Fluctuations in the canonical ensemble

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The particle number and energy fluctuations in a system of charged particles are studied in the canonical ensemble for nonzero net values of the conserved charge. In the thermodynamic limit, the fluctuations in the canonical ensemble differ from those in the grand canonical ensemble. A system with several species of particles is considered. We calculate the quantum statistical effects that can be taken into account for the canonical ensemble fluctuations in the infinite-volume limit. The fluctuations of the particle numbers in the pion-nucleon gas are considered in the canonical ensemble as an example of the system with two conserved charges—baryonic number and electric charge.

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I. INTRODUCTION AND OUTLINE

The statistical model approach turns out to be rather successful in describing data on particle production in relativistic nuclear collisions (see, e.g., Ref. [1] and the recent review [2]). This stimulates further investigation of the properties of statistical models. In particular, the applicability of various statistical ensembles is an interesting issue. In nucleus-nucleus ($A + A$) collisions one prefers to use the grand canonical ensemble (GCE) because it is the most convenient one from a technical point of view. The canonical ensemble (CE) [3–8] or even the microcanonical ensemble (MCE) [9] have been used to describe the pp, $p\bar{p}$, and e^+e^- collisions when a small number of secondary particles are produced. In all these cases, the statistical systems are far away from the thermodynamic limit, so the statistical ensembles are not equivalent and exact charge conservation or both energy and charge conservation laws have to be taken into account. The CE is relevant for systems with a large number of produced particles (e.g., a large number of resultant pions or large nucleon number in $p + A$ collisions) but a small (less than or equal to one) number of carriers of conserved charges, such as strange hadrons [6], antibaryons [7], or charmed hadrons [8]. This may happen not only in elementary pp, $p\bar{p}$ and e^+e^-, but also in $p + A$ or even $A + A$ collisions. The statistical mechanics of quarks and gluons were investigated in the CE in Ref. [10]. The exact conservation of baryon number, electric charge, strangeness as well as non-Abelian charges such as total isospin and color leads to significant reduction of the energy, entropy, and particle number densities for systems with a volume smaller than 5 fm^3. The CE and MCE effects for thermodynamical observables are irrelevant and the GCE formulation becomes valid in the thermodynamical limit when the system volume V tends to infinity. All statistical ensembles then become thermodynamically equivalent.[1]

The analysis of the fluctuations is a useful tool to study the properties of the system created during high-energy particle and nuclear collisions (see the review papers [12] and references therein). A method to subtract the "trivial" geometrical fluctuations and check the thermal equilibration in $A + A$ collisions was suggested in Ref. [13]. It was pointed out that, for the hadronic system produced in thermodynamic equilibrium, the temperature and multiplicity fluctuations are related, respectively, to the heat capacity [14,15] and compressibility [16]. An extensive discussion of the equilibrium fluctuations can be found in Ref. [17]. It was suggested [18] that studying the event-by-event fluctuations in $A + A$ collisions may help to discover the QCD critical point, namely, the point at which a line of the first-order phase transition separating hadronic matter from quark-gluon plasma comes to an end. The resulting signature of the fluctuations in the vicinity of the QCD critical point should be their nonmonotonic dependence on the collision energy. A nonmonotonic dependence of the fluctuations on the collision energy could also appear as the result of dynamical fluctuations in thermodynamical parameters of the initial state [19].

In textbooks on statistical mechanics, the discussion is usually limited to the nonrelativistic cases so that, in the CE, the particle number is just fixed and does not fluctuate. The CE and MCE formalisms were used to calculate the level density of atomic nuclei in Ref. [20]. Interesting applications of statistical models within CE were developed in intermediate-energy $A + A$ collisions to describe multifragmentation phenomena (see, e.g., Ref. [21] and references therein). Fluctuations in the size of nuclear fragments were discussed in both GCE and CE. Sometimes, the results obtained in the CE and GCE were different. However, the total number of nucleons is still fixed in the CE, and fluctuations of this number in the CE are absent by definition.

By contrast, in the relativistic case, relevant for the statistical description of hadron production in high-energy collisions, the situation is completely different. Only the net conserved charges are fixed (the average values of net charges in the GCE, or exact ones in the CE), but the numbers of

[1]Possible thermodynamic nonequivalence between the CE and MCE for infinite systems in the vicinity of first-order phase transitions are discussed in Ref. [11].

negatively and positively charged particles fluctuate in both GCE and CE. These fluctuations, which are different in GCE and CE, are the subject of our investigation here.

Multiplicity distribution and isospin fluctuations were calculated in Ref. [22] for the CE in which additive quantum numbers as well as total isospin are strictly conserved. The system of fixed number of pions N was transformed into different charge states π^+, π^-, π^0 with $n_+ + n_- + n_0 = N$. Under these conditions the role of exact conservation of total isospin and its projection was studied as well as its dependence on the total number of pions N.

The question of the applicability of different statistical ensembles for particle number fluctuations has been addressed in our recent papers [23,24]. Fluctuations in the particle number have been calculated in the CE [23] and MCE [24] and compared to those in the GCE. It has been shown that these fluctuations are different for various statistical ensembles in the particular case of a relativistic ideal gas with a vanishing net charge. The particle number fluctuations have been found to be suppressed in the CE and MCE in comparison to the GCE. This suppression remains valid in the thermodynamic limit as well, so the well-known equivalence of all statistical ensembles applies to averaged quantities, but it does not apply to the fluctuations. In particular, the fluctuations of negatively and positively charged particles are suppressed in the CE [23] vis à vis the fluctuations in the GCE.

In Ref. [23] we studied the CE for one-particle species and a zero net value of the conserved charge, neglecting the effects of quantum statistics. In the present paper we extend our study; in high-energy $p + p$ and $A + A$ collisions the resultant system has some positive values for the baryon number and electric charge. Moreover, many different species of hadrons are created. We study the CE average particle numbers (Sec. II), their fluctuations (Sec. III), and energy fluctuations (Sec. IV) in systems with nonzero net charge and various particle species. As the electric charge of hadrons can be both ± 1 and ± 2, we consider the CE system of singly and doubly charged particles in Sec. V. The effects of Bose and Fermi statistics are studied in the thermodynamic limit in Sec. VI. We also calculate in Sec. VII the CE particle number fluctuations for an ideal pion-nucleon gas, which is an example of a system with two conserved charges—baryonic number and electric charge. We summarize our consideration and formulate conclusions in Sec. VIII.

II. THE GCE AND CE PARTITION FUNCTIONS AND MEAN PARTICLE MULTIPLICITIES

Let us start with the multispecies system of $+1$ and -1 charged particles. In applications of the statistical approach to hadron production in high-energy collisions, the conserved charge under consideration can be the electric charge and baryonic number, or strangeness and charm, which are also conserved in the strong interactions. In this section we calculate the average numbers of positively and negatively charged particles in the CE. These results are not new (see Refs. [4,6–8]) and are presented in our paper for completeness. In the case of a Boltzmann ideal gas (i.e., one in which the interactions and quantum statistics effects are neglected) the partition function in the GCE reads

$$
\begin{aligned}
Z_{\text{g.c.e.}}(V,T,\mu) &= \sum_{N_{1+},N_{1-}=0}^{\infty} \cdots \sum_{N_{j+},N_{j-}=0}^{\infty} \cdots \frac{(\lambda_{1+}z_1)^{N_{1+}}}{N_{1+}!} \\
&\quad \times \frac{(\lambda_{1-}z_1)^{N_{1-}}}{N_{1-}!} \cdots \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \cdots \\
&= \prod_j \sum_{N_{j+},N_{j-}=0}^{\infty} \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \\
&= \prod_j \exp(\lambda_{j+}z_j + \lambda_{j-}z_j) \\
&= \exp\left[2z\cosh\left(\frac{\mu}{T}\right)\right],
\end{aligned} \tag{1}
$$

where j numerates the species, $\lambda_{j\pm} = \exp(\pm\mu/T)$, z_j is a single-particle partition function given by

$$
\begin{aligned}
z_j &= \frac{g_j V}{2\pi^2} \int_0^\infty k^2 dk \exp\left[-\frac{\left(k^2+m_j^2\right)^{1/2}}{T}\right] \\
&= \frac{g_j V}{2\pi^2} T\, m_j^2\, K_2\left(\frac{m_j}{T}\right),
\end{aligned} \tag{2}
$$

