Boltzmann equation in classical and quantum field theory

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Improving upon the previous treatment by Mueller and Son, we derive the Boltzmann equation that results from a classical scalar field theory. This is obtained by starting from the corresponding quantum field theory and taking the classical limit with particular emphasis on the path integral and perturbation theory. A previously overlooked Van Vleck determinant is shown to control the tadpole type of self-energy that can still appear in the classical perturbation theory. Further comments on the validity of the approximations and possible applications are also given.

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I. INTRODUCTION

There are many situations in nature where classical field theory is the most direct and efficient way of describing the system under study. This happens when the occupation number becomes large enough that the quantum commutator becomes irrelevant. Such situations include most of macroscopic electromagnetic systems and the dense systems created in colliding heavy ions. In studying heavy-ion collisions, both the particle degrees of freedom and the classical field degrees of freedom becomes relevant: The final state of a collision consists mostly of the particle degrees of freedom, whereas the initial state of two approaching nuclei can be efficiently described by a classical non-Abelian gauge field [1–3]. A natural framework which encompasses both elements is the coupled system of classical field equation and the Boltzmann type of kinetic equation with the mean field [4].

An important problem in this framework is the conversion of classical field degrees of freedom into particle degrees of freedom and their subsequent thermalization. In this respect, the equivalence of the classical field theory and the Boltzmann equation proposed by Mueller and Son [5] is significant because it has a potential of providing a consistent framework for the thermalization in heavy-ion collisions. The current article improves on the work of Mueller and Son in the following aspects.

First, going from quantum to classical many body theory almost always involves the Wigner function [4,6,7]. In Ref. [5], this point was overlooked. In this work, we show that the Wigner functional does appear in the formulation because of the fact that the density operator is in general nondiagonal.

Second, the Feynman rules used in Ref. [5] for the classical field are a mixture of quantum and classical ones: The vertex rule is classical but the propagators are quantum mechanical. We show that consistent formulation involves only classical vertices and propagators.

Third, formulation of classical path integral involves a functional determinant related to the Jacobi field. Therefore, perturbation theory based on classical path integral involves ghost fermions [8,9]. It turned out that the classical perturbation theory as formulated in Ref. [5] still contains a quantum effect in the form of the tadpole self-energy. The ghost contribution is shown to compensate this remaining quantum effect, thus making the theory consistently classical.

Despite these gaps in the formulation, the final result in Ref. [5] is essentially correct in the $f \gg 1$ limit. The goal of this article is to provide a more consistent derivation of the kinetic equation corresponding to the classical scalar field starting from the quantum scalar field. We take particular care in treating the path integral and the perturbation theory in a consistent manner. We also comment on the validity of this classical method and possible use in the context of understanding the thermalization in heavy-ion collisions.

II. CLASSICAL PATH INTEGRAL FROM QUANTUM PATH INTEGRAL

To be specific, consider a real scalar field theory defined by the following Hamiltonian:

$$H = \int d^3x \left[\frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi) \right]. \tag{1}$$

The equations of motion are as follows:

$$\dot{\phi} = \frac{\delta H}{\delta \pi} = \pi \tag{2}$$

$$\dot{\pi} = -\frac{\delta H}{\delta \phi} \equiv E(\phi), \tag{3}$$

where we defined $E(\phi) = \nabla^2\phi - V'(\phi)$ for notational convenience.

In many publications since the 1980's, E. Gozzi and his collaborators have been extensively studying the properties of classical path integrals (see Refs. [8,9] and references therein). The starting point in these studies is the following transition probability between two points in the phase space:

$$P\left[\varphi_f, \pi_f^\varphi; t_f \middle| \varphi_i, \pi_i^\varphi; t_i\right] = \delta\left[\varphi_f - \varphi_c\left(t_f \middle| \varphi_i, \pi_i^\varphi\right)\right] \times \delta\left[\pi_f^\varphi - \pi_c^\varphi\left(t_f \middle| \varphi_i, \pi_i^\varphi\right)\right]. \tag{4}$$

Here φ and π^φ are the generalized coordinate and momentum and φ_c and π_c^φ are the solutions of Eqs. (2) and (3) with the boundary conditions $\varphi_c(t_i|\varphi_i, \pi_i^\varphi) = \varphi_i$ and $\pi_c^\varphi(t_i|\varphi_i, \pi_i^\varphi) = \pi_i^\varphi$. The evolution of the initial phase-space

density $\rho_{\rm cl}[\varphi, \pi^\varphi; t_i]$ is given by the following expression:

$$\rho_{\rm cl}[\varphi, \pi^\varphi; t] = \int [d\varphi_i]\big[d\pi_i^\varphi\big] \times P\big[\varphi, \pi^\varphi; t\,\big|\varphi_i, \pi_i^\varphi; t_i\big]\,\rho_{\rm cl}\big[\varphi_i, \pi_i^\varphi; t_i\big]. \quad (5)$$

In this article we use the square bracket notation $\int[d\phi]$ to indicate a functional integral at a fixed time and the curly D notation $\int \mathcal{D}\phi = \int \prod_{i=1}^{N}[d\phi_i]$ to indicate a functional integral over space and time. We also implicitly absorb any normalization constants into the definition of $[d\phi]$.

To turn Eq. (5) into a path integral, we use the fact that transition probabilities satisfy the following:

$$\int dy\, P[z|y]P[y|x] = P[z|x]. \quad (6)$$

Dividing the time between the final time $t_f = t_{N+1}$ and the initial time $t_i = t_0$ into $N+1$ equal intervals we obtain

$$P[\phi_{N+1}, \pi_{N+1}|\phi_0, \pi_0] = \int \prod_{i=1}^{N}[d\phi_i][d\pi_i] \prod_{j=0}^{N} P[\phi_{j+1}, \pi_{j+1}|\phi_j, \pi_j], \quad (7)$$

where we have suppressed the time arguments in P.

For sufficiently small $\Delta t = (t_f - t_i)/(N+1)$, we should be able to solve the classical equations of motion approximately. The simplest finite difference method is the Euler method based on the following approximation of the time derivative:

$$\dot f(t_k) \approx \frac{f(t_k) - f(t_{k-1})}{\Delta t} + O(\Delta t) \quad \text{(backward Euler)} \quad (8)$$

or

$$\dot f(t_k) \approx \frac{f(t_{k+1}) - f(t_k)}{\Delta t} + O(\Delta t) \quad \text{(forward Euler)}. \quad (9)$$

We use here the backward Euler method for φ

$$\varphi_{k+1} = \varphi_k + \pi_{k+1}^\varphi\,\Delta t + O(\Delta t^2) \quad (10)$$

and the forward Euler method for π^φ

$$\pi_{k+1}^\varphi = \pi_k^\varphi + E(\varphi_k)\Delta t + O(\Delta t^2) \quad (11)$$

so that φ_{k+1} can be expressed solely in terms of φ_k and π_k^φ. This is, of course, not the only choice. Different discretization method in general leads to different form of the discretized path integral although they are all equivalent in the $\Delta t \to 0$ limit.

