\mathbf{r}]\right)\mathbf{R}\right\}\eta^{E,\mathbf{P}}(\mathbf{r},t), \quad (2.8)$$

where $\mathbf{P}$ is the total pseudomomentum which corresponds to the total momentum in the field-free case [25], $t=t_e-t_p$ and $\mathbf{r}=\mathbf{r}_e-\mathbf{r}_p$ are, respectively, the relative internal time and space position, while T and $\mathbf{R}$ refer to the center-of-mass time and space coordinates.

Thus, substituting in Eq. (2.2) the Fourier-transform according to the rule

$$G^{e/p}(\mathbf{r},\mathbf{r}',t-t') = \int G^{e/p}(\mathbf{r},\mathbf{r}',\varepsilon)\exp[-i\varepsilon(t-t')]\frac{d\varepsilon}{2\pi}, \quad (2.9)$$

we obtain the equation

$$\eta^{E,\mathbf{P}}(\boldsymbol{r},t) = \frac{ie^2}{2\pi}\int \exp[-i\varepsilon(t-t')]G^e(\mathbf{r},\mathbf{R}'+\mathbf{r}',\varepsilon)\gamma^\mu \times G^p(0,\mathbf{R}',E-\varepsilon)\gamma^\nu D_{\mu\nu}(\mathbf{r}',t') \times \exp\left\{i\left(\mathbf{P}+\frac{e}{2}[\mathbf{B}\times\mathbf{r}']\right)\mathbf{R}'\right\} \times \eta^{E,\mathbf{P}}(\mathbf{r}',t')d\varepsilon d\mathbf{r}'d\mathbf{R}'dt'. \quad (2.10)$$

We will assume that the vector potential (2.4) is specified in the rest frame of the neutron, which is not supposed to be affected by a magnetic field. Thus, the neutron wave function is given by

$$\psi_n = u_n e^{-im_n t}, \quad u_n = \begin{pmatrix} w_n \\ 0 \end{pmatrix} (w_n^* w_n = 1), \quad (2.11)$$

where $p_n=(m_n,0)$, w_n is the two-component neutron spinor. The neutrino is supposed to be massless and not interacting with a magnetic field. Therefore its wave function can be presented as follows:

$$\psi_\nu = u_\nu e^{-i(\varepsilon_\nu t-\mathbf{p}_\nu\mathbf{r})}, \quad u_\nu = \frac{1}{\sqrt{2}}\begin{pmatrix} v_\nu \\ -v_\nu \end{pmatrix} (v_\nu^* v_\nu = 1), \quad (2.12)$$

where $p_\nu=(\varepsilon_\nu,\mathbf{p}_\nu)$, $\varepsilon_\nu=|\mathbf{p}_\nu|$, and v_ν is the two-component eigenspinor of the spin-projection operator, $(\boldsymbol{\sigma}\mathbf{p}_\nu)v_\nu=-\varepsilon_\nu v_\nu$.

Accounting for Eqs. (2.8), (2.11), and (2.12), the matrix element (2.1) takes the form

$$T_{fi} = \frac{G}{\sqrt{2}}(2\pi)^4\delta(m_n-\varepsilon_\nu-E)\delta^{(3)}(\mathbf{p}_\nu-\mathbf{P})\bar{\eta}^{E,\mathbf{P}}(0,0)\gamma_\mu \times (1+\alpha\gamma_5)u_n\gamma^\mu(1+\gamma_5)u_\nu. \quad (2.13)$$

Using the rules[1] $|2\pi\delta(E)|^2=2\pi\delta(E)$ and $|(2\pi)^3\delta^{(3)}(\mathbf{p})|^2=(2\pi)^3\delta^{(3)}(\mathbf{p})$, we get the rate of the decay process (1.2) as

$$w_b = \frac{G^2}{2}\sum\nolimits^{(pe^-)}\int(2\pi)^4\delta(m_n-\varepsilon_\nu-E) \times \delta^{(3)}(\mathbf{p}_\nu-\mathbf{P})|\mathcal{M}|^2\frac{d\mathbf{p}_\nu}{(2\pi)^3}\frac{d\mathbf{P}}{(2\pi)^3}, \quad (2.14)$$

where

$$\mathcal{M} = \bar{\eta}^{E,\mathbf{P}}(0,0)\gamma_\mu(1+\alpha\gamma_5)u_n\gamma^\mu(1+\gamma_5)u_\nu \quad (2.15)$$

is a reduced matrix element, the sum $\sum^{(pe^-)}$ runs over all bound states of the (pe^-) system and the average over neutron spin states is assumed.

III. THE BETHE-SALPETER WAVE FUNCTION

The key element that determines the decay rate (2.14) is the wave function of the bound (pe^-) system. We solve the Bethe-Salpeter equation (2.10) neglecting the retardation effects [27] (this amounts to the nonrelativity of internal motion), so that the photon propagator in the Coulomb gauge assumes the form

$$D_{\mu\nu}(\mathbf{r},t) = -\delta_\mu^0\delta_\nu^0\frac{\delta(t)}{r}. \quad (3.1)$$

Inserting Eq. (3.1) in the Bethe-Salpeter equation (2.10), we obtain

$$\eta^{E,\mathbf{P}}(\mathbf{r},0) = -\frac{ie^2}{2\pi}\int G^e(\mathbf{r},\mathbf{R}'+\mathbf{r}',\varepsilon)\gamma^0 \times G^p(0,\mathbf{R}',E-\varepsilon)\gamma^0\frac{1}{r'} \times \exp\left\{i\left(\mathbf{P}+\frac{e}{2}[\mathbf{B}\times\mathbf{r}']\right)\mathbf{R}'\right\} \times \eta^{E,\mathbf{P}}(\mathbf{r}',0)d\varepsilon d\mathbf{r}'d\mathbf{R}'. \quad (3.2)$$