and $z \equiv \sum_j z_j$. The V, T, and, μ are, respectively, the system volume, temperature, and chemical potential connected with the conserved charge Q. The g_j and m_j are, respectively, the degeneracy factors and masses for the jth particle species, and K_2 is the modified Hankel function. The CE partition function is obtained by an explicit introduction of the charge conservation constraint, $\sum_j (N_{j+} - N_{j-}) = Q$ for each microscopic state of the system:

$$
\begin{aligned}
Z_{\text{c.e.}}(V,T,Q) &= \sum_{N_{1+},N_{1-}=0}^{\infty} \cdots \sum_{N_{j+},N_{j-}=0}^{\infty} \cdots \frac{(\lambda_{1+}z_1)^{N_{1+}}}{N_{1+}!} \\
&\quad \times \frac{(\lambda_{1-}z_1)^{N_{1-}}}{N_{1-}!} \cdots \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \cdots \\
&\quad \times \delta[(N_{1+} + \cdots + N_{j+} + \ldots - N_{1-} - \ldots \\
&\quad - N_{j-} - \ldots) - Q] \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} \prod_j \sum_{N_{j+},N_{j-}=0}^{\infty} \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \\
&\quad \times \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \exp[i(N_{j+} - N_{j-} - Q)\phi] \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\Bigg[-i\,Q\,\phi + \sum_j z_j(\lambda_{j+}e^{i\phi} \\
&\quad + \lambda_{j-}e^{-i\phi})\Bigg] = I_Q(2z).
\end{aligned} \tag{3}
$$

Parameters λ_{j+} and λ_{j-} in the CE (3) are auxiliary parameters only introduced to calculate the mean number and the fluctuations of positively and negatively charged particles. They are set to one in the final formulas. In Eq. (3) the integral representations of the δ-Kronecker symbol and the modified

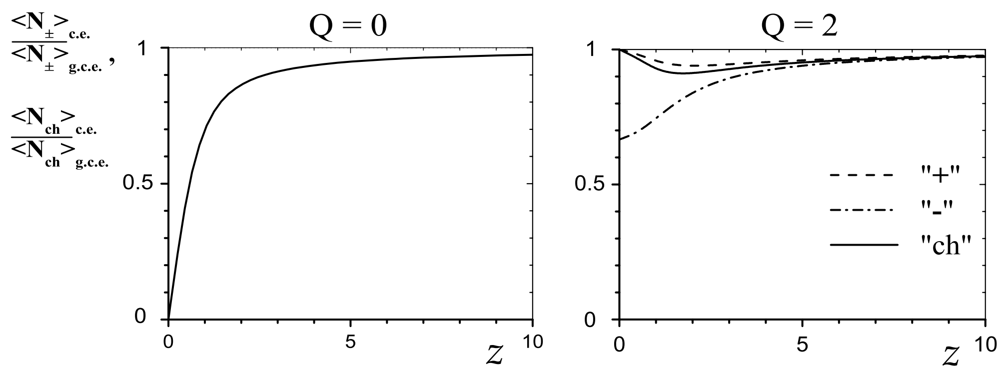

FIG. 1. The ratios of the mean particle numbers in the CE to those in the GCE as functions of z for $Q = 0$ and $Q = 2$.

Bessel function were used [25]:

$$\delta(n) = \int_0^{2\pi} \frac{d\phi}{2\pi} \exp(in\phi),$$
$$I_Q(2z) = \int_0^{2\pi} \frac{d\phi}{2\pi} \exp[-i\,Q\,\phi + 2z\,\cos\phi]. \quad (4)$$

The averages of N_{j+} and N_{j-} in both the GCE and CE can be presented as follows:

$$\langle N_{j\pm}\rangle = \left(\lambda_{j\pm}\frac{\partial \ln Z}{\partial \lambda_{j\pm}}\right) = a_{\pm} z_j, \quad (5)$$

where $a_\pm$ in Eq. (5) is

$$a_{\pm}^{\text{g.c.e.}} = \exp\left(\pm\frac{\mu}{T}\right), \quad a_{\pm}^{\text{c.e.}} = \frac{I_{Q\mp1}(2z)}{I_Q(2z)}, \quad (6)$$

for the GCE and CE, respectively. In the final expressions one should put $\lambda_{j\pm} = \exp(\pm\mu/T)$ and $\lambda_{j\pm} = 1$ for the GCE and CE.

The average numbers of N_+ and N_- are equal to

$$\langle N_\pm\rangle = \left\langle \sum_j N_{j\pm}\right\rangle = a_\pm \sum_j z_j = a_\pm z. \quad (7)$$

The mean net charge in the GCE is equal to

$$Q = \langle N_+\rangle_{\text{g.c.e.}} - \langle N_-\rangle_{\text{g.c.e.}} = 2\sinh\left(\frac{\mu}{T}\right) z, \quad (8)$$

which leads to a simple relation that connects the values of Q and μ:

$$\exp\left(\frac{\mu}{T}\right) = \frac{Q}{2z} + \sqrt{1 + \left(\frac{Q}{2z}\right)^2} \equiv y + \sqrt{1+y^2}; \quad (9)$$

thus

$$\langle N_\pm\rangle_{\text{g.c.e.}} = z(y + \sqrt{1+y^2})^{\pm1}, \quad (10)$$

where $y \equiv Q/2z = \sinh(\mu/T)$.

In the CE an exact charge conservation is imposed on each microscopic state; thus it is evidently fulfilled also for the average values:

$$\langle N_+\rangle_{\text{c.e.}} - \langle N_-\rangle_{\text{c.e.}} = z\frac{I_{Q-1}(2z)}{I_Q(2z)} - z\frac{I_{Q+1}(2z)}{I_Q(2z)} = Q, \quad (11)$$

as in fact can be easily seen from the identity $I_{n-1}(x) - I_{n+1}(x) = 2nI_n(x)/x$ [25].

The ratios of $\langle N_\pm\rangle$ calculated in the CE and in the GCE,

$$\frac{\langle N_\pm\rangle_{\text{c.e.}}}{\langle N_\pm\rangle_{\text{g.c.e.}}} = \frac{I_{Q\mp1}(2z)}{I_Q(2z)} \cdot (y + \sqrt{1+y^2})^{\mp1}, \quad (12)$$

are shown in Fig. 1 for $Q = 0$ and $Q = 2$. There is a strong canonical suppression effect, $\langle N_\pm\rangle_{\text{c.e.}} \ll \langle N_\pm\rangle_{\text{g.c.e.}}$, for small systems ($z \ll 1$), and the canonical and grand canonical ensembles become equivalent, $\langle N_\pm\rangle_{\text{c.e.}} = \langle N_\pm\rangle_{\text{g.c.e.}}$, in the thermodynamic limit $z \to \infty$. One can see that the CE suppression effect is reduced for a nonzero net charge of the system as compared to a system with a zero net charge. In Fig. 2 the ratios (12) as functions of $Q = 1, 2, \ldots$ are shown at fixed positive values of $y = Q/2z$, which correspond to the fixed positive net charge number densities ($Q = 0$ corresponds to $y = 0$ and is presented in Fig. 1). Small values of y mean large z (e.g., for $y = 0.1$ shown in Fig. 2 one finds "large" $z = 5$ at $Q = 1$); thus the system is already close to the thermodynamic limit. Because of this the canonical suppression is small and it is the same for positive and negative particles. The case of large y differs (e.g., for $y = 2$ shown in Fig. 2 the values of z are "small" at small Q: $z = 0.25$ at $Q = 1$). The canonical suppression effect becomes strong for negative particles at small Q. However, the canonical suppression at large y is negligible for the average value of positive particle number as it should be approximately equal to Q.