For small enough Δt then,

$$P\big[\varphi_{j+1}, \pi_{j+1}^\varphi\big|\varphi_j, \pi_j^\varphi\big] = \delta\big[\varphi_{j+1} - \varphi_j - \pi_{j+1}^\varphi\Delta t\big] \times \delta\big[\pi_{j+1}^\varphi - \pi_j^\varphi - E(\varphi_j)\Delta t\big]. \quad (12)$$

With this form, it is easy to check that the preservation of probability

$$\int [d\varphi_{N+1}]\big[d\pi_{N+1}^\varphi\big]\,\rho_{\rm cl}\big[\varphi_{N+1}, \pi_{N+1}^\varphi; t_{N+1}\big] = \int [d\varphi_0]\big[d\pi_0^\varphi\big]\,\rho_{\rm cl}\big[\varphi_0, \pi_0^\varphi; t_0\big] \quad (13)$$

is trivially satisfied. Using dummy variables χ and π^χ to express the δ functionals, we finally get the classical path integral for the evolution of the density functional as follows:

$$\begin{aligned}\rho_{\rm cl}\big[\varphi_{N+1}, \pi_{N+1}^\varphi; t_{N+1}\big] &= \int \prod_{j=0}^{N}[d\varphi_j] \prod_{k=1}^{N+1}\big[d\pi_k^\chi\big] \\ &\times \exp\left\{ i \sum_{j=0}^{N} \pi_{j+1}^\chi\Big[\varphi_{j+1} - \varphi_j - \pi_{j+1}^\varphi\Delta t)\Big]\right\} \\ &\times \int \prod_{l=0}^{N}\big[d\pi_l^\varphi\big] \prod_{k=0}^{N}[d\chi_k]\, \exp\left\{ -i \sum_{k=0}^{N} \chi_k \right. \\ &\left. \times \big[\pi_{k+1}^\varphi - \pi_k^\varphi - E(\varphi_k)\Delta t\big]\right\} \rho_{\rm cl}\big[\varphi_0, \pi_0^\varphi; t_0\big]. \end{aligned} \quad (14)$$

How closely can we reproduce the classical path integral Eq. (14) from the corresponding quantum path integral? To answer this question, we need to go back to the basics of deriving a quantum mechanical path integral. Given a Hamiltonian operator $\hat H$, we would like to rewrite the matrix element of the evolution operator, $\langle\phi_f|\hat U(t_f, t_i)|\phi_i\rangle$, as a path integral. Dividing the time interval into many small segments and using the fact that $\hat U(t, t')\hat U(t', t'') = \hat U(t, t'')$, we can write

$$\langle\phi_{N+1}|\hat U(t_{N+1}, t_0)|\phi_0\rangle = \langle\phi_{N+1}|\hat U(t_{N+1}, t_N) \times \hat U(t_N, t_{N-1})\cdots\hat U(t_1, t_0)|\phi_0\rangle. \quad (15)$$

There are many ways of inserting the resolutions of identity to turn this expression into a path integral just as there are many ways of discretizing the classical equation of motion. The prescription that most closely resembles Eqs. (10) and (11) turned out to be inserting

$$1 = \int [d\phi_k][d\pi_k]|\phi_k\rangle\langle\phi_k|\pi_k\rangle\langle\pi_k| \quad (16)$$

between $\hat U(t_{k+1}, t_k)$ and $\hat U(t_k, t_{k-1})$ for all $1 \leqslant k \leqslant N$ and inserting

$$1 = \int [d\pi_{N+1}]|\pi_{N+1}\rangle\langle\pi_{N+1}| \quad (17)$$

between $\langle\phi_{N+1}|$ and $\hat U(t_{N+1}, t_N)$. In this way, we get the following:

$$\langle\phi_{N+1}|\hat U(t_{N+1}, t_0)|\phi_0\rangle = \int \prod_{k=1}^{N}[d\phi_k] \int \prod_{k=1}^{N+1}[d\pi_k]$$

$$\times \exp\left\{ i \sum_{k=1}^{N+1} \pi_k(\phi_k - \phi_{k-1}) - i \sum_{k=1}^{N+1} H(\pi_k, \phi_{k-1})\Delta t \right\}. \tag{18}$$

The time evolution of a density operator needs two such path integrals because the matrix element

$$\begin{aligned}\langle \phi_f|\hat{\rho}(t_f)|\tilde{\phi}_f\rangle &= \langle \phi_f|\hat{U}(t_f,t_i)\hat{\rho}(t_i)\hat{U}(t_i,t_f)|\tilde{\phi}_f\rangle \\ &= \int [d\phi_i][d\tilde{\phi}_i]\, \langle \phi_f|\hat{U}(t_f,t_i)|\phi_i\rangle \\ &\quad \times \langle \phi_i|\hat{\rho}(t_i)|\tilde{\phi}_i\rangle \langle \tilde{\phi}_i|\hat{U}(t_i,t_f)|\tilde{\phi}_f\rangle \end{aligned} \tag{19}$$

involves two matrix elements of the evolution operator $\hat{U}$. To bring out classical features more clearly, define

$$\varphi = (\phi + \tilde{\phi})/2 \tag{20}$$

$$\chi = \phi - \tilde{\phi}, \tag{21}$$

where ϕ belongs to the path integral representation of $\langle \phi_f|\hat{U}(t_f,t_i)|\phi_i\rangle$ and $\tilde{\phi}$ belongs to the path integral representation of $\langle \tilde{\phi}_i|\hat{U}(t_i,t_f)|\tilde{\phi}_f\rangle$.