To carry out an integration over ε in the kernel of this equation, we expand the electron and proton propagators into series over Dirac eigenstates in a magnetic field keeping only the positive energy pole contributions [28]:

$$G^{e/p}(\mathbf{r},\mathbf{r}',\varepsilon) = \sum_\kappa \frac{\psi_{e/p}^{(\kappa)}(\mathbf{r})\bar{\psi}_{e/p}^{(\kappa)}(\mathbf{r}')}{\varepsilon-\varepsilon_\kappa+i0}. \quad (3.3)$$

If z axis is directed along the magnetic field vector $\mathbf{B}$, then $\varepsilon_\kappa=\sqrt{m_{e/p}^2+p_z^2+2neB}$ is the energy of the Dirac eigenstate specified by a set of the quantum numbers $\kappa=\{n,p_z,j_z,s\}$, where the discrete numbers $n=0,1,2,\ldots$ denote the Landau levels, p_z is the longitudinal momentum, j_z is the z component of the total angular momentum, and $s=\pm1$ is equivalent to the spin quantum number in the field-free case (see, for instance, [29]). In cylindrical space coordinates, the Dirac

[1] The normalization length and time are set to unity.

wave functions in Eq. (3.3) can be presented as the products of longitudinal and transverse parts

$$\psi^{(\kappa)}_{e/p}(\mathbf{r}) = \exp(ip_z z)\lambda^{(\tilde{\kappa})}_{e/p}(p_z, \boldsymbol{\rho})\ (\boldsymbol{\rho} \equiv \mathbf{r}_\perp), \tag{3.4}$$

where $\lambda^{(\tilde{\kappa})}_{e/p}(p_z, \boldsymbol{\rho})$ is the transverse spinor labeled by a set of the quantum numbers $\tilde{\kappa} = \{n, j_z, s\}$. The explicit forms of the transverse electron and proton spinors are

$$\lambda_e^{(n,j_z,+1)}(p_z,\boldsymbol{\rho}) = \frac{1}{\sqrt{2\varepsilon_{\kappa_e}}}\begin{pmatrix} \sqrt{m_e+\varepsilon_{\kappa_e}}\phi_{n-1,j_z-1/2}(\boldsymbol{\rho}) \\ 0 \\ \frac{p_z}{\sqrt{m_e+\varepsilon_{\kappa_e}}}\phi_{n-1,j_z-1/2}(\boldsymbol{\rho}) \\ i\sqrt{\frac{2neB}{m_e+\varepsilon_{\kappa_e}}}\phi_{n,j_z+1/2}(\boldsymbol{\rho}) \end{pmatrix},$$

$$\lambda_e^{(n,j_z,-1)}(p_z,\boldsymbol{\rho}) = \frac{1}{\sqrt{2\varepsilon_{\kappa_e}}}\begin{pmatrix} 0 \\ \sqrt{m_e+\varepsilon_{\kappa_e}}\phi_{n,j_z+1/2}(\boldsymbol{\rho}) \\ -i\sqrt{\frac{2neB}{m_e+\varepsilon_{\kappa_e}}}\phi_{n-1,j_z-1/2}(\boldsymbol{\rho}) \\ \frac{-p_z}{\sqrt{m_e+\varepsilon_{\kappa_e}}}\phi_{n,j_z+1/2}(\boldsymbol{\rho}) \end{pmatrix},$$

$$\lambda_p^{(n,j_z,+1)}(p_z,\boldsymbol{\rho}) = \frac{1}{\sqrt{2\varepsilon_{\kappa_p}}}\begin{pmatrix} \sqrt{m_p+\varepsilon_{\kappa_p}}\phi_{n,j_z-1/2}(\boldsymbol{\rho}) \\ 0 \\ \frac{p_z}{\sqrt{m_p+\varepsilon_{\kappa_p}}}\phi_{n,j_z-1/2}(\boldsymbol{\rho}) \\ -i\sqrt{\frac{2neB}{m_p+\varepsilon_{\kappa_p}}}\phi_{n-1,j_z+1/2}(\boldsymbol{\rho}) \end{pmatrix},$$

$$\lambda_p^{(n,j_z,-1)}(p_z,\boldsymbol{\rho}) = \frac{1}{\sqrt{2\varepsilon_{\kappa_p}}}\begin{pmatrix} 0 \\ \sqrt{m_p+\varepsilon_{\kappa_p}}\phi_{n-1,j_z+1/2}(\boldsymbol{\rho}) \\ i\sqrt{\frac{2neB}{m_p+\varepsilon_{\kappa_p}}}\phi_{n,j_z-1/2}(\boldsymbol{\rho}) \\ \frac{-p_z}{\sqrt{m_p+\varepsilon_{\kappa_p}}}\phi_{n-1,j_z+1/2}(\boldsymbol{\rho}) \end{pmatrix}, \tag{3.5}$$

where the transverse Landau orbitals are given by [21,29]

$$\phi_{n_\rho,m}(\boldsymbol{\rho}) = \sqrt{\frac{n_\rho!}{2\pi a_B^2(n_\rho-m)!}}\exp(im\varphi) \times \exp\left(-\frac{\rho^2}{4a_B^2}\right)\left(\frac{\rho^2}{2a_B^2}\right)^{-m/2} L_{n_\rho}^{(-m)}\left(\frac{\rho^2}{2a_B^2}\right) \times (n_\rho = 0,1,2,\ldots,\quad n_\rho - m = 0,1,2,\ldots) \tag{3.6}$$

with $L_{n_\rho}^{(-m)}$ being a generalized Lagguerre polynomial.