For small systems ($z \ll 1$) using the series expansion [25]

$$I_n(2z) = \frac{z^n}{n!} + \frac{z^{n+2}}{(n+1)!} + O(z^{n+4}), \quad (13)$$

one finds for $Q = 0$

$$\langle N_\pm\rangle_{\text{c.e.}} \simeq z^2 \ll \langle N_\pm\rangle_{\text{g.c.e.}} = z, \quad (14)$$

and for $Q \geqslant 1$

$$\langle N_+\rangle_{\text{c.e.}} \simeq Q, \quad \langle N_+\rangle_{\text{c.e.}} \simeq \langle N_+\rangle_{\text{g.c.e.}}; \quad (15)$$

$$\langle N_-\rangle_{\text{c.e.}} \simeq \frac{z^2}{Q+1}, \quad \langle N_-\rangle_{\text{c.e.}} \simeq \frac{Q}{Q+1}\langle N_-\rangle_{\text{g.c.e.}}. \quad (16)$$

In the large-volume limit ($V \to \infty$ corresponds also to $z \to \infty$) the mean quantities in the CE and GCE are equal. This result is referred to as an equivalence of the canonical

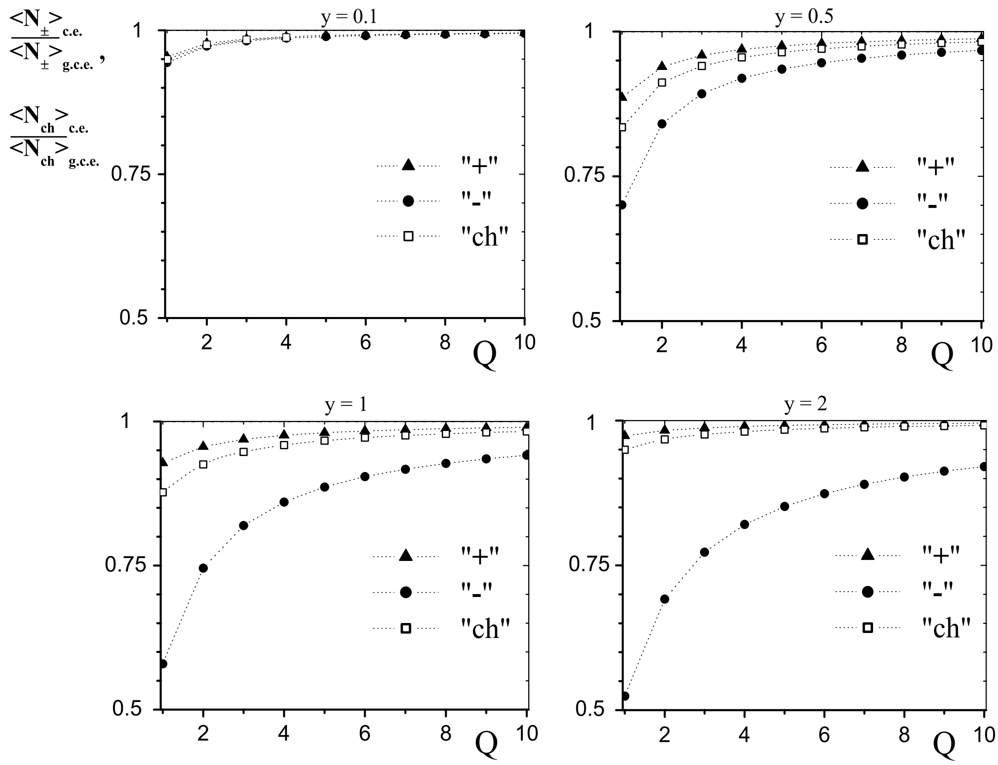

FIG. 2. The ratios of the mean particle numbers in the CE to those in the GCE as functions of $Q = 1, 2, 3, \ldots$ for fixed values of $y = Q/2z$.

and grand canonical ensembles. Using the uniform limit of the modified Bessel function [25]

$$\lim_{n\to\infty} I_n(nx) = \frac{1}{\sqrt{2\pi n}} \frac{\exp(\eta n)}{(1+x^2)^{1/4}} \left[1 + O\left(\frac{1}{n}\right)\right], \qquad (17)$$

where

$$\eta = \sqrt{1+x^2} + \log \frac{x}{1+\sqrt{1+x^2}}, \qquad (18)$$

one can easily find

$$\langle N_\pm \rangle_{\text{c.e.}} \simeq z(y + \sqrt{1+y^2})^{\pm 1} = \langle N_\pm \rangle_{\text{g.c.e.}}. \qquad (19)$$

(Note that fixed Q at $z \to \infty$ means a zero value of the net charge density and $y = 0$.)

The total multiplicity of charged particles is defined as $N_{\text{ch}} = N_+ + N_-$. Its average in the GCE and CE reads

$$\langle N_{\text{ch}} \rangle_{\text{g.c.e.}} \equiv \langle N_+ + N_- \rangle_{\text{g.c.e.}} = \langle N_+ \rangle_{\text{g.c.e.}} + \langle N_- \rangle_{\text{g.c.e.}} = 2z \cosh\left(\frac{\mu}{T}\right), \qquad (20)$$

$$\langle N_{\text{ch}} \rangle_{\text{c.e.}} \equiv \langle\, N_+ + N_- \,\rangle_{\text{c.e.}} = \langle N_+ \rangle_{\text{c.e.}} + \langle N_- \rangle_{\text{c.e.}} = z\left[\frac{I_{Q-1}(2z)}{I_Q(2z)} + \frac{I_{Q+1}(2z)}{I_Q(2z)}\right]. \qquad (21)$$

III. THE SCALED VARIANCE

A useful measure of the fluctuations of any variable X is the ratio of its variance $V(X) \equiv \langle X^2 \rangle - \langle X \rangle^2$ to its mean value $\langle X \rangle$, referred to here as the scaled variance:

$$\omega^X \equiv \frac{\langle X^2 \rangle - \langle X \rangle^2}{\langle X \rangle}. \qquad (22)$$

Note that $\omega^X = 1$ for the Poisson distribution. To study the fluctuations of charged particle numbers, the second moments of the multiplicity distribution have to be calculated. One finds

$$\left\langle N_{j\pm}^2 \right\rangle = \frac{1}{Z}\left[\lambda_{j\pm}\frac{\partial}{\partial \lambda_{j\pm}}\left(\lambda_{j\pm}\frac{\partial Z}{\partial \lambda_{j\pm}}\right)\right] = a_\pm z_j + b_\pm z_j^2, \qquad (23)$$

$$\langle N_{j+} N_{j-} \rangle = \frac{\lambda_{j+}\lambda_{j-}}{Z}\frac{\partial^2 Z}{\partial \lambda_{j+}\partial \lambda_{j-}} = z_j^2, \qquad (24)$$

where $a_\pm$ is given by Eq. (6) and

$$b_\pm^{\text{g.c.e.}} = \exp\left(\pm\frac{2\mu}{T}\right) = \left(a_\pm^{\text{g.c.e.}}\right)^2, \quad b_\pm^{\text{c.e.}} = \frac{I_{Q\mp 2}(2z)}{I_Q(2z)}, \qquad (25)$$

in the GCE and CE, respectively. The scaled variances $\omega^{j\,\pm}$ and $\omega^{j\,\text{ch}}$ are equal to

$$\omega^{j\,\pm} \equiv \frac{\left\langle N_{j+}^2 \right\rangle - \langle N_{j+} \rangle^2}{\langle N_{j+} \rangle} = 1 - z_j\left(a_\pm - \frac{b_\pm}{a_\pm}\right), \qquad (26)$$

$$\omega^{j\,\text{ch}} \equiv \frac{\langle (N_{j+} + N_{j-})^2 \rangle - \langle N_{j+} + N_{j-} \rangle^2}{\langle N_{j+} + N_{j-} \rangle} = 1 + z_j\left[\frac{b_+ + b_- + 2}{a_+ + a_-} - (a_+ + a_-)\right]. \qquad (27)$$

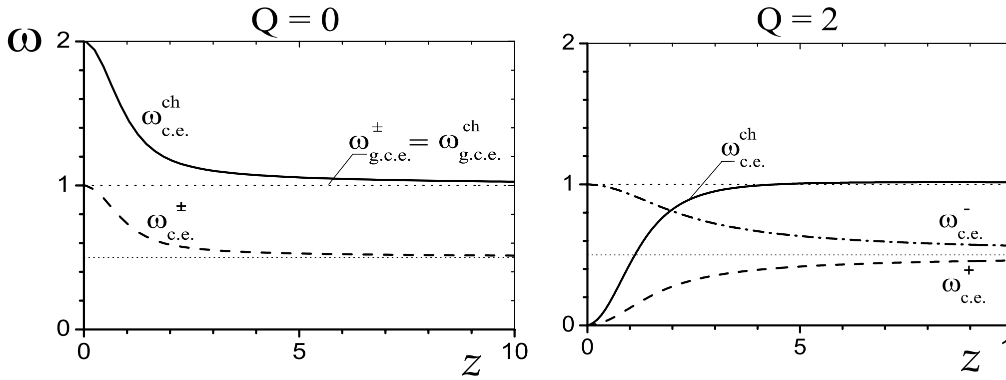

FIG. 3. The scaled variances $\omega^{\pm}_{\text{c.e.}}$ (29) and $\omega^{\text{ch}}_{\text{c.e.}}$ (30) as functions of z for fixed values of the conserved charge Q.