Introducing the Wigner functional $\rho_W(\varphi, \pi^\varphi)$ as

$$\langle \varphi + \chi/2|\hat{\rho}|\varphi - \chi/2\rangle = \int [d\pi^\varphi]\rho_W[\varphi, \pi^\varphi]e^{i\chi\pi^\varphi} \tag{22}$$

we get

$$\begin{aligned}&\rho_W\left[\varphi_{N+1}, \pi^\varphi_{N+1}\right] \\ &= \int [d\varphi_0][d\chi_0]\, \rho_W\left[\varphi_0, \pi^\varphi_0\right] \int \prod_{k=1}^{N}[d\varphi_k] \int \prod_{k=1}^{N}[d\chi_k] \\ &\quad \times \int \prod_{k=0}^{N}\left[d\pi^\varphi_k\right] \int \prod_{k=1}^{N+1}\left[d\pi^\chi_k\right] \\ &\quad \times \exp\left\{ i \sum_{k=0}^{N} \pi^\chi_{k+1}\left(\varphi_{k+1} - \varphi_k - \pi^\varphi_{k+1}\Delta t\right)\right\} \\ &\quad \times \exp\left\{ -i \sum_{k=0}^{N} \chi_k[\pi^\varphi_{k+1} - \pi^\varphi_k - E(\varphi_k)\Delta t] + O(\chi^3)\right\}. \end{aligned} \tag{23}$$

Note that without the Wigner transformation of the initial and the final distributions, the exponents cannot be arranged in the above finite-difference form that includes the end points.

The difference between Eq. (23) and the classical expression Eq. (14) is the $O(\chi^3)$ term inside the exponential. (There are no terms even in χ in this expression because the exponent must change sign under the exchange $\phi \leftrightarrow \tilde{\phi}$ or, equivalently, $\chi \to -\chi$.) If we drop the $O(\chi^3)$ terms, then we have exactly the same *evolution kernel* as the classical theory. However, even in this case, this does not mean that ρ_W will be truly classical. If the initial Wigner functional contains quantum information, then ρ_W at any later times will still contain it.

In perturbation theory, dropping $O(\chi^3)$ terms means ignoring any Feynman diagrams that may contain a loop or loops made up of the *off-shell* propagators. The validity of such operation will be shortly discussed in the next section where we discuss the Feynman rules.

To summarize this section, the omission of the terms nonlinear in χ results in the classical *evolution* of the initial distribution. The crucial difference between our analysis here and that of Ref. [5] is the appearance of the Wigner functional that depends both on the coordinate (φ) and the momentum (π^φ). This is important in the perturbation theory because the propagators strongly depend on the initial distribution. We therefore discuss the propagators and the Feynman rules next.

III. PROPAGATORS AND FEYNMAN RULES

Having obtained the *classical* path integral from a quantum one, we can now derive Feynman rules for the classical perturbation theory following the same procedure as in the quantum case. To do so, consider the generating functional with the source terms $J_\varphi\varphi - J_\chi\chi$ and the restrictions that the source terms vanish at the end points t_i and t_f. Carrying out π^χ integrals and χ_0 integral results in the following:

$$\delta\left[\frac{\varphi_1 - \varphi_0}{h} - \pi^\varphi_0 - E(\varphi_0)\Delta t\right] \prod_{k=0}^{N} \delta\left[\varphi_{k+1} - \varphi_k - \pi^\varphi_{k+1}\Delta t\right].$$

We can then carry out all π^φ_k integrals to get the following:

$$\begin{aligned}\mathcal{Z}[J_\varphi, J_\chi] &= \int \prod_{k=0}^{N+1}[d\varphi_k] \int \prod_{k=1}^{N}[d\chi_k] \\ &\quad \times \rho_W[\varphi_0, (\varphi_1 - \varphi_0)/\Delta t - E(\varphi_0)\Delta t] \\ &\quad \times \exp\left\{ -i \sum_{k=1}^{N} \chi_k \left[\frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{\Delta t} \right.\right. \\ &\quad \left.\left. - E(\varphi_k)h + J^\chi_k \Delta t\right] + i \sum_{k=1}^{N} J^\varphi_k \varphi_k \Delta t \right\} \\ &= \int \mathcal{D}\varphi\, \mathcal{D}\chi \\ &\quad \times \rho_W[\varphi_i, \dot{\varphi}_i] \exp\left[i \int \chi \left(-\partial^2\varphi - m^2\varphi \right.\right. \\ &\quad \left.\left. - \frac{\lambda}{3!}\varphi^3 - J_\chi \right) + i \int J_\varphi\varphi \right], \end{aligned} \tag{24}$$

where we switched to the continuum notation for simplicity.

It is tempting now to develop a perturbation theory from this expression. But we are not done yet. Note that the initial distribution depends on the value of the field and its time derivative. But the integration is over the field values only. To treat $\rho_W[\varphi_i, \dot{\varphi}_i]$ as a proper weight we need to transform the measure to include integration over $\dot{\varphi}_i$. In classical field theory, this can be most easily done by noting that to solve a second-order differential equation requires specifying either the values of the field and its first time derivative at the initial time or the values of the field at the initial and the final time. Therefore, barring caustic points, there is a one-to-one correspondence between the value of φ at t_f and the value of $\dot{\varphi}$ at t_i. The Jacobian of this transformation is as follows:

$$\mathcal{J} = \left|\frac{\delta\varphi_f}{\delta\dot{\varphi}_i}\right| = \left|\frac{\delta^2 S}{\delta\varphi_i\delta\varphi_f}\right|^{-1}, \tag{25}$$

where S is the action integral. This Jacobian is related to the Van Vleck determinant as follows:

$$\mathcal{J}[\varphi] = |\mathrm{Det}(\partial^2 + m^2 + (\lambda/2)\varphi^2)|. \tag{26}$$

Changing the variable φ_f to $\dot{\varphi}_i$ then gives the following:

$$\begin{aligned}\mathcal{Z}[J_\varphi, J_\chi] &= \int [d\pi_i][d\varphi_i]\rho_W[\varphi_i, \pi_i] \int_{\varphi_i,\pi_i} \mathcal{D}\varphi\mathcal{D}\chi\, \mathcal{J}[\varphi] \\ &\times \exp\left[i \int \chi \left(-\partial^2\varphi - m^2\varphi - \frac{\lambda}{3!}\varphi^3 - J_\chi \right) \right. \\ &\left. + i \int J_\varphi \varphi \right],\end{aligned} \tag{27}$$

where we have renamed $\dot{\varphi}_i = \pi_i$ and there is no $[d\varphi_f][d\chi_f]$ integral. One can easily check that in the limit $J_\varphi \to 0$, $\mathcal{Z} \to 1$. This is as it should be because $\mathcal{Z}$ reduces to a trace of a density operator in this limit.