Using Eqs. (3.3) and (3.4), we have

$$G^{e/p}(\mathbf{r},\mathbf{r}',\varepsilon) = \frac{1}{2\pi}\sum_{\tilde{\kappa}}\int \frac{\lambda^{(\tilde{\kappa})}_{e/p}(p_z,\boldsymbol{\rho})\bar{\lambda}^{(\tilde{\kappa})}_{e/p}(p_z,\boldsymbol{\rho}')}{\varepsilon - \sqrt{\varepsilon_{\tilde{\kappa}}^2 + p_z^2} + i0} \times \exp[ip_z(z-z')]dp_z, \tag{3.7}$$

where $\varepsilon_{\tilde{\kappa}} = \sqrt{m_{e/p}^2 + 2neB}$ is the energy of the transverse motion. Inserting Eq. (3.7) in Eq. (3.2) and integrating over ε, we receive

$$\eta^{E,\mathbf{P}}(\mathbf{r},0) = -\frac{e^2}{2\pi}\sum_{\tilde{\kappa}_e,\tilde{\kappa}_p}\int \frac{\exp[iq_z(z-z')]\lambda_e^{(\tilde{\kappa}_e)}(q_z,\boldsymbol{\rho})\lambda_p^{(\tilde{\kappa}_p)}(P_z-q_z,0)}{E - \sqrt{\varepsilon_{\tilde{\kappa}_e}^2 + q_z^2} - \sqrt{\varepsilon_{\tilde{\kappa}_p}^2 + (P_z-q_z)^2}} \times \lambda_e^{(\tilde{\kappa}_e)*}(q_z,\boldsymbol{\rho}'+\mathbf{R}'_\perp)\lambda_p^{(\tilde{\kappa}_p)*}(P_z-q_z,\mathbf{R}'_\perp)\frac{1}{\sqrt{\rho'^2+z'^2}} \times \exp\left\{i\left(\mathbf{P}_\perp + \frac{e}{2}[\mathbf{B}\times\boldsymbol{\rho}']\right)\mathbf{R}'_\perp\right\}\eta^{E,\mathbf{P}}(\mathbf{r}',0)dq_z d\boldsymbol{\rho}' dz' d\mathbf{R}'_\perp, \tag{3.8}$$

where the sum over $\tilde{\kappa}_p$ involves only the transverse proton spinors with $j_z = \pm 1/2$ [according to Eqs. (3.5) and (3.6) the transverse proton spinors with $j_z \neq \pm 1/2$ vanish at $\boldsymbol{\rho} = 0$].

We consider below the following two cases of field strengths: (1) $B_{\rm cr} < B \ll B'_{\rm cr}$ and (2) $B'_{\rm cr} < B \ll B_p$. For each of these cases we develop the corresponding approximation to the Bethe-Salpeter equation (3.8) taking into account the kinematical regime of the decay process (1.2).

A. The case $B_{\rm cr} < B \ll B'_{\rm cr}$

Recall that in the case $B > B_{\rm cr}$ the energy conservation law dictates that the electron in the usual decay process (1.1) can occupy only the lowest Landau level. Considering the bound-state decay process (1.2) in the case $B > B_{\rm cr}$, we note that [as follows from Eq. (3.8)] the electron can (virtually) occupy not only the lowest Landau level. However, the probability for the electron to occupy the excited Landau levels is vanishingly small due to the following fact. Virtual electron transitions between the lowest and excited Landau levels are induced by the Coulomb electron-proton interaction, which in the present case of field strengths is much weaker than the interaction of the electron with a magnetic field. Thus, the electron contribution from excited Landau levels to the wave function of the bound (pe^-) system is negligible. Therefore, one can leave in Eq. (3.8) only those electron terms that correspond to the lowest Landau level (this amounts to the adiabatic approximation). In accordance with Eq. (3.5), the transverse electron spinors for the lowest Landau level ($n = 0$) are given by

$$\lambda_e^{(0,j_z,-1)}(p_z,\boldsymbol{\rho}) = u(p_z)\phi_{0,m}(\boldsymbol{\rho}), \quad m = j_z + \frac{1}{2} = 0,-1,-2,\ldots, \quad u(p_z) = \begin{pmatrix} 0 \\ \sqrt{\frac{m_e+\varepsilon_{\kappa_e}}{2\varepsilon_{\kappa_e}}} \\ 0 \\ \frac{-p_z}{\sqrt{2\varepsilon_{\kappa_e}(m_e+\varepsilon_{\kappa_e})}} \end{pmatrix}, \tag{3.9}$$

where $\varepsilon_{\kappa_e} = \sqrt{m_e^2 + p_z^2}$. In addition, the nonrelativistic approximation applies to the proton transverse spinors and energies:

$$\lambda_p^{(n,1/2,+1)}(p_z, \boldsymbol{\rho}) = u_p^{(+)}\phi_{n,0}(\rho),$$
$$\lambda_p^{(n,-1/2,-1)}(p_z, \boldsymbol{\rho}) = u_p^{(-)}\phi_{n-1,0}(\rho) \quad [\phi_{-1,0}(\rho) \equiv 0],$$
$$\varepsilon_{\kappa_p} = m_p + n\omega_p + \frac{p_z^2}{2m_p},$$
$$u_p^{(+)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u_p^{(-)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{3.10}$$

where $\omega_p = eB/m_p$ is the Larmor frequency for the proton.

Taking into account Eqs. (3.9) and (3.10) and using $\phi_{n,0}(0) = \sqrt{eB/2\pi}$, we arrive at

$$\eta^{E_\pm,\mathbf{P}}(\mathbf{r}, 0) = f(\mathbf{r})u_p^{(\pm)}, \tag{3.11}$$

where $E_- = E_+ + \omega_p$ and

$$f_k(\mathbf{r}) = -\frac{e^2}{2\pi}\sqrt{\frac{eB}{2\pi}}\sum_{n,m}\phi_{0,m}(\boldsymbol{\rho}) \times \int \frac{\exp[iq_z(z-z')]u_k(q_z)u_l(q_z)}{E_+ - E_e(q_z) - E_p(n, P_z - q_z)}\phi_{0,m}^*(\boldsymbol{\rho}' + \mathbf{R}'_\perp) \times \phi_{n,0}(R'_\perp)\frac{1}{\sqrt{\rho'^2 + z'^2}}\exp\left\{i\left(\mathbf{P}_\perp + \frac{e}{2}[\mathbf{B}\times\boldsymbol{\rho}']\right)\mathbf{R}'_\perp\right\} \times f_l(\mathbf{r}')dq_z d\boldsymbol{\rho}' dz' d\mathbf{R}'_\perp. \tag{3.12}$$