Equations (26) and (27) describe the particle number fluctuations of a given species j. One can establish a general rule for how to calculate the fluctuations of $N_{\pm} = \sum_j N_{j\pm}$ and $N_{\text{ch}} = N_+ + N_-$. To do this one should set $\lambda_{1\pm} = \lambda_{2\pm} = \cdots = \lambda_{\pm}$ in Eqs. (1) and (3) and differentiate with respect to $\lambda_{\pm}$ in Eqs. (5) and (23) to get $\langle N^n_{\pm}\rangle$ $(n = 1, 2)$. This eventually results in a substitution of z instead of z_j in all final formulas for the averages and fluctuations. One obtains

$$\omega^{\pm}_{\text{g.c.e.}} = \omega^{\text{ch}}_{\text{g.c.e.}} = 1. \tag{28}$$

$$\omega^{\pm}_{\text{c.e.}} = 1 - z\left[\frac{I_{Q\mp 1}(2z)}{I_Q(2z)} - \frac{I_{Q\mp 2}(2z)}{I_{Q\mp 1}(2z)}\right], \tag{29}$$

$$\omega^{\text{ch}}_{\text{c.e.}} = 1 + z\left[\frac{I_{Q-2}(2z) + I_{Q+2}(2z) + 2I_Q(2z)}{I_{Q-1}(2z) + I_{Q+1}(2z)} - \frac{I_{Q-1}(2z) + I_{Q+1}(2z)}{I_Q(2z)}\right]. \tag{30}$$

The scaled variances $\omega^{\pm}_{\text{c.e}}$ and $\omega^{\text{ch}}_{\text{c.e}}$ calculated with Eqs. (29) and (30) are shown in Fig. 3 for $Q = 0$, $Q = 2$ and in Fig. 4 for fixed positive values of y. Using the asymptotic behavior of the modified Bessel function for $z \to 0$, Eq. (13), and $z, Q \to \infty$ with $y = Q/2z = \text{constant}$, Eqs. (17) and (18), one can easily find the limits of the scaled variances, for both a given particle species j and the

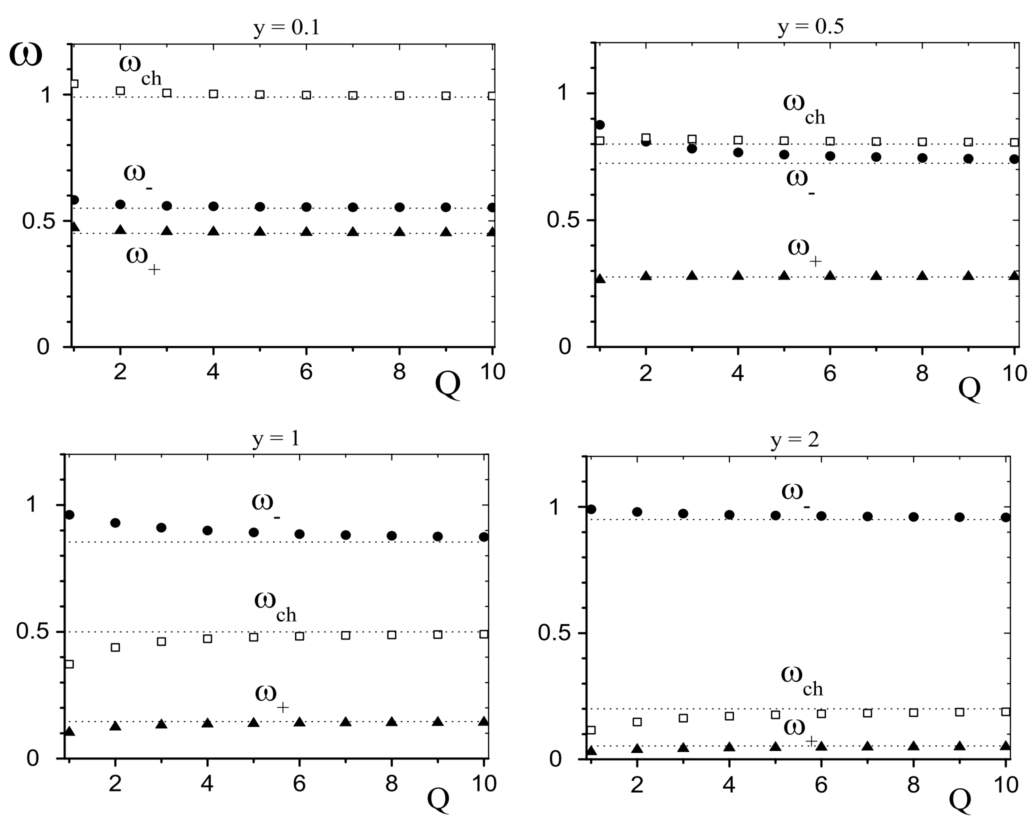

FIG. 4. The scaled variances $\omega^{\pm}_{\text{c.e.}}$ (29) and $\omega^{\text{ch}}_{\text{c.e.}}$ (30) as functions of $Q = 1, 2, 3, \ldots$ for fixed values of $y = Q/2z$.

sum over all particle species:

1. A small-system limit $z \to 0$ gives for $Q = 0$

$$\omega^{j\,+}_{\text{c.e.}} = \omega^{-}_{\text{c.e.}} \simeq 1 - \frac{z_j z}{2}, \quad \omega^{+}_{\text{c.e.}} = \omega^{-}_{\text{c.e.}} \simeq 1 - \frac{z^2}{2}, \tag{31}$$

$$\omega^{j\,\text{ch}}_{\text{c.e.}} \simeq 1 + \frac{z_j}{z} - z_j z, \qquad \omega^{\text{ch}}_{\text{c.e.}} \simeq 2 - z^2, \tag{32}$$

and for $Q \geqslant 1$

$$\omega^{j\,+}_{\text{c.e.}} \cong 1 - \frac{z_j}{z} + \frac{z_j z}{Q(Q+1)}, \quad \omega^{+}_{\text{c.e.}} \cong \frac{z^2}{Q(Q+1)}, \tag{33}$$

$$\omega^{j-}_{\text{c.e.}} \cong 1 - \frac{z_j z}{(Q+1)(Q+2)}, \quad \omega^{-}_{\text{c.e.}} \cong 1 - \frac{z^2}{(Q+1)(Q+2)}, \tag{34}$$

$$\omega^{j\,\text{ch}}_{\text{c.e.}} \cong 1 - \frac{z_j}{z} + \frac{4 z_j z}{Q(Q+1)}, \quad \omega^{\text{ch}}_{\text{c.e.}} \cong \frac{4 z^2}{Q(Q+1)}. \tag{35}$$

2. A large-system limit $z \to \infty$ gives for fixed Q (note again that fixed Q in the thermodynamic limit $z \to \infty$ means a zero value of the net charge density and leads, therefore, to $y = 0$)

$$\omega^{j\,\pm}_{\text{c.e.}} \simeq 1 - \frac{z_j}{2z} + \frac{z_j}{8z^2} \mp \frac{Q z_j}{4z^2}, \quad \omega^{\pm}_{\text{c.e.}} \simeq \frac{1}{2} + \frac{1}{8z} \mp \frac{Q}{4z}, \tag{36}$$

$$\omega^{j\,\text{ch}}_{\text{c.e.}} \simeq 1 + \frac{z_j}{4z^2}, \qquad \omega^{\text{ch}}_{\text{c.e.}} \simeq 1 + \frac{1}{4z}, \tag{37}$$

and for fixed $Q/2z = y$

$$\omega^{j\,\pm}_{\text{c.e.}} \simeq 1 - \frac{z_j}{2z} \mp \frac{z_j}{2z}\frac{y}{\sqrt{1+y^2}}, \quad \omega^{\pm}_{\text{c.e.}} \simeq \frac{1}{2} \mp \frac{y}{2\sqrt{1+y^2}}, \tag{38}$$

$$\omega^{j\,\text{ch}}_{\text{c.e.}} \simeq 1 - \frac{z_j}{z}\frac{y^2}{1+y^2}, \qquad \omega^{\text{ch}}_{\text{c.e.}} \simeq \frac{1}{1+y^2}. \tag{39}$$