Note that without the Jacobian $\mathcal{J}$, the $J_\varphi \to 0$ limit will not in general yield 1. This is because performing χ integral produces $\delta[(\partial^2\varphi + m^2\varphi + (\lambda/3!)\varphi^3) + J_\chi]$, which is then integrated over φ to yield $1/\mathcal{J}$. The exact value of the Jacobian, however, depends on how one discretizes the equation of motion. This may sounds peculiar, but this phenomenon is not new. In stochastic dynamics, it is well known that this kind of functional Jacobian depends on the discretization prescription [10]. Furthermore, the Jacobian can be made constant if one chooses a particular prescription. This was also noticed in Ref. [11]. Later in this section, we comment more on the role of the determinant. For now we simply set it to a constant and ignore it. As shown shortly, this amounts to the following diagrammatic rule:

(i) Omit diagrams containing tadpoles.

To get the rest of the Feynman rules, let us examine the two main ingredients of perturbation theory: propagators and interaction vertices. The forms of the interaction vertices are fully determined by the interaction Lagrangian. In the case of the classical $\lambda\phi^4$ theory, the corresponding Feynman rule is:

(ii) Assign $-i\lambda$ to each $\lambda\varphi^3\chi$ vertex.

To determine propagators, one must specify what the *unperturbed state* is. In the quantum theory, this is the perturbative vacuum that can fluctuate into particle-antiparticle pairs. Therefore, the expectation value $\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$ is nonzero even if $\langle 0|\hat{\phi}(x)|0\rangle = 0$. conversely, in the classical vacuum, both of these quantities are zero because the classical vacuum cannot fluctuate. It is literally a state where nothing exists.

Propagators are determined by the free field limit. Following the derivation in the last section, it is easy to see that without the self-interaction, the generating function quickly reduces to the following:

$$\begin{aligned}\mathcal{Z}[J_\varphi, J_\chi] = \int [d\pi_i][d\varphi_i]\rho_W[\varphi_i, \pi_i] \int_{\varphi_i,\pi_i} \mathcal{D}\varphi\mathcal{J}[\varphi] \\ \times \delta[-\partial^2\varphi - m^2\varphi + J_\chi] \exp\left[i \int J_\varphi\varphi \right]\end{aligned} \tag{28}$$

regardless of whether it is the quantum case or the classical case. Yet for both cases, the combination $\varphi = (\phi + \tilde{\phi})/2$ satisfies the classical equation of motion whose solution is given by the following:

$$\begin{aligned}\varphi = \varphi_i \cos[E_k(t - t_i)] + \pi_i \frac{\sin[E_k(t - t_i)]}{E_k} \\ + \int_{t_i}^{t} dt' G_{\mathrm{ret}}(t - t') J_\chi(t').\end{aligned} \tag{29}$$

The fact that φ satisfies the classical equation of motion even in the fully quantum case is somewhat unexpected. However, it shows explicitly that the form of the initial distribution functional ρ_W plays an essential role in distinguishing the quantum and the classical cases.

Let us consider the purely classical case first. In equilibrium, the density functional is given by the following classical Boltzmann factor:

$$\begin{aligned}\rho_{\mathrm{cl}}[\varphi_i, \pi_i] &= \frac{e^{-\beta H_{\mathrm{cl}}}}{Z_{\mathrm{cl}}} \\ &= \frac{1}{Z_{\mathrm{cl}}} \exp\left\{ -\beta \sum_k \left[\pi_i(k)^2 + E_k^2 \varphi_i(k)^2 \right]/2 \right\},\end{aligned} \tag{30}$$

where $\pi_i(k)$ and $\varphi_i(k)$ are the Fourier components of π_i and φ_i respectively. Using the above classical solution and carrying out the φ_i and π_i integral then yield the following:

$$\begin{aligned}\mathcal{Z}_{\mathrm{cl}}[J_\varphi, J_\chi] = \exp\left\{ -\sum_k \frac{T}{2E_k^2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' J_\varphi(t) \right. \\ \left. \times \cos[E_k(t - t')] J_\varphi(t') + i \int J_\varphi G_{\mathrm{ret}} J_\chi \right\},\end{aligned} \tag{31}$$

where we have again suppressed the momentum indices for J_φ and J_χ. The resulting Feynman rules are as follows:

(iii) In equilibrium, the propagators are

$$G_{\mathrm{cl}}^{\varphi\varphi}(p) = (T/E_p) 2\pi\delta(p^2 - m^2) \tag{32}$$

$$G_{\mathrm{cl}}^{\varphi\chi}(p) = \frac{i}{p^2 - m^2 + i\epsilon p_0} \tag{33}$$

$$G_{\mathrm{cl}}^{\chi\varphi}(p) = \frac{i}{p^2 - m^2 - i\epsilon p_0} \tag{34}$$

$$G_{\mathrm{cl}}^{\chi\chi}(p) = 0 \tag{35}$$

The same set of rules were also used in Refs. [12–15] as the classical limit of the quantum rules.

The $\varphi\chi$ propagator is the response function that controls generation of classical field by a source. Hence it must be there be it classical or quantum [cf. Eq. (29)]. However, in the $T \to 0$ limit, $\varphi\varphi$ propagator vanishes because of the fact that classical vacuum does not fluctuate. Note also that the classical equilibrium density function is T/E_k just as in the case of electromagnetic waves in a cavity. Therefore, if one is to extend this formalism to an out-of-equilibrium situation, one should use the following:

(iii′) In nonequilibrium,

$$G_{\rm cl}^{\varphi\varphi}(p, X) = f_{\rm cl}(p, X) 2\pi\, \delta(p^2 - m^2), \tag{36}$$

where $f_{\rm cl}(p, X)$ is the density function to be determined and $G_{\rm cl}^{\varphi\varphi}(p, X)$ is the Wigner transformed two-point function

$$G_W(p, X) = \int d^4 r e^{ir^\mu p_\mu} G(x, y), \tag{37}$$

with $r = x - y$ and $X = (x + y)/2$.

The appearance of the on-shell δ function in $G_{\rm cl}^{\varphi\varphi}$ propagator deserves deeper consideration. Suppose that the initial $\rho[\varphi_i, \chi_i]$ is actually independent of χ_i. That is, $\hat{\rho}$ is diagonal in ϕ. In that case, the Wigner functional contains a δ functional $\delta[\pi_i]$. For an illustration, suppose

$$\rho'_W[\varphi_i, \pi_i] = \frac{1}{Z'} \exp\left[-\beta \sum_k E_k^2 \varphi_i(k)^2 \Big/ 2\right] \delta[\pi_i]. \tag{38}$$

In that case, carrying out φ_i and π_i^φ integrals results in the following:

$$\mathcal{Z}'[J_\varphi, J_\chi] = \exp\left[-\sum_k \frac{T}{2E_k^2} \left\{\int_{t_i}^{t_f} dt\, J_\varphi(t) \cos[E_k(t - t_i)]\right\}^2 + i \int J_\varphi G_{\rm ret} J_\chi\right]. \tag{39}$$

Because there is no compensating sine term, the $\varphi\varphi$ Green function no longer exhibits time-translation invariance. The $\varphi\varphi$ correlator is given by the following:

$$G^{\varphi\varphi}(t, t') = \frac{T}{E_k^2} \cos[E_k(t - t_i)] \cos[E_k(t' - t_i)]. \tag{40}$$

Because this is not a function of $t - t'$, this form of the propagator *does not* conserve energy at each vertex. The usual momentum space Feynman diagram techniques will not work.