Here

$$E_e(q_z) = \sqrt{m_e^2 + q_z^2},$$
$$E_p(n, P_z - q_z) = m_p + n\omega_p + \frac{(P_z - q_z)^2}{2m_p},$$

$k, l = 1, 2, 3, 4$ stand for the spinor indices and a sum over the repeating spinor index l is assumed. To carry out an integration over $\mathbf{R}'_\perp$ in the kernel of Eq. (3.12), we use an expansion of the undisplaced Landau state over an infinite set of the displaced Landau orbitals [29]

$$\phi_{n,m}(\boldsymbol{\rho}) = \exp\left(i\frac{e}{2}[\mathbf{B}\times\boldsymbol{\rho}]\boldsymbol{\rho}_0\right)\sqrt{\frac{2\pi}{eB}} \times \sum_{m'}\phi_{n-m',m-m'}(\boldsymbol{\rho}_0)\phi_{n,m'}(\boldsymbol{\rho} - \boldsymbol{\rho}_0). \tag{3.13}$$

Involving the following integral:

$$\int \exp(i\mathbf{P}_\perp\boldsymbol{\rho})\phi_{n-m',m-m'}^*(\boldsymbol{\rho})\phi_{n-N,0}(\rho)d\boldsymbol{\rho} = \frac{2\pi}{eB}\phi_{n-m',N-m'}^*(\boldsymbol{\rho}_C)\phi_{n-m,N-m}(\boldsymbol{\rho}_C), \tag{3.14}$$

where $\boldsymbol{\rho}_C = e[\mathbf{P}_\perp \times \mathbf{B}]/(eB)^2$, and introducing

$$g(\mathbf{r}) = \exp\left(-\frac{i}{2}\mathbf{P}_\perp\mathbf{r}\right)f(\mathbf{r} - \boldsymbol{\rho}_C), \tag{3.15}$$

we get

$$g_k(\mathbf{r}) = -\frac{e^2}{2\pi}\sum_m \phi_{0,m}(\boldsymbol{\rho}) \times \int \frac{\exp[iq_z(z-z')]u_k(q_z)u_l(q_z)}{E_+ - E_e(q_z) - E_p(|m|, P_z - q_z)}\phi_{0,m}^*(\boldsymbol{\rho}') \times \frac{1}{\sqrt{(\boldsymbol{\rho}' - \boldsymbol{\rho}_C)^2 + z'^2}}g_l(\mathbf{r}')dq_z d\boldsymbol{\rho}' dz'. \tag{3.16}$$

Let us seek a solution to this equation in a form of the following expansion:

$$g(\mathbf{r}) = \sum_m h(m; z)\phi_{0,m}(\boldsymbol{\rho}). \tag{3.17}$$

Inserting it in Eq. (3.16) we obtain an infinite set of the coupled integral equations

$$h_k(m; z) = \frac{1}{2\pi}\sum_{m'}\int \frac{\exp[iq_z(z-z')]u_k(q_z)u_l(q_z)}{E_+ - E_e(q_z) - E_p(|m|, P_z - q_z)} \times V_{mm'}(\boldsymbol{\rho}_C, z')h_l(m'; z')dq_z dz'. \tag{3.18}$$

Here the one-dimensional potential functions $V_{mm'}(\boldsymbol{\rho}_C, z)$ are given by

$$V_{mm'}(\boldsymbol{\rho}_C, z) = -e^2\int \frac{\phi_{0,m}^*(\boldsymbol{\rho})\phi_{0,m'}(\boldsymbol{\rho})}{\sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}_C)^2 + z^2}}d\boldsymbol{\rho}. \tag{3.19}$$

Examining their properties, we deduce that in the case $\mathbf{P}_\perp = 0$ the infinite set (3.18) becomes decoupled

$$h_k(m; z) = \frac{1}{2\pi}\int \frac{\exp[iq_z(z-z')]u_k(q_z)u_l(q_z)}{E_+ - E_e(q_z) - E_p(|m|, P_z - q_z)} \times V_m(z')h_l(m; z')dq_z dz', \tag{3.20}$$

where

$$V_m(z) = -e^2\int \frac{|\phi_{0,m}(\boldsymbol{\rho})|^2}{\sqrt{\rho^2 + z^2}}d\boldsymbol{\rho} = -e^2\exp\left(\frac{z^2}{4a_B^2}\right)\left(\frac{z^2}{2a_B^2}\right)^{\frac{|m|}{2}-\frac{1}{4}} W_{-\frac{|m|}{2}-\frac{1}{4},\frac{|m|}{2}+\frac{1}{4}}\left(\frac{z^2}{2a_B^2}\right) \tag{3.21}$$

with $W_{\mu,\nu}$ being a Whittaker function [30].

In the nonrelativistic limit, Eq. (3.20) is equivalent to the one-dimensional Schrödinger equation for a hydrogen atom in the adiabatic approximation

$$\left[\frac{1}{2\mu}\frac{d^2}{dz^2} - V_m(z) + E_m^\parallel\right]\psi_m(z) = 0, \tag{3.22}$$

where $\mu = (m_e + m_p)/2m_e m_p$ is the reduced mass and

$$E_m^\parallel = E_+ - m_p - m_e - \frac{P_z^2}{2(m_e + m_p)} - |m|\omega_p,$$
$$\psi_m(z) = \exp\left(\frac{iP_z m_e z}{m_p}\right)h_2(m; z). \tag{3.23}$$

Here it should be pointed out that the use of the nonrelativistic approximation (3.22) for the treatment of the bound states in the case of a magnetic field with strength $B \gtrsim B_{\rm cr}$ can be justified for the following reasons [31]: (1) the shape of the transverse electron wave function determined by the Landau orbital (3.6) in the relativistic theory is the same as in the nonrelativistic theory; (2) the electron remains nonrelativistic in the z direction as long as the binding energy $|E^{\parallel}| \ll m_e$.