As one sees from Eqs. (3) and (4) the scaled variances reach their asymptotic values very quickly. In Fig. 3 the scaled variances for $Q = 0$ and $Q = 2$ can be compared (for $Q = 0$ see details in Ref. [23]). One notices that their values at $z \to \infty$ are the same, but the behavior at small values of z is different. Namely, if $Q \geqslant 1$ the fluctuations of positively charged particles are very small at small z, whereas the fluctuations of negatively charged particles have a Poisson width. This can be easily understood because for small volumes the average number of positive particles is approximately equal to Q [see Eq. (15)] and the fluctuations of N_+ are small. However, at small z and fixed Q the average number of negatively charged particles is much smaller than Q [see Eq. (16)] and the fluctuations of N_- are not affected by the conservation law. Similar physical reasons explain the behavior of the fluctuations at nonzero charge density in the thermodynamic limit. Figure 4 shows the following features of the asymptotic values of $\omega^{+}_{\text{c.e.}}$ and $\omega^{-}_{\text{c.e.}}$ at $Q \gg 1$. When the charge density becomes larger (y increases) $\omega^{+}_{\text{c.e.}}$ decreases and tends to 0 at $y \to \infty$, whereas $\omega^{-}_{\text{c.e.}}$ increases and tends to 1 at $y \to \infty$. The physical reasons for this are seen from Eq. (19), which at $y \gg 1$ gives $\langle N_+\rangle_{\text{c.e.}} \simeq z2y = Q$ and $\langle N_-\rangle_{\text{c.e.}} \simeq z(2y)^{-1} = Q(4y^2)^{-1} \ll Q$. Therefore, at $y \gg 1$ an exact charge conservation keeps N_+ close to its average value Q and makes the fluctuations of N_+ in the CE small. Under the same conditions, $\langle N_-\rangle_{\text{c.e.}}$ is much smaller than Q; thus the fluctuations of N_- are not affected by the CE suppression effects and have a Poisson form. These features of the CE differ strikingly from those in the GCE. The GCE scaled variances (28) are equal to 1 for N_-, N_+, and N_{ch}, and this remains valid for all values of the system net charge or net charge density.

IV. ENERGY FLUCTUATIONS

The partition function in the GCE and CE is equal to $Z \equiv \sum \exp(-E/T)$, where the sum over microstates includes the summation (integration) over particle momenta and summation over number of particles and over different particle species. Each microscopic state has the weight factor $\prod_j \exp[(\mu N_{j+} - \mu N_{j-})/T]$ in the GCE (1) and $\delta[Q - \sum_j (N_{j+} - N_{j-})]$ in the CE (3). To calculate the average energy and its fluctuations it is convenient to rewrite the partition function as $Z = \sum \exp[\sum_j (\beta_{j+} E_{j+} + \beta_{j-} E_{j-})/T]$, where β_{j+} and β_{j-} are the auxiliary parameters and $\beta_{j+} = \beta_{j-} = \beta \equiv 1/T$ in the final formulas. It then follows that

$$\langle E_{j\pm}\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta_{j\pm}} = -a_\pm z_j' \equiv \langle \varepsilon_j\rangle \langle N_{j\pm}\rangle, \tag{40}$$

$$\langle E_{i\pm} E_{j\pm}\rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta_{i\pm}\beta_{j\pm}} = a_\pm z_j'' \delta_{ij} + b_\pm z_i' z_j', \tag{41}$$

$$\langle E_{i+} E_{j-}\rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta_{i+}\beta_{j-}} = z_i' z_j', \tag{42}$$

where $z_j' = \partial z_j/\partial\beta$, $z_j'' = \partial^2 z_j/\partial\beta^2$, and z_j, $a_\pm$, and $b_\pm$ are given by Eqs. (2), (6), and (25), respectively. In Eq. (40) we have introduced the average value of one-particle energy $\langle \varepsilon_j\rangle \equiv -z_j'/z_j$. By introducing also $\langle \varepsilon_j^2\rangle \equiv z_j''/z_j$ the energy fluctuations can be then presented as follows:

$$W^{j\pm} \equiv \frac{\langle E_{j\pm}^2\rangle - \langle E_{j\pm}\rangle^2}{\langle E_{j\pm}\rangle} = \frac{\langle \varepsilon_j^2\rangle - \langle \varepsilon_j\rangle^2}{\langle \varepsilon_j\rangle} + \langle \varepsilon_j\rangle \omega^{j\pm}, \tag{43}$$

where $\omega^{j\pm}$ is given by Eq. (26). Introducing the total energies $E_\pm \equiv \sum_j E_{j\pm}$ and $E_{\text{ch}} \equiv \sum_j (E_{j+} + E_{j-})$, one finds

$$W^{\pm} \equiv \frac{\langle E_\pm^2\rangle - \langle E_\pm\rangle^2}{\langle E_\pm\rangle} = \frac{\langle \varepsilon^2\rangle - \langle \varepsilon\rangle^2}{\langle \varepsilon\rangle} + \langle \varepsilon\rangle \omega^{\pm}, \tag{44}$$

$$W^{\text{ch}} \equiv \frac{\langle E_{\text{ch}}^2\rangle - \langle E_{\text{ch}}\rangle^2}{\langle E_{\text{ch}}\rangle} = \frac{\langle \varepsilon^2\rangle - \langle \varepsilon\rangle^2}{\langle \varepsilon\rangle} + \langle \varepsilon\rangle \omega^{\text{ch}}, \tag{45}$$

where $\langle \varepsilon\rangle \equiv \sum_j z_j \langle \varepsilon_j\rangle/z$, $\langle \varepsilon^2\rangle \equiv \sum_j \langle \varepsilon_j^2\rangle z_j/z$. Equations (44) and (45) are valid in both GCE and CE. The energy fluctuations

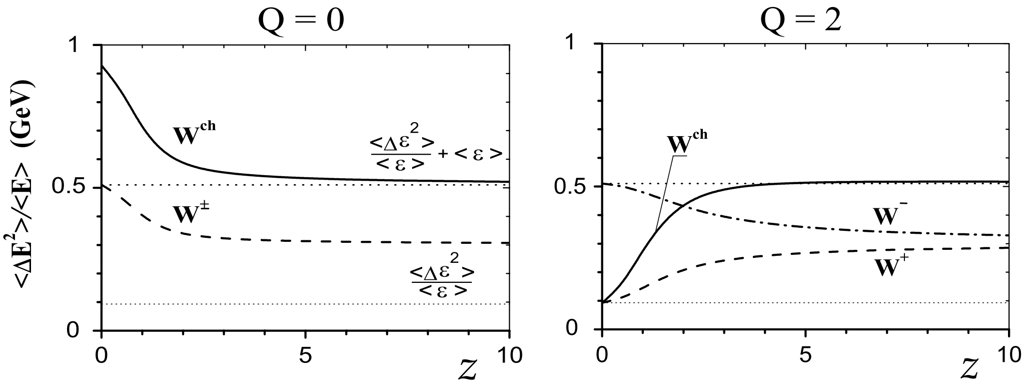

FIG. 5. The CE energy fluctuations W^+, W^-, and W^{ch} in an ideal pion gas at temperature $T = 120$ MeV.

consist of two terms. The first term takes into account the fluctuations of one-particle energies; the second one accounts for the fluctuations of the number of particles. The fluctuations of the number of particles are relatively more important than the fluctuations of one-particle energies. Indeed, the maximal value of the first term on the right-hand side of Eqs. (44) and (45) is equal to $\langle\varepsilon\rangle/3$ for particles with $m/T \to 0$, and it decreases and goes to zero at $m/T \to \infty$. However, the second term on the right-hand side of Eqs. (44) and (45) is equal to $\langle\varepsilon\rangle$ for any system in the GCE. The value of $(\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2)/\langle\varepsilon\rangle$ in Eqs. (44) and (45) is the same for "+"and "−" particles, and in both the GCE and CE. However, the values of ω's are different in the GCE and CE. Moreover, $\omega^+_{\mathrm{c.e.}}$, $\omega^-_{\mathrm{c.e.}}$, and $\omega^{\mathrm{ch}}_{\mathrm{c.e.}}$ differ from each other for the nonzero net charge Q. Therefore, the scaled variances of the energy fluctuations are different in the GCE and CE, and in the CE the values of W^+, W^- and W^{ch} differ from each other and depend on the net charge of the system. An example of the energy fluctuations W^+, W^-, and W^{ch} for an ideal pion gas with $Q = 0$ and $Q = 2$ is presented in Fig. 5. It shows that the dependences of the energy fluctuations W^+, W^-, and W^{ch} on z in the CE resemble those for $\omega^+_{\mathrm{c.e.}}$, $\omega^-_{\mathrm{c.e.}}$, and $\omega^{\mathrm{ch}}_{\mathrm{c.e.}}$ shown in Fig. 3.