Our ability to use Feynman rules with the usual conservation four-momentum at each vertex depends on whether we get a function only of $t - t'$ for the $\varphi\varphi$ correlator. In equilibrium, the propagators depend only on the difference of the two coordinates. Hence, the total four-momentum is conserved at each vertex. In nonequilibrium situations, the propagators depend both on the difference $r = x - y$ and the sum $X = (x + y)/2$. Wigner transformation assigns 'momentum' to the difference as done in Eq. (37). However, unless the X dependence of $G_W(p, X)$ is slow, the momenta that enter each vertex are not conserved. For the derivation of the Boltzmann equation, this slow dependence is therefore essential.

The above Feynman rules are used to calculate the perturbative corrections in the classical field theory. Just as loops appear in the quantum Feynman diagrams, loops appear in the classical Feynman diagrams as well [5]. This looks like a violation of the theorem that states the classical field theory corresponds to the sum of all tree diagrams. However, this theorem is derived in the context of the vacuum theory. It does not necessarily apply to the in-medium case. Furthermore, the theorem is derived considering only the uncut diagrams where as the in-medium theory necessarily involves cut diagrams (that include $\varphi\varphi$ propagators). Nevertheless, if this theorem is not to be violated, we must at least show that the vacuum loops do not appear in our classical theory.

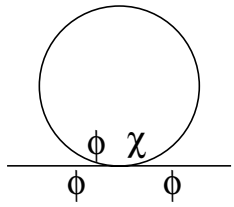

FIG. 1. Tadpole diagram in the classical perturbation theory.

That this is indeed the case can be shown as follows. A vacuum loop in a given diagram must not contain the density function $f_{\rm cl}$. Because $f_{\rm cl}$ appears only in the $\varphi\varphi$ propagator, any potential vacuum loop diagram must consist entirely of $\chi\varphi$ and $\varphi\chi$ propagators. Now with only the $\lambda\chi\varphi^3$ vertex, χ is conserved in the sense that it cannot split into multiple χ's. Hence one can follow χ as if it is a fermion. What flows along this line is the time since the $\varphi\chi$ propagator is a retarded one and the $\chi\varphi$ propagator is an advanced one. Therefore, a loop made of $\varphi\chi$ and/or $\chi\varphi$ propagators implies existence of a time loop which is impossible. It follows that there is no vacuum loop contribution in the classical theory if the loop involves more than a single propagator.

The only remaining possibility is a single loop formed by a $\varphi\chi$ propagator as shown in Fig. 1 that corresponds to the following:

$$\Sigma_{1-\rm loop} = \frac{\lambda}{2} G_{\rm ret}(t = 0). \tag{41}$$

The value of this expression is actually ambiguous as it involves $\theta(0)$. This is where the Van Vleck determinant Eq. (26) plays its role. To determine the role of $\mathcal{J}$ in perturbation theory, we may exponentiate the determinant using Fadeev-Popov type of ghost fields c and $\bar{c}$

$$\mathrm{Det}(\partial^2 + m^2 + (\lambda/2)\varphi^2) \to \int \mathcal{D}c\mathcal{D}\bar{c}\, \exp\left[i \int \bar{c}(\partial^2 + m^2 + (\lambda/2)\varphi^2)c\right]. \tag{42}$$

Because there is no in-medium part for the ghost field, the corresponding propagators should be as follows:

$$G_{\rm Q}^{c\bar{c}} = \frac{i}{p^2 - m^2 + i\epsilon p_0} \tag{43}$$

$$G_{\rm Q}^{\bar{c}c} = \frac{i}{p^2 - m^2 - i\epsilon p_0}, \tag{44}$$

with the rule that a ghost loop gets an additional factor of (-1).

First of all, ghost propagators must form a loop because no external line can be a ghost and there can be no ghost excitations in the medium. However, because the ghost propagator is a retarded one, a ghost loop made of two or more ghost lines must vanish. A loop made up of a single ghost line is then the only possible nonzero diagram

$$\Sigma_{1-\rm loop}^{\rm ghost} = -\frac{\lambda}{2} G_{\rm ret}(t = 0). \tag{45}$$

This exactly cancels the tadpole contribution in Eq. (41). Thus the ghost saves the classicality of the perturbation theory by removing the ambiguity of defining $\theta(0)$. If we define $\theta(0) = 0$, then the ghost field contribution disappears and the Van Vleck determinant can be set to a constant. Again, as mentioned earlier, this is a known phenomenon. In the study of stochastic dynamics, it is well known that how one discretizes can change the form of Van Vleck determinant even making it a constant [10]. In Ref. [5], because all loops that are formed with $\varphi\chi$ propagators and/or $\chi\varphi$ propagators are set to zero, one might say that $\theta(0)$ was implicitly defined to be zero in that article.

To determine the validity of the above classical perturbation theory as a approximation of the underlying quantum theory, let us now consider the quantum case. The form of the free field Wigner function can be easily obtained from the ground-state wave function of the simple harmonic oscillator in the Euclidean space (for instance, see Ref. [16])

$$\rho_{\rm Q}[\varphi_i, \pi_i] = \frac{1}{Z_Q} \exp\left\{-\sum_k \frac{\tanh(E_k\beta/2)}{E_k}\left[\pi_i(k)^2 + E_k^2\varphi_i(k)^2\right]\right\}. \tag{46}$$

In the small β or large $T = 1/\beta$ limit, this goes over to the classical case. Again using the classical solution and carrying out the φ_i and π_i integral yield, we obtain the following:

$$\mathcal{Z}_Q[J_\varphi, J_\chi] = \exp\left\{-\sum_k \frac{\coth(E_k\beta/2)}{4E_k} \times \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' J_\varphi(t)\cos[E_k(t-t')] \times J_\varphi(t') + i\int J_\varphi G_{\rm ret} J_\chi\right\}. \tag{47}$$