The one-dimensional Schrödinger equation (3.22) has been studied in Refs. [12–14,20]. Here we outline its basic properties that are important for our purposes. The eigensolutions of Eq. (3.22) can be categorized in two distinct classes: (1) the states having no node and (2) the states having node(s) in their wave functions. The states having no node in their wave functions are tightly bound. For each m there is one such state, which is the most tightly bound (the ground state) in the case $m = 0$. The ground-state energy $E^{\parallel} \ll -13.6$ eV, i.e., it is much lower than that of a hydrogen atom in the field-free case. On the contrary, the states having nodes in their wave functions are weakly bound. For example, in the case $m = 0$ the state with one node in its wave function has about the same energy as the ground state of a hydrogen atom in the field-free case, $E^{\parallel} \simeq -13.6$ eV, whereas the states with more than one node in their wave functions have the higher energies ($E^{\parallel} > -13.6$ eV).

B. The case $B'_{\rm cr} < B \ll B_p$

When considering the influence of very strong magnetic fields on the neutron β decay, one should be careful about the effect of a magnetic field on the anomalous magnetic moments of the neutron and proton. In particular, it is known (see, for instance, Ref. [7]) that the interplay between the anomalous magnetic moments of the neutron and proton shifts the energies of these particles making the neutron stable in the case $B \sim 1.5 \times 10^{18}$ G. However, if the field strength is $B \lesssim 1.5 \times 10^{18}$ G the effect of the nucleon anomalous magnetic moments is still subdominant. Note that the corresponding shift of the electron energy due to the electron anomalous magnetic moment vanishes in such a strong field [32,33] (see also Ref. [7]). In what follows, we neglect the effects connected with the anomalous magnetic moments, assuming that the field strength does not exceed the value $B \sim 1.5 \times 10^{18}$ G.

In the case of a superstrong magnetic field ($B'_{\rm cr} < B \ll B_p$) both the electron and the proton in the usual decay process (1.1) can occupy only the lowest Landau level. Using the same arguments as in the preceding subsection with regard to the electron's dynamics in the bound-state decay process (1.2), we again arrive at Eq. (3.18). However, noticing that the (virtual) proton transitions between Landau levels induced by the Coulomb electron-proton interaction are almost completely suppressed, we can neglect the couplings between different values of m in Eq. (3.18) and consider only the case $m = 0$. Thereby, instead of the infinite set of coupled equations (3.18), we obtain the single integral equation

$$h_k(0; z) = \frac{1}{2\pi} \int \frac{\exp[iq_z(z - z')] u_k(q_z) u_l(q_z)}{E_+ - E_e(q_z) - E_p(0, P_z - q_z)} \times V_{00}(\boldsymbol{\rho}_C, z') h_l(0; z') dq_z dz'. \qquad (3.24)$$

The corresponding Schrödinger equation is

$$\left[\frac{1}{2\mu}\frac{d^2}{dz^2} - V_{00}(\boldsymbol{\rho}_C, z) + E_0^{\parallel}(\boldsymbol{\rho}_C)\right]\psi_0(\boldsymbol{\rho}_C, z) = 0, \qquad (3.25)$$

where $\boldsymbol{\rho}_C$ plays a role of parameter and

$$E_0^{\parallel}(\boldsymbol{\rho}_C) = E_+ - m_p - m_e - \frac{P_z^2}{2(m_e + m_p)},$$
$$\psi_0(\boldsymbol{\rho}_C, z) = \exp\left(\frac{iP_z m_e z}{m_p}\right) h_2(0; z). \qquad (3.26)$$

Note that if $\rho_C \ll a_B$, we have

$$V_{00}(\boldsymbol{\rho}_C, z) \approx V_0(z) = -\frac{e^2}{a_B}\sqrt{\frac{\pi}{2}}\text{erfc}\left(\frac{|z|}{\sqrt{2}a_B}\right)\exp\left(\frac{z^2}{2a_B^2}\right) = \begin{cases} -\sqrt{\frac{\pi}{2}}\frac{e^2}{a_B}, & |z| \to 0, \\ -\frac{e^2}{|z|}, & |z| \to \infty, \end{cases} \qquad (3.27)$$

where erfc(y) is the complementary error function [30].

IV. THE DECAY RATE

In this section, using the results of Sec. III for the wave function of the bound (pe^-) system, we derive the bound-state decay rates in the particular cases, namely, $B_{\rm cr} < B \ll B'_{\rm cr}$ and $B'_{\rm cr} < B \ll B_p$. Then we utilize the obtained bound-state decay rates in the extrapolation procedure, when treating the general case $B_e \lesssim B \ll B_p$.

A. The case $B_{\rm cr} < B \ll B'_{\rm cr}$

Using Eqs. (3.11), (3.15), and (3.17) we perform an integration over $\mathbf{P}$ in Eq. (2.14) and obtain for the bound-state decay rate

$$w_b = \frac{G^2}{2}\sum_\tau \int 2\pi \{\delta[m_n - \varepsilon_\nu - E_{+,\tau}(\mathbf{p}_\nu)]|\mathcal{M}_{+,\tau}(\mathbf{p}_\nu)|^2 + \delta[m_n - \varepsilon_\nu - E_{-,\tau}(\mathbf{p}_\nu)]|\mathcal{M}_{-,\tau}(\mathbf{p}_\nu)|^2\}\frac{d\mathbf{p}_\nu}{(2\pi)^3}. \qquad (4.1)$$

Here the indices τ label the solutions of Eq. (3.18) with corresponding eigenenergies $E_{\pm,\tau}(\mathbf{p}_\nu)$ $[E_{-,\tau}(\mathbf{p}_\nu) = E_{+,\tau}(\mathbf{p}_\nu) + \omega_p]$. In accordance with Eq. (2.15), the reduced matrix elements are given by

$$\mathcal{M}_{\pm,\tau}(\mathbf{p}_\nu) = \sum_m \phi^*_{0,m}(\boldsymbol{\rho}_\nu)\big[\bar{u}_p^{(\pm)}\gamma_\mu(1 + \alpha\gamma_5)u_n\big] \times [\bar{h}_\tau(m; 0)\gamma^\mu(1 + \gamma_5)u_\nu], \qquad (4.2)$$

where $\boldsymbol{\rho}_\nu = e[\mathbf{p}_\nu \times \mathbf{B}]/(eB)^2$.