The upper horizontal dotted line in Fig. 5 shows the value of $\langle\varepsilon^2\rangle/\langle\varepsilon\rangle$ that corresponds to $W^+ = W^- = W^{\mathrm{ch}}$ in the GCE. The lower horizontal dotted line in Fig. 5 shows the value of $(\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2)/\langle\varepsilon\rangle$ that indicates the fluctuations of one-pion energy.

V. SINGLY AND DOUBLY CHARGED PARTICLES

In the following sections we consider the extension of the CE formalism. First, let us study the system of particles and antiparticles with charges ± 1 and ± 2. The CE partition function reads

$$Z_{\mathrm{c.e.}}(V,T,Q) = \sum_{N_+,N_-,\widetilde{N}_+,\widetilde{N}_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!}\frac{(\lambda_- z)^{N_-}}{N_-!}\frac{(\widetilde{\lambda}_+\widetilde{z})^{\widetilde{N}_+}}{\widetilde{N}_+!} \times \frac{(\widetilde{\lambda}_-\widetilde{z})^{\widetilde{N}_-}}{\widetilde{N}_-!}\delta[(N_+ - N_- + 2\widetilde{N}_+ - 2\widetilde{N}_-) - Q]$$

$$= \int_0^{2\pi}\frac{d\phi}{2\pi}\exp[-i\,Q\,\phi + z(\lambda_+ e^{i\phi} + \lambda_- e^{-i\phi}) + \widetilde{z}(\widetilde{\lambda}_+ e^{i2\phi} + \widetilde{\lambda}_- e^{-i2\phi})]$$

$$= \sum_{k=-\infty}^{\infty} I_{Q-2k}(2z) I_k(2\widetilde{z}), \tag{46}$$

where we have used the relation $\exp[x(t+\frac{1}{t})] = \sum_{k=-\infty}^{\infty} t^k I_k(2x)$. The z and $\tilde{z}$ in Eq. (46) are the one-particle partition functions for charges ± 1 and ± 2, respectively. In terms of variables $c_{m\pm}$ ($m = 1, 2, 4$)

$$c_{m\pm} = \frac{\sum_{k=-\infty}^{\infty} I_{Q\mp m-2k}(2z) I_k(2\widetilde{z})}{\sum_{k=-\infty}^{\infty} I_{Q-2k}(2z) I_k(2\widetilde{z})}, \tag{47}$$

one finds

$$\langle N_\pm\rangle_{\mathrm{c.e.}} = c_{1\pm} z, \qquad \langle \widetilde{N}_\pm\rangle_{\mathrm{c.e.}} = c_{2\pm}\widetilde{z}, \tag{48}$$

$$\langle N_\pm^2\rangle_{\mathrm{c.e.}} = c_{1\pm} z + c_{2\pm} z^2, \quad \langle \widetilde{N}_\pm^2\rangle_{\mathrm{c.e.}} = c_{2\pm}\widetilde{z} + c_{4\pm}\widetilde{z}^2. \tag{49}$$

From Eqs. (48) and (49) it follows that

$$\omega^\pm_{1\,\mathrm{c.e.}} \equiv \frac{\langle N_\pm^2\rangle_{\mathrm{c.e.}} - \langle N_\pm\rangle^2_{\mathrm{c.e.}}}{\langle N_\pm\rangle_{\mathrm{c.e.}}} = 1 - z\left(c_{1\pm} - \frac{c_{2\pm}}{c_{1\pm}}\right), \tag{50}$$

$$\omega^\pm_{2\,\mathrm{c.e.}} \equiv \frac{\langle \widetilde{N}_\pm^2\rangle_{\mathrm{c.e.}} - \langle \widetilde{N}_\pm\rangle^2_{\mathrm{c.e.}}}{\langle \widetilde{N}_\pm\rangle_{\mathrm{c.e.}}} = 1 - \widetilde{z}\left(c_{2\pm} - \frac{c_{4\pm}}{c_{2\pm}}\right), \tag{51}$$

$$\omega^\pm_{\mathrm{c.e.}} \equiv \frac{\langle (N_\pm + \widetilde{N}_\pm)^2\rangle_{\mathrm{c.e.}} - \langle N_\pm + \widetilde{N}_\pm\rangle^2_{\mathrm{c.e.}}}{\langle N_\pm + \widetilde{N}_\pm\rangle_{\mathrm{c.e.}}} = 1 + \frac{z^2 c_{2\pm} + \widetilde{z}^2 c_{4\pm} + 2z\widetilde{z}c_{3\pm}}{zc_{1\pm} + \widetilde{z}c_{2\pm}} - (zc_{1\pm} + \widetilde{z}c_{2\pm}). \tag{52}$$

To illustrate the specific features of the considered system we present the CE results in the case $Q = 0$. In the GCE all ω's are still equal to 1. For $\langle Q\rangle_{\mathrm{g.c.e.}} = 0$ one also has $\langle N_\pm\rangle_{\mathrm{g.c.e.}} = z$ and $\langle \widetilde{N}_\pm\rangle_{\mathrm{g.c.e.}} = \widetilde{z}$. To calculate Eqs. (48) and (52) for finite values of z and $\widetilde{z}$ one can effectively use

Eq. (47). In the thermodynamic limit $V \to \infty$ both $z \to \infty$ and $\widetilde{z} \to \infty$. In this case it is convenient to return to the integral representation in Eq. (46) and use it also for the derivatives of the CE partition function with respect to $\lambda_\pm$ and $\widetilde{\lambda}_\pm$. Using the saddle point method to calculate the ϕ-integrals one finds then for $z, \widetilde{z} \gg 1$,

$$\langle N_\pm \rangle_{\text{c.e.}} \simeq z\left[1 - \frac{1}{4(z+4\widetilde{z})}\right], \quad \langle \widetilde{N}_\pm \rangle_{\text{c.e.}} = \widetilde{z}\left(1 - \frac{1}{z+4\widetilde{z}}\right); \tag{53}$$

$$\omega^\pm_{1\text{c.e.}} \simeq 1 - \frac{z}{2(z+4\widetilde{z})}, \qquad \omega^\pm_{2\text{c.e.}} \simeq 1 - \frac{2\widetilde{z}}{z+4\widetilde{z}},$$
$$\omega^\pm_{\text{c.e.}} \simeq 1 - \frac{(z+2\widetilde{z})^2}{2(z+\widetilde{z})(z+4\widetilde{z})}. \tag{54}$$

From these formulas one finds that $\omega^\pm_{\text{c.e.}} \simeq 0.5$ if $\widetilde{z}/z$ is either much smaller or much larger than 1. The scaled variance $\omega^\pm_{\text{c.e.}}$ has a maximum at $z = 2\widetilde{z}$. At this point one finds $\omega^\pm_{\text{c.e.}} = 5/9$, $\omega^\pm_{1\,\text{c.e.}} = 5/6$, and $\omega^\pm_{2\,\text{c.e.}} = 2/3$.

VI. QUANTUM STATISTICS EFFECTS

It is instructive to apply a different technique [17] to calculate the fluctuations of the thermodynamical quantities with the exact conservation laws imposed. This method allows one to find the values of the CE fluctuations in the thermodynamic limit and include the effects of quantum statistics.