The propagators are then given by

$$G_{\rm Q}^{\varphi\varphi} = [1/2 + n_{\rm BE}(E_p)]2\pi\delta(p^2 - m^2) \tag{48}$$

$$G_{\rm Q}^{\varphi\chi} = \frac{i}{p^2 - m^2 + i\epsilon p_0} \tag{49}$$

$$G_{\rm Q}^{\chi\varphi} = \frac{i}{p^2 - m^2 - i\epsilon p_0} \tag{50}$$

$$G_{\rm Q}^{\chi\chi} = 0, \tag{51}$$

where $n_{\rm BE}(E_p) = 1/(e^{E_p/T} - 1)$ is the Bose-Einstein distribution. The corresponding nonequilibrium propagator is

$$G_{\rm Q}^{\varphi\varphi} = (1/2 + f_{\rm Q})2\pi\delta(p^2 - m^2), \tag{52}$$

where $f_{\rm Q}$ is now the density function to be determined. In Ref. [5], these quantum form of propagators are used with the classical field Feynman rules (i) and (ii) above. If $f_Q \gg 1/2$, there is very little difference between this set of quantum propagators and the set of classical propagators. However, one of the points made in Ref. [5] was that keeping the 1/2 term in Eq. (52) makes the classical kinetic equation to match up with the quantum one up to the next-to-leading order in f. In view of the fact that this is actually mixing quantum and classical descriptions, it deserves a more in-depth study. We do so in the next section.

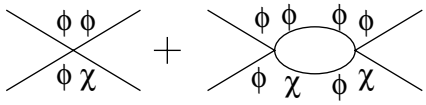

FIG. 2. The first order vertex correction for the classical field.

Because we have explicit forms of the quantum and the classical propagators now, we can discuss the validity of the classical perturbation expansion. The differences between the classical and the quantum Feynman rules are as follows (excluding the tadpole diagrams):

(i) The quantum $\varphi\varphi$ propagator has $1/2 + f$, whereas the classical one has just f.
(ii) The quantum interaction includes $\lambda\chi^3\varphi/24$ term, whereas this is missing in the classical case.

The first of these items indicates that the classical approximation is valid in the large f limit so that the appearance of $1/2$ does not make a difference. But if f is too big, then potentially perturbative corrections can be as large or larger than the leading-order contribution. To be concrete, consider the vertex correction depicted in Fig. 2. Compared to the bare vertex, the correction term is smaller by a factor of $O(\lambda f_{\rm cl})$. Therefore, the *classical* perturbation theory is valid when $f_{\rm cl} \gg 1$ but also $\lambda f_{\rm cl} \ll 1$. This is the same conclusion reached in Ref. [5] but in a heuristic way.

The next question we should ask is how big the size of the quantum correction is. To have an estimate, consider substituting a $\lambda\varphi^3\chi$ vertex with a $\lambda\chi^3\varphi$ vertex in a classical perturbation theory diagram. Because there is no $\chi\chi$ propagator, the only way this substitution is possible is when the inserted vertices are connected with three φ field and one χ field as depicted in Fig. 3. In the large f limit, the quantum contribution is $O(1/f^2)$ smaller than the purely classical contribution. Therefore at a given order in λ, the quantum correction is always $O(1/f^2)$ smaller than the classical one. Again, this is the same conclusion as in Ref. [5].

These estimates also indicate that there is a limit that we can trust the classical perturbation theory depending on the relative size of $1/f$ and λ. Suppose that $f = O(1/\sqrt{\lambda})$. In the small λ limit, this fulfills $f \gg 1$ as well as $\lambda f \ll 1$. However, because $1/f^2 = O(\lambda)$, only the leading order classical perturbative correction is reliable. The leading order (in λ) quantum

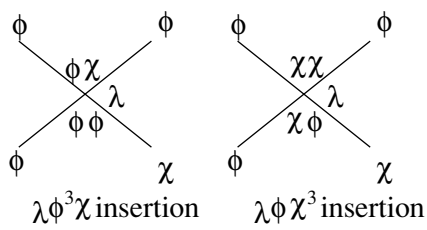

FIG. 3. Substituting $\lambda\varphi^3\chi$ with $\lambda\chi^3\varphi$.

correction in this case is as big as the second-order classical correction.

IV. COMPARISON OF QUANTUM AND CLASSICAL BOLTZMANN EQUATIONS

Having derived the Feynman rules, the derivation of Boltzmann equation can now proceed as in Ref. [5] for the $\lambda\phi^4$ theory. For convenience, here we show the results from Ref. [5]. The collision term of the Boltzmann equation is given by the following:

$$\begin{aligned} C = \frac{\lambda^2}{2}\int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^5 2^4 \omega(k_1)\omega(k_2)\omega(k_3)\omega(p)} \\ \times \delta[\omega(p)+\omega(k_1)-\omega(k_2)-\omega(k_3)] \\ \times \delta(\mathbf{p}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\ \times \Big\{ F(k_1)F(k_2)F(k_3) + F(p)F(k_2)F(k_3) \\ - F(p)F(k_1)F(k_2) - F(p)F(k_1)F(k_3) \Big\}, \end{aligned} \tag{53}$$

where we used $F(p) = f(p) + 1/2$ to simplify and combine the terms in Ref. [5].

As shown in Sec. III, quantum and classical propagators have different forms. The above expression [Eq. (53)] uses the quantum form of propagators but otherwise ignores quantum corrections. If we are in a purely classical regime, F in the above expression must be changed to f. The resulting Boltzmann equation then contains only the terms cubic in f. This is appropriate because it is easy to verify that the equilibrium distribution resulting from having only the cubic terms is $f(E) = T/E$. This form leads to the Rayleigh-Jeans catastrophe, which Planck cured by inventing quantum mechanics.

Conversely, if we keep the above form, the steady-state solution is

$$f(E) = \frac{T}{E} - \frac{1}{2}, \tag{54}$$

which may be recognized as the first two terms of the expansion of the Bose-Einstein distribution in the large T limit. Therefore in the $T \gg E$ limit, this form is a better approximation of the true quantum distribution than the Rayleigh-Jeans form. However, unlike the Rayleigh-Jeans form, this form of the distribution function becomes negative when $T < 2E$, which is clearly unphysical. Therefore mixing the quantum propagators and the classical vertices does not necessarily improve the purely classical result.