To evaluate the reduced matrix element (4.2), we note that for the field strengths under consideration one has $\rho_\nu^{\max}/a_B = p_\nu^{\max} a_B \lesssim 1$, where $p_\nu^{\max}$ is the maximal possible value for the neutrino's momentum in the process (1.2). This allows us to put $\rho_\nu = 0$ in Eq. (4.2), so that we get

$$w_b = \frac{eBG^2}{2}\sum_\tau \int \{\delta[\Delta - \varepsilon_\nu - m_e - E_\tau^{\|}(p_{\nu,z})]|\mathcal{M}_{+,\tau}(\mathbf{p}_\nu)|^2 + \delta\big[\Delta-\varepsilon_\nu-m_e-\omega_p-E_\tau^{\|}(p_{\nu,z})]|\mathcal{M}_{-,\tau}(\mathbf{p}_\nu)|^2\}\frac{d\mathbf{p}_\nu}{(2\pi)^3}. \quad (4.3)$$

Here

$$\mathcal{M}_{+,\tau}(\mathbf{p}_\nu) = \sqrt{2}\big[h_{\tau,2}^{p_{\nu,z}*}(0) - h_{\tau,4}^{p_{\nu,z}*}(0)\big] \times [(1+\alpha)\omega_{n,1}v_{\nu,2} - 2\alpha\omega_{n,2}v_{\nu,1}],$$
$$\mathcal{M}_{-,\tau}(\mathbf{p}_\nu) = \sqrt{2}\big[h_{\tau,2}^{p_{\nu,z}*}(0) - h_{\tau,4}^{p_{\nu,z}*}(0)\big](1-\alpha)\omega_{n,2}v_{\nu,2}, \quad (4.4)$$

where in accordance with Eq. (3.18) the wave functions $h_\tau^{p_{\nu,z}}$ are the solutions of the equation

$$h_i^{p_{\nu,z}}(z) = \frac{1}{2\pi} \times \int \frac{\exp[iq_z(z-z')]u_i(q_z)u_l(q_z)}{E^{\|}(p_{\nu,z}) + m_e + m_p - E_e(q_z) - E_p(0, p_{\nu,z}-q_z)} \times V_0(z')h_l^{p_{\nu,z}}(z')dq_z dz'. \quad (4.5)$$

In the nonrelativistic approximation

$$E_\tau^{\|}(p_{\nu,z}) = E_\tau^{\|} + \frac{p_{\nu,z}^2}{2(m_e+m_p)}, \qquad h_{\tau,2}^{p_{\nu,z}}(0) = \psi_\tau(0), \quad h_{\tau,4}^{p_{\nu,z}}(0) = 0, \quad (4.6)$$

where ψ_τ are the solutions of the Schrödinger equation (3.22) for $m = 0$ with energies $E_{0,\tau}^{\|} \equiv E_\tau^{\|}$. The index $\tau = 0, 1, \ldots$ corresponds to the number of nodes in the wave function $\psi_\tau(z)$.

Using Eq. (4.6) and neglecting ω_p and $p_\nu^2/2(m_e+m_p)$ with respect to $\Delta - m_e - E^{\|}$, we obtain (after averaging over neutron spin states) the following result for the bound-state decay rate:

$$w_b = \frac{eBG^2}{8\pi^2}(1+3\alpha^2)\sum_\tau |\psi_\tau(0)|^2(\Delta - m_e - E_\tau^{\|})^2, \quad (4.7)$$

where the sum runs over even states ($\tau = 0, 2, \ldots$). The odd states ($\tau = 1, 3, \ldots$) do not contribute to the bound-state decay rate because their wave functions vanish at the origin [$\psi_\tau(0) \equiv 0$].

Let us derive the ratio of the bound-state decay rate w_b [see Eq. (4.7)] and the continuum-state decay rate w_c, i.e., the decay rate of the usual process (1.1). To facilitate an accurate estimation of the ratio w_b/w_c, we calculate w_c under the same assumptions that have been made to obtain Eq. (4.7), i.e., we assume $\rho_\nu = 0$ and ω_p, $p_\nu^2/2(m_e+m_p) \ll \Delta - m_e$. Thus, we get

$$\frac{w_b}{w_c} = \frac{\pi\sum_\tau |\psi_\tau(0)|^2}{Cm_e}\left(\frac{\Delta - E_\tau^{\|}}{m_e} - 1\right)^2, \quad (4.8)$$

where

$$C = \frac{1}{3}\left(\frac{\Delta^2}{m_e^2} + 2\right)\sqrt{\frac{\Delta^2}{m_e^2} - 1} - \frac{\Delta}{m_e}\text{arccosh}\left(\frac{\Delta}{\mathrm{m_e}}\right). \quad (4.9)$$

It is useful to compare the result (4.8) with that in the field-free case (see also Refs. [1,2]):

$$\frac{w_b}{w_c} = \frac{2\pi^2\sum_n |\psi_n(0)|^2}{C_0 m_e^3}\left(\frac{\Delta - \varepsilon_n}{m_e} - 1\right)^2, \quad (4.10)$$

where

$$C_0 = \frac{1}{60}\left(\frac{2\Delta^4}{m_e^4} - \frac{9\Delta^2}{m_e^2} - 8\right) \times \sqrt{\frac{\Delta^2}{m_e^2} - 1} + \frac{\Delta}{4m_e}\text{arccosh}\left(\frac{\Delta}{m_e}\right), \quad (4.11)$$

$n = 1, 2, \ldots$ is the principal quantum number, ψ_n [$|\psi_n(0)|^2 = (4\pi n^3 a_0^3)^{-1}$] and $\varepsilon_n = -(2a_0 n^2)^{-1}$ are, respectively, the wave function and the binding energy of the *ns* state of a hydrogen atom. It is seen that both expressions (4.8) and (4.10) have a similar structure. However, the important difference consists in that ψ_τ are the one-dimensional wave functions while ψ_n are the three-dimensional ones. This difference is also reflected in the appearance of the factor m_e^{-1} in the right-hand side of Eq. (4.8) instead of the factor m_e^{-3} occurring in the field-free case (4.10).