The ideal quantum gas of identical Bose or Fermi particles and antiparticles can be characterized by the occupation numbers $n_p^\pm$ of the one-particle states labeled by momenta p. The GCE average values and fluctuations are [26]

$$\langle n_p^\pm \rangle_{\text{g.c.e.}} = \frac{1}{\exp[(\sqrt{p^2+m^2} \mp \mu)/T] - \gamma}, \tag{55}$$

$$\left\langle \Delta n_p^{\pm 2} \right\rangle_{\text{g.c.e.}} \equiv \langle (n_p^\pm)^2 \rangle_{\text{g.c.e.}} - \langle n_p^\pm \rangle^2_{\text{g.c.e.}}$$
$$= \langle n_p^\pm \rangle_{\text{g.c.e.}} (1 + \gamma \langle n_p^\pm \rangle_{\text{g.c.e.}}) \equiv v_p^{\pm 2}, \tag{56}$$

respectively, where γ is equal to $+1$ and -1 for Bose and Fermi statistics, respectively. ($\gamma = 0$ corresponds to the Boltzmann approximation.) These expressions are microscopic in a sense that they describe the average values and fluctuations of a single mode with momentum p. However, the average values of all macroscopic quantities of the system can be determined through the average occupation numbers of these single modes. The fluctuations of the macroscopic observables can be written in terms of the microscopic correlator $\langle \Delta n_p^\alpha \Delta n_k^\beta \rangle_{\text{g.c.e.}}$, where α, β are $+$ and(or) $-$. This correlator can be presented as

$$\left\langle \Delta n_p^\alpha \Delta n_k^\beta \right\rangle_{\text{g.c.e.}} = v_p^{\alpha 2} \delta_{pk} \delta_{\alpha\beta}, \tag{57}$$

owing to the statistical independence of different quantum levels and different charge states in the GCE. The variances of the total number of positively (negatively) charged particles $N_\alpha = \sum_p n_p^\alpha$ are equal to

$$\left\langle \Delta N_\alpha^2 \right\rangle_{\text{g.c.e.}} \equiv \left\langle N_\alpha^2 \right\rangle_{\text{g.c.e.}} - \langle N_\alpha \rangle^2_{\text{g.c.e.}}$$
$$= \sum_{p,k} \left(\left\langle n_p^\alpha n_k^\alpha \right\rangle_{\text{g.c.e.}} - \left\langle n_p^\alpha \right\rangle_{\text{g.c.e.}} \left\langle n_k^\alpha \right\rangle_{\text{g.c.e.}} \right)$$
$$= \sum_{p,k} \left\langle \Delta n_p^\alpha \Delta_k^\alpha \right\rangle_{\text{g.c.e.}} = \sum_p v_p^{\alpha 2}.$$

We have previously assumed that the quantum p levels are nondegenerate. In fact each level should be further specified by the projection of a particle spin. Thus, each p level splits into $g = 2j+1$ sublevels. It will be assumed that the p summation includes all these sublevels too. The degeneracy factor enters explicitly when the summation over discrete levels is substituted by the integration in the thermodynamic limit:

$$\sum_p \ldots = \frac{gV}{2\pi^2} \int_0^\infty p^2 dp \ldots .$$

The scaled variance $\omega^\alpha_{\text{g.c.e.}}$ in the thermodynamical limit $V \to \infty$ reads

$$\omega^\alpha_{\text{g.c.e.}} \equiv \frac{\left\langle N_\alpha^2 \right\rangle_{\text{g.c.e.}} - \langle N_\alpha \rangle^2_{\text{g.c.e.}}}{\langle N_\alpha \rangle_{\text{g.c.e.}}} = \frac{\sum_{p,k} \left\langle \Delta n_p^\alpha \Delta n_k^\alpha \right\rangle_{\text{g.c.e.}}}{\sum_p \left\langle n_p^\alpha \right\rangle_{\text{g.c.e.}}}$$
$$= \frac{\sum_p v_p^{\alpha 2}}{\sum_p \left\langle n_p^\alpha \right\rangle_{\text{g.c.e.}}} \simeq \frac{\int_0^\infty p^2 dp\, v_p^{\alpha 2}}{\int_0^\infty p^2 dp \left\langle n_p^\alpha \right\rangle_{\text{g.c.e.}}}. \tag{58}$$

Equation (58) corresponds to the particle number fluctuations in the GCE. To illustrate the role of quantum statistics let us consider the case of $\mu = 0$ (i.e., $\langle Q \rangle_{\text{g.c.e.}} = 0$, where $Q \equiv \sum_{p,\alpha} q^\alpha n_p^\alpha$). In the formulas that follow we assume that $q^+ = 1$ and $q^- = -1$, however, these formulas are valid for any values of q^+ and $q^- = -q^+$. From Eqs. (56) and (58) one finds $\omega^{\pm\text{Boltz}}_{\text{g.c.e.}} = 1$ in the Boltzmann limit ($\gamma = 0$), $\omega^{\pm\text{Bose}}_{\text{g.c.e.}} > 1$ for the Bose particles ($\gamma = 1$), and $\omega^{\pm\text{Fermi}}_{\text{g.c.e.}} < 1$ for the Fermi particles ($\gamma = -1$). The strongest quantum effects correspond to $m/T \to 0$:

$$\omega^{\pm\text{Boltz}}_{\text{g.c.e.}} = 1, \quad \omega^{\pm\text{Bose}}_{\text{g.c.e.}} = \frac{\pi^2}{6\,\zeta(3)} \simeq 1.368,$$
$$\omega^{\pm\text{Fermi}}_{\text{g.c.e.}} = \frac{\pi^2}{9\,\zeta(3)} \simeq 0.912. \tag{59}$$

The scaled variance $\omega^{\text{ch}}_{\text{g.c.e.}}$ for all charged particles can be easily obtained from (58) by replacing $\sum_p$ by $\sum_{p\alpha}$, and one finds

$$\omega^{\text{ch Boltz}}_{\text{g.c.e.}} = \omega^{\pm\text{Boltz}}_{\text{g.c.e.}}, \quad \omega^{\text{ch Bose}}_{\text{g.c.e.}} = \omega^{\pm\text{Bose}}_{\text{g.c.e.}}, \quad \omega^{\text{ch Fermi}}_{\text{g.c.e.}} = \omega^{\pm\text{Fermi}}_{\text{g.c.e.}}. \tag{60}$$

The formula for the microscopic correlator is modified if we impose the exact charge conservation in our equilibrated system. For this purpose we introduce the equilibrium probability distribution $W(n_p^\alpha)$ of the occupation numbers. As a first step we assume that each n_p^α fluctuates independently according to the Gauss distribution law with the average value $\langle n_p^\alpha \rangle_{\text{g.c.e.}}$

(55) and the mean square deviation $v_p^{\alpha 2}$ (56):

$$W(n_p^\alpha) \propto \prod_{p,\alpha} \exp\left[-\frac{\left(\Delta n_p^\alpha\right)^2}{2v_p^{\alpha 2}}\right]. \tag{61}$$

To justify this assumption (see Ref. [17] one can consider the sum of n_p^α in small momentum volume $(\Delta p)^3$ with the center at p. At fixed $(\Delta p)^3$ and $V \to \infty$ the average number of particles inside $(\Delta p)^3$ becomes large. Each particle configuration inside $(\Delta p)^3$ consists of $(\Delta p)^3\, V/(2\pi)^3 \gg 1$ statistically independent terms, each with average value $\langle n_p^\alpha\rangle_{\text{g.c.e.}}$ (55) and variance $v_p^{\alpha 2}$ (56). From the central limit theorem it follows that the probability distribution for the fluctuations inside $(\Delta p)^3$ should be Gaussian. In fact, we always convolve n_p^α with some smooth function of p, so instead of writing the Gaussian distribution for the sum of n_p^α in $(\Delta p)^3$ we can use it directly for n_p^α.