Let us now consider why keeping the 1/2 term in the propagator somewhat mysteriously reproduce the quantum statistical result up to the next-to-leading order terms in f. Suppose we have a scattering process that involves m initial particles (k_i) and n final particles (p_j). The quantum Boltzmann equation must contain the following combination of the density functions:

$$\begin{aligned} C_Q = \prod_{i=1}^{m} f_Q(k_i) \prod_{j=1}^{n} [1 + f_Q(p_j)] \\ - \prod_{j=1}^{n} f_Q(p_j) \prod_{i=1}^{m} [1 + f_Q(k_i)] \end{aligned} \tag{55}$$

to be consistent with the equilibrium Bose-Einstein density. Expanding in terms of f_Q's (we are in the large f_Q limit), we have the following:

$$\begin{aligned} C_Q = \sum_{s=1}^{n}\prod_{i=1}^{m} f_Q(k_i) \prod_{j=1,j\neq s}^{n} f_Q(p_j) \\ + \sum_{s=1}^{n}\sum_{t=1}^{n}\theta(s>t)\prod_{i=1}^{m} f_Q(k_i) \prod_{j=1,j\neq s,j\neq t}^{n} f_Q(p_j) \\ - \sum_{s=1}^{m}\prod_{i=1,i\neq s}^{m} f_Q(k_i)\prod_{j=1}^{n} f_Q(p_j) \\ - \sum_{s=1}^{m}\sum_{t=1}^{m}\theta(s>t) \prod_{i=1,i\neq s,i\neq t}^{m} f_Q(k_i) \\ \times \prod_{j=1}^{n} f_Q(p_j) \\ + o(f^{m+n-3}), \end{aligned} \tag{56}$$

where we defined $\theta(\text{condition}) = 1$ if the condition is met and $\theta(\text{condition}) = 0$ otherwise. Conversely the classical counterpart must contain the following:

$$\begin{aligned} C_{\rm cl} = \sum_{s=1}^{n}\prod_{i=1}^{m} f_{\rm cl}(k_i) \prod_{j=1,j\neq s}^{n} f_{\rm cl}(p_j) \\ - \sum_{s=1}^{m}\prod_{i=1,i\neq s}^{m} f_{\rm cl}(k_i)\prod_{j=1}^{n} f_{\rm cl}(p_j) \end{aligned} \tag{57}$$

to be consistent with the Rayleigh-Jeans form of the equilibrium distribution. It is clear that the leading order terms match between the quantum and the classical cases. Now let $f_{\rm cl} \to f_{\rm cl} + 1/2$ to get the following:

$$\begin{aligned} C'_{\rm cl} = \sum_{s=1}^{n}\prod_{i=1}^{m}[f_{\rm cl}(k_i)+1/2] \prod_{j=1,j\neq s}^{n} [f_{\rm cl}(p_j)+1/2] \\ - \sum_{s=1}^{m}\prod_{i=1,i\neq s}^{m} [f_{\rm cl}(k_i)+1/2]\prod_{j=1}^{n}[f_{\rm cl}(p_j)+1/2]. \end{aligned} \tag{58}$$

Expanding up to the next order in $f_{\rm cl}$ yields the following:

$$\begin{aligned} C'_{\rm cl} = \sum_{s=1}^{n}\prod_{i=1}^{m} f_{\rm cl}(k_i) \prod_{j=1,j\neq s}^{n} f_{\rm cl}(p_j) \\ - \sum_{s=1}^{m}\prod_{i=1,i\neq s}^{m} f_{\rm cl}(k_i)\prod_{j=1}^{n} f_{\rm cl}(p_j) \\ + \frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\theta(s\neq t)\prod_{i=1}^{m} f_{\rm cl}(k_i) \prod_{j=1,j\neq s,j\neq t}^{n} f_{\rm cl}(p_j) \end{aligned}$$

$$-\frac{1}{2}\sum_{s=1}^{m}\sum_{t=1}^{m}\theta(s\neq t)\prod_{i=1,i\neq s,i\neq t}^{m} f_{\rm cl}(k_i)\prod_{j=1}^{n} f_{\rm cl}(p_j) + o(f^{m+n-3}), \tag{59}$$

which can be recognized to be the same as Eq. (56) once we use the fact that $\theta(s>t)$ is equivalent to $\theta(s\neq t)/2$ under the sums. Thus the *prescription*, $f_{\rm cl}\to f_{\rm cl}+1/2$ does produce the first two terms in the quantum Boltzmann equation for any general scattering process. This also explains why the same kind of consideration also seems to work for hot quantum chromodynamics [17]. Although it is curious that this prescription can indeed get the next-to-leading order term in f right, this is irrelevant in getting the right equilibrium limit because that requires retaining all the terms in f.

V. SEPARATION OF HARD AND SOFT MODES

In view of the discussion in the last section, the better application of the statistical mechanics of the classical field is not to get the final equilibrium distribution[1] but to use it as an intermediate stage where the field degrees of freedom and the particle degrees of freedom intermix. In this section, we illustrate how this may be achieved. A more detailed analysis of this important problem, however, is out of the scope of the present article and will be reported in later publications [18].

As shown in the last section, the quantum perturbative correction becomes the same size as the classical perturbative correction when $f\sim 1$. As indicated by the equilibrium form of f, $f(k)$ becomes small as k becomes larger. Therefore, the classical field description is usually appropriate only for the soft modes.

With the price of introducing a cutoff Λ (where f becomes $\sim$1), the quantum field should then be split into the soft and hard modes:

$$\varphi=\varphi_s+\varphi_h \quad\text{and}\quad \chi=\chi_s+\chi_h, \tag{60}$$

where the soft modes (with the subscript s) have $|\mathbf{k}|<\Lambda$ and the hard modes (with the subscript h) have $|\mathbf{k}|\geqslant\Lambda$. The soft modes should then be treated as a classical field and the hard modes should be kept as a quantum field.