B. The case $B'_{\rm cr} < B \ll B_p$

By analogy with the previous subsection, in the case of a magnetic field with strength in the range $B'_p < B \ll B_p$ we have

$$w_b = \frac{G^2}{2}\sum_\tau \int 2\pi\delta[m_n - \varepsilon_\nu - E_{+,\tau}(\mathbf{p}_\nu)]|\mathcal{M}_{+,\tau}(\mathbf{p}_\nu)|^2\frac{d\mathbf{p}_\nu}{(2\pi)^3}, \quad (4.12)$$

where the reduced matrix elements are given by

$$\mathcal{M}_{+,\tau}(\mathbf{p}_\nu) = \sqrt{\frac{eB}{2\pi}}\exp\left(-\frac{p_{\nu,\perp}^2}{4eB}\right)[\bar{u}_p^{(+)}\gamma_\mu(1+\alpha\gamma_5)u_n] \times [\bar{h}_\tau(0;0)\gamma^\mu(1+\gamma_5)u_\nu] \quad (4.13)$$

with h_τ being the solutions of Eq. (3.24). Following the approximate procedure developed in the preceding subsection, we deduce that

$$w_b = \frac{eBG^2}{8\pi^2}(1+2\alpha+5\alpha^2) \times \sum_\tau |\psi_\tau(0)|^2(\Delta - m_e - E_\tau^{\|})^2, \quad (4.14)$$

where ψ_τ and $E_\tau^{\|}$ are specified in Eq. (4.6). And for the ratio of bound-state and continuum-state decay rates we obtain

$$\frac{w_b}{w_c} = \frac{\pi\sum_\tau |\psi_\tau(0)|^2}{Cm_e}\left(\frac{\Delta - E_\tau^{\|}}{m_e} - 1\right)^2. \quad (4.15)$$

It is remarkable that this expression is identical with the one obtained in the case $B_{\rm cr} < B \ll B'_{\rm cr}$ [see Eq. (4.8)]. We utilize this fact below, when estimating the ratio w_b/w_c in the general case $B_e \lesssim B \ll B_p$.

C. The case $B_e \lesssim B \ll B_p$

A smooth dependence of w_c on the field strength B [8,9] combined with the identity of the expressions (4.8) and (4.15) allows us to extrapolate the obtained results for the ratio w_b/w_c in the whole range of considered field strengths. Specifically, we assume the expression (4.8) [(4.15)] obtained in the case $B_{\rm cr} < B \ll B'_{\rm cr}$ ($B'_{\rm cr} < B \ll B_p$) to be valid for estimating the ratio w_b/w_c in the general case $B_e \lesssim B \ll B_p$, which incorporates the above particular ranges of field strengths.

Let us note that in the field-free case ($B = 0$) the relative contribution of excited states to the ratio w_b/w_c [see Eq. (4.10)] is about 20% [2,3]. Taking into account that, in contrast to the field-free situation, in the presence of a strong magnetic field the excited states are very weakly bound in comparison with the ground state, we neglect their contribution to the ratio w_b/w_c. The ground-state wave function $\psi(z)$ can be approximated as follows [12,34]:

$$\psi(z) = (2\mu|E^{\parallel}|)^{1/4} \exp(-\sqrt{2\mu|E^{\parallel}|}|z|), \tag{4.16}$$

where $E^{\parallel}$ is the ground-state energy. Thus we get ($\mu \simeq m_e$)

$$\frac{w_b}{w_c} = \frac{\pi}{C}\sqrt{\frac{2|E^{\parallel}|}{m_e}}\left(\frac{\Delta - E^{\parallel}}{m_e} - 1\right)^2. \tag{4.17}$$

In the field-free case the corresponding result is given by (accounting only for the ground-state contribution)

$$\frac{w_b}{w_c} = \frac{\pi}{C_0}\sqrt{\frac{2|\varepsilon|}{m_e}}\frac{|\varepsilon|}{m_e}\left(\frac{\Delta - \varepsilon}{m_e} - 1\right)^2, \tag{4.18}$$

where $\varepsilon = -e^4 m_e/2$ is the ground-state energy of the hydrogen atom.

Figure 1 displays the numerical results for the ratio w_b/w_c calculated in accordance with Eq. (4.17) using two asymptotic formulas for $E^{\parallel}$: (1) the well-known formula from Ref. [35]

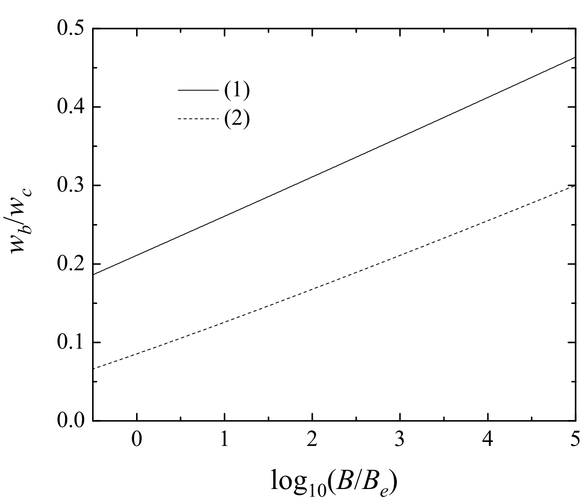

FIG. 1. The estimate for the ratio of bound-state and continuum-state decay rates in the case of a strong magnetic field. The cases, where the asymptotic values for $E^{\parallel}$ are derived according to Ref. [35] [see Eq. (4.19)] and Ref. [20] [see Eq. (4.20)], are represented by the solid and dashed lines, respectively.