The average value of the conserved charge $Q = \sum_{p,\alpha} q^\alpha n_p^\alpha$ is regulated in the GCE by the chemical potential μ. If we impose exact charge conservation, $\Delta Q = \sum_{p,\alpha} q^\alpha \Delta n_p^\alpha = 0$, the distribution (61) will be modified as

$$\begin{aligned} W(n_p^\alpha) &\propto \prod_{p,\alpha} \exp\left[-\frac{\left(\Delta n_p^\alpha\right)^2}{2v_p^{\alpha 2}}\right] \delta\left(\sum_{p,\alpha} q^\alpha \Delta n_p^\alpha\right) \\ &\propto \int_{-\infty}^{\infty} d\lambda \prod_{p,\alpha} \exp\left[-\frac{\left(\Delta n_p^\alpha\right)^2}{2v_p^{\alpha 2}} + i\lambda q^\alpha \Delta n_p^\alpha\right]. \end{aligned} \tag{62}$$

It is convenient to generalize distribution (62) to $W(n_p^\alpha, \lambda)$ using further the integration along the imaginary axis in the λ plane. After completing squares one gets

$$W\left(n_p^\alpha, \lambda\right) \propto \prod_{p,\alpha} \exp\left[-\frac{\left(\Delta n_p^\alpha - \lambda v_p^{\alpha 2} q^\alpha\right)^2}{2v_p^{\alpha 2}} + \frac{\lambda^2}{2} v_p^{\alpha 2} q^{\alpha 2}\right], \tag{63}$$

and the average values are now calculated as

$$\langle \ldots \rangle = \frac{\int_{-i\infty}^{i\infty} d\lambda \int_{-\infty}^{\infty} \prod_{p,\alpha} dn_p^\alpha \ldots W\left(n_p^\alpha, \lambda\right)}{\int_{-i\infty}^{i\infty} d\lambda \int_{-\infty}^{\infty} \prod_{p,\alpha} dn_p^\alpha W\left(n_p^\alpha, \lambda\right)}. \tag{64}$$

Equation (64) gives the CE averaging in the thermodynamic limit $V \to \infty$. One easily finds

$$\begin{aligned} &\left\langle\left(\Delta n_p^\alpha - v_p^{\alpha 2}\lambda q^\alpha\right)\left(\Delta n_k^\beta - v_k^{\beta 2}\lambda q^\beta\right)\right\rangle = \delta_{pk}\delta_{\alpha\beta} v_p^{\alpha 2}, \\ &\langle \lambda^2\rangle = -\left(\sum_{p,\alpha} v_p^{\alpha 2} q^{\alpha 2}\right)^{-1}, \quad \left\langle\left(\Delta n_p^\alpha - v_p^{\alpha 2}\lambda q^\alpha\right)\lambda\right\rangle = 0, \end{aligned}$$

so it follows that

$$\begin{aligned} \left\langle \Delta n_p^\alpha \Delta n_k^\beta\right\rangle &= \delta_{pk}\delta_{\alpha\beta} v_p^{\alpha 2} - v_p^{\alpha 2} q^\alpha v_k^{\beta 2} q^\beta \langle\lambda^2\rangle \\ &\quad + \left\langle \Delta n_p^\alpha \lambda\right\rangle v_k^{\beta 2} q^\beta + \left\langle \Delta n_k^\beta \lambda\right\rangle v_p^{\alpha 2} q^\alpha \\ &= \delta_{pk}\delta_{\alpha\beta} v_p^{\alpha 2} + v_p^{\alpha 2} q^\alpha v_k^{\beta 2} q^\beta \langle\lambda^2\rangle \\ &= \delta_{pk}\delta_{\alpha\beta} v_p^{\alpha 2} - \frac{v_p^{\alpha 2} q^\alpha v_k^{\beta 2} q^\beta}{\sum_{p,\alpha} v_p^{\alpha 2} q^{\alpha 2}}. \end{aligned} \tag{65}$$

By means of Eq. (65) we obtain

$$\begin{aligned} \omega_{\text{c.e.}}^\alpha \equiv \frac{\left\langle N_\alpha^2\right\rangle - \langle N_\alpha\rangle^2}{\langle N_\alpha\rangle} &= \frac{\sum_p v_p^{\alpha 2}}{\sum_p \left\langle n_p^\alpha\right\rangle_{\text{g.c.e.}}} \\ &\quad - \frac{\left(\sum_p v_p^{\alpha 2} q^\alpha\right)^2}{\sum_p \left\langle n_p^\alpha\right\rangle_{\text{g.c.e.}} \sum_{p,\alpha} v_p^{\alpha 2} q^{\alpha 2}}. \end{aligned} \tag{66}$$

Equation (55) leads to $v_p^{\alpha 2} = \langle n_p^\alpha\rangle_{\text{g.c.e.}}$ in the Boltzmann approximation; thus from Eq. (66) one finds $[y \equiv Q/2z = \sinh(\mu/T)]$

$$\omega_{\text{c.e.}}^\alpha = 1 - \frac{\exp(\alpha\mu/T)}{\exp(\mu/T) + \exp(-\mu/T)} = \frac{1}{2} - \alpha\frac{y}{2\sqrt{1+y^2}}, \tag{67}$$

which coincides with Eq. (38). The formula for $\omega_{\text{c.e.}}^{\text{ch}}$ can be obtained from (66) after replacing $\sum_p$ by $\sum_{p,\alpha}$, and it is the same as Eq. (39). At $\mu = 0$ from Eq. (66) we find the CE scaled variances:

$$\begin{aligned} &\omega_{\text{c.e.}}^{\pm\text{Boltz}} = \frac{1}{2}, \quad \omega_{\text{c.e.}}^{\pm\text{Bose}} = \frac{\pi^2}{12\zeta(3)} \simeq 0.684, \\ &\omega_{\text{c.e.}}^{\pm\text{Fermi}} = \frac{\pi^2}{18\zeta(3)} \simeq 0.456, \end{aligned} \tag{68}$$

$$\begin{aligned} &\omega_{\text{c.e.}}^{\text{ch Boltz}} = 2\,\omega_{\text{g.c.e.}}^{\pm\text{Boltz}}, \quad \omega_{\text{c.e.}}^{\text{ch Bose}} = 2\,\omega_{\text{c.e.}}^{\pm\text{Bose}}, \\ &\omega_{\text{c.e.}}^{\text{ch Fermi}} = 2\,\omega_{\text{c.e.}}^{\pm\text{Fermi}}. \end{aligned} \tag{69}$$

As seen from Eqs. (60) and (69) the scaled variance of positively (negatively) charged particles with Bose or Fermi statistics in the CE is half those of the corresponding scaled variances in the GCE. Therefore, the CE suppression of the particle number fluctuations in the thermodynamic limit works at $\mu = 0$ in quantum systems similar to that in the Boltzmann case. This result can be rephrased in another way: The Bose enhancement and Fermi suppression of the GCE fluctuations remain the same in the CE for $\omega_{\text{c.e.}}^{\pm}$ at $\mu = 0$ in the thermodynamic limit. Equation (70) demonstrates that the scaled variances of all charged particles in the CE for any statistics are a factor of 2 larger than the corresponding scaled variances for (negative) positive particles, whereas in the GCE these scaled variances presented by Eq. (60) are equal to each other.

Comparing Eq. (65) and Eq. (57) one notices changes of the microscopic correlator because of exact charge conservation. Namely, in the CE the fluctuations of each mode are reduced) [i.e., the $\langle(\Delta n_p^\alpha)^2\rangle$ calculated from Eq. (65) is smaller than that in Eq. (56)] and anticorrelations between different modes $p \neq k$ and the same charge states $\alpha = \beta$ appear. These two changes of the microscopic correlator result in a suppression of the CE scaled variances $\omega_{\text{c.e.}}^\alpha$ compared with the GCE ones $\omega_{\text{g.c.e.}}^\alpha$ [compare Eq. (66)] and Eq. (58)]. Therefore, the fluctuations of both N_- and N_+ are always suppressed in the CE. As we have seen in previous sections the behavior of N_{ch} fluctuations in the CE can be more complicated. This occurs because of the correlations of different modes $p \neq k$ for the different charge states $\alpha = -\beta$ [i.e., the second term on the right-hand side of Eq. (65) is positive for $\alpha = -\beta$].

Exact charge conservation should also lead to the canonical suppression of $\langle n_p^\alpha \rangle$, and this should result in canonical suppression effects for $\langle N_\alpha \rangle$. They are, however, absent in the present formulation; thus formula (65) for the microscopic correlator is not sufficient to calculate $\langle N