To be concrete, consider the $\lambda\phi^4$ theory with the following Hamiltonian

$$H=\int d^3x\left(\frac{\pi^2}{2}+\frac{(\nabla\phi)^2}{2}+\frac{m^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4\right). \tag{61}$$

The continuum version of Eq. (23) is

$$\rho_W\left[\varphi_{N+1},\pi^{\varphi}_{N+1}\right] = \int[d\varphi_0][d\chi_0]\,\rho_W\left[\varphi_0,\pi^{\varphi}_0\right]\int\mathcal{D}\varphi\int\mathcal{D}\chi\int\mathcal{D}\pi^{\varphi} \times\int\mathcal{D}\pi^{\chi}\exp\left\{i\int\pi^{\chi}\left(\dot{\varphi}-\pi^{\varphi}\right) -i\int\left[\chi\left(\dot{\pi^{\varphi}}-\nabla^2\varphi+m^2\varphi+\frac{\lambda}{3!}\varphi^3\right)+\frac{\lambda}{4!}\chi^3\varphi\right]\right\}. \tag{62}$$

Separating the hard and the soft modes, the interaction term becomes the following:

$$V=\frac{\lambda}{3!}(\chi_h+\chi_s)(\varphi_h+\varphi_s)^3+\frac{\lambda}{4!}(\varphi_h+\varphi_s)(\chi_h+\chi_s)^3. \tag{63}$$

Keeping only the terms that contain the most number of φ_s yields

$$V=\frac{\lambda}{4!}\left(4\chi_s\varphi_s^3+4\chi_h\varphi_s^3\right). \tag{64}$$

These terms are linear in χ_s and χ_h. Hence, the equations of motion are

$$-\partial^2\varphi_s-m^2\varphi_s-\frac{\lambda}{3!}\varphi_s^3=0 \tag{65}$$

and

$$-\partial^2\varphi_h-m^2\varphi_h=\frac{\lambda}{3!}\varphi_s^3. \tag{66}$$

These equations describe a system where the classical field evolves independent of the particles but the *quantum mechanical* free particles are produces by a classical source. From the analogous problem in QED (generation of photons from a classical source), it is an easy matter to show that the spectrum of hard modes at the end of evolution is

$$f_h(E_k,\mathbf{k}_h)=\frac{\lambda^2}{36}\left|J(E_k,\mathbf{k}_h)\right|^2, \tag{67}$$

where

$$J(E_k,\mathbf{k}_h)=\int d^3x\int_{t_0}^{\infty}dt e^{-i\mathbf{k}_h\cdot\mathbf{x}+iE_kt}[\varphi_s(t,x)]^3. \tag{68}$$

This is, of course, in addition to the hard modes that already existed in the initial state.

If the cutoff Λ remains constant at all times, this is the complete solution for the hard spectrum provided that we can solve the soft mode classical equation of motion either perturbatively or nonperturbatively. In reality, it is not as simple because of the fact that the cutoff Λ should be a functional of φ_s, too. In the color glass condensate approach to the heavy-ion collisions [19], a similar conversion of field degrees of freedom to particle degrees of freedom was performed to get partons out of classical non-Abelian gauge field. The line of argument given here may provide a firmer ground for such a treatment. One can conceive that a kind of "renormalization group" equation in the manner of Ref. [20] should exist for the cutoff. This will then enable us to describe the whole system in a consistent fashion. This topic is currently under investigation.

VI. SUMMARY AND DISCUSSION

In this article, we showed that one can indeed go from the statistical mechanics of quantum field theory to the statistical mechanics of particles via classical field theory. Along the way, we encountered several subtleties that had to be carefully

[1]This was also recognized in Ref. [5], where it was implied that their formalism is for the system evolving toward equilibrium but not applicable near the kinetic equilibrium.

dealt with. In this regard, we improve upon the treatment given in Ref. [5].

Going from quantum field theory to the classical field theory involves the Wigner functional, which plays the role of initial density function. In other words, the probability of an initial state depends on both the value of the field and the value of the first time derivative. The form of the initial Wigner functional in turn determines the propagators. In addition, our ability to use the Feynman diagrams with energy-momentum conservation at each vertex depends very much on the fact that time translation invariance is maintained. For this to work at least approximately, the Wigner transformed propagators $G_W(p, X)$ [cf., Eq. (37)] cannot depend strongly on $X = (x + y)/2$. This in turn implies that the density matrix at time t must also be approximately of the form

$$\ln \rho \sim E_k^2 \varphi_k^2 + \pi_k^2 \sim H_k, \tag{69}$$

where φ_k and π_k are Fourier components of the corresponding fields at t and H_k is the corresponding free field Hamiltonian. This is a rather strong condition that is in general valid only near equilibrium. Whether this can be reconciled with the classical field limit is not yet clear and requires further study.

Going from the quantum field theory to the classical field theory also involves a Jacobian in the form of the Van Vleck determinant. The role of this Jacobian turned out to be to cancel the single remaining quantum effect still left in the classical perturbation theory in the form of the tadpole self-energy.

It must be also emphasized that to define propagators, one must define what is meant by the *unperturbed vacuum*. Pure classical vacuum cannot fluctuate as it is literally a state where nothing exists. Conversely, the quantum vacuum fluctuates all the time. The importance of this distinction is particularly apparent when one considers the free field theory. The form of the path integral for the classical and the quantum theory in this case is exactly the same except for the form of the initial density functional. To be consistent, the classical initial density functional must conform to the classical vacuum property and the quantum density functional must conform to the quantum vacuum property. Mixing of the two formalisms leads to an interesting conclusion that the classical field theory can reproduce the results of *quantum statistical* Boltzmann equation up to the next leading order in the density (Ref. [5]). The correct Boltzmann equation obtained from classical perturbation theory yields the Rayleigh-Jeans form of the equilibrium distribution function as it must.

As for the validity of the classical perturbation theory, in addition to the two conditions already mentioned we also argued that there is a limit that a purely classical perturbation theory makes sense. For instance, we have shown that if $f \sim 1/\sqrt{\lambda}$, the leading-order quantum correction is as big as the second-order classical correction. And hence, going beyond the first-order classical perturbation theory does not make much sense. Quantum effect must be considered after the leading order.

We would also like comment on the meaning of *thermalization* here. It is well known that an isolated system *as a whole* cannot thermalize. A simple example is an eigenstate of the total Hamiltonian, which, by definition, is stationary. This is in contrast to the Boltzmann equation, where the stationary solution is guaranteed to be the equilibrium distribution. The crucial difference of course is whether one is interested in the system as a whole, or just a *part* of the system be it in the momentum space or the coordinate space. It is therefore perhaps more natural to apply the current formalism to a particular sector of the system, say the soft modes, and regard the hard spectrum as particle degrees of freedom.[2]

In summary, in this article, we reexamined the derivation of the Boltzmann equation from the classical field theory as advocated by Mueller and Son [5]. We pointed out a few subtle points in manipulating functional integrals and developing the perturbation theory and showed how to deal with them. With our improvement, the framework advocated by Mueller and Son has a potential to be a very useful tool in investigating dense many-body systems, for instance, the problem of converting field degrees of freedom to particle degrees of freedom. Further study in this line is continuing and will be reported in later publications.

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[2] This is different from similar separation of hard and soft modes previously considered in the *equilibrium setting*. For instance, see Ref. [21].

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