(see also Refs. [12,13,15])

$$E^{\parallel} = \varepsilon \ln^2\left(\frac{eB}{2m_e|\varepsilon|}\right) \tag{4.19}$$

and (2) the recent formula from Ref. [20]

$$E^{\parallel} = \varepsilon \ln^2\left[\frac{c_\infty eB/2m_e|\varepsilon|}{\ln^2(eB/2m_e|\varepsilon|)}\right], \tag{4.20}$$

where $c_\infty = 0.2809$. It is seen that both asymptotic formulas give the same order of magnitude for the ratio w_b/w_c. In both cases the results exhibit a linear dependence on $\log_{10}(B/B_e)$. However, the results obtained on a basis of Eq. (4.19) are approximately two times larger in magnitude than those obtained on a basis of Eq. (4.20). Due to the fact that the asymptotic formula (4.20) yields more accurate values for the ground-state energy $E^{\parallel}$ than the formula (4.19) does (see Ref. [20] for details), we can conclude that the estimate for the ratio w_b/w_c using Eq. (4.20) is more realistic.

V. SUMMARY AND CONCLUSIONS

In summary, we have considered and analyzed theoretically the neutron decay into a bound (pe^-) state and an antineutrino in the presence of a magnetic field with strength $B \gtrsim 10^{13}$ G. The amplitude of the bound-state decay process has been formulated within the framework of the standard model of weak interactions. For the description of the bound (pe^-) system we have employed the Bethe-Salpeter equation. The approximations to the Bethe-Salpeter wave function of the bound (pe^-) state have been specified depending on the field strength B. We have derived the bound-state decay rate w_b in two particular cases, namely, $B_{\rm cr} < B \ll B'_{\rm cr}$ and $B'_{\rm cr} < B \ll B_p$. In both cases, the identical expressions for the ratio of bound-state and continuum-state decay rates w_b/w_c have been obtained. We have estimated the ratio w_b/w_c in the general case $B_e \lesssim B \ll B_p$ using two asymptotic formulas [20,35] for the ground-state energy of the hydrogen atom in a strong magnetic field. For both asymptotic formulas a logarithmiclike behavior $w_b/w_c \simeq a \log_{10}(B/B_e) + b$, where a and b are positive constants, has been determined.

The numerical estimate for the ratio of bound-state and continuum-state decay rates has been performed. It has been found that in contrast to the field-free case, where the bound-state decay mode is suppressed by a factor of about 4×10^{-6} [1–3] as compared with the usual (continuum-state) decay mode, in the presence of a strong magnetic field $B \gtrsim B_e$ the ratio w_b/w_c is of the order 0.1–0.4. This remarkable finding can be important for the physics of supernovae and neutron stars, where magnetic fields with strength $B \gtrsim 10^{13}$ G may exist. In particular, the high value of the neutron bound-state decay rate w_b can influence the nucleon balance in protoneutron stars and the hydrogen fraction in the atmosphere of neutron stars and magnetars. The high value of w_b can affect the neutrino spectra from astrophysical objects with strong magnetic fields, because in the neutron bound-state β decay process the energy distribution of antineutrino is peaked about $\Delta - m_e$. However, for estimating these possible astrophysical effects one should consider a more involved problem, namely, the bound-state β decay of a neutron moving in the presence of a strong magnetic field and dense matter (for instance, such as a neutron star). This implies modifications of the photon, electron, and proton propagators in the Bethe-Salpeter equation (2.2) due to many-particle effects [36]. In addition, an electric field induced in the rest frame of a neutron (due to its transverse motion in the matter rest frame) should be accounted for, because for the transverse neutron velocities $v_\perp \gg a_B|E^\parallel|$ the induced electric fields are as strong as to pull the hydrogen atom apart. Note, however, that in dense matter an electric field can be strongly screened by the surrounding medium. The screening also modifies the electron-proton interaction. In the context of the present analysis the latter factor becomes appreciable if $a_B \gtrsim l_s$, where l_s is a screening length for the medium. Since $l_s \sim n_c^{-1/3}$, where n_c is a density of charged particles (electrons or protons) in matter, we obtain the following criterion $a_B \gtrsim n_c^{-1/3}$ (or $B \lesssim n_c^{2/3}/e$) which indicates the situation where the role of screening cannot be neglected (such situation can be encountered, for example, in the interiors of a protoneutron star).

Let us remark that the high value of the bound-state decay rate w_b can be also important for cosmological applications. If the controversial hypothesis of strong magnetic fields influencing the usual β decay process in the early universe [7] is realistic, then one has a right to expect that the neutron β decay into a bound state of the (pe^-) system might have substantial consequences for big-bang nucleosynthesis and the production of light elements in the early universe.

Note that in the terrestrial laboratory environment the strongest magnetic field that can be produced is of the order of 10^7 G (see, for example, [37]) which is much lower than B_e. However, the results of our present analysis can be used even in the case of such fields if the neutron β decay takes place in semiconducting or dielectric media, where a small effective mass for the electron and a large dielectric constant reduce the Coulomb force relative to the magnetic force [31].

Finally, it is straightforward to generalize the results obtained in this work to the bound-state β decay of a nucleus with charge $Z-1$ in a strong magnetic field. This is realized by replacing the nucleon mass defect Δ with the corresponding value for the nucleus under consideration and taking into account that the atomic energy and radius are given by $\varepsilon = -Z^2e^4m_e$ and $a_0 = (Ze^2m_e)^{-1}$, respectively. After performing such a procedure, it can be deduced from Eqs. (4.17), (4.19), and/or (4.20) that $w_b/w_c \simeq Z[a\log_{10}(Z^{-2}B/B_e) + b]$, where the positive constants a and b do not depend on Z. Thus, the dependence of the ratio w_b/w_c on the nuclear charge is different from that in the absence of a magnetic field, in which case $w_b/w_c \propto Z^3$.

ACKNOWLEDGMENTS

We are very grateful to Yuri V. Popov for fruitful discussions on the problem of neutron bound-state β decay. We are also thankful to Vladimir L. Kauts for valuable comments.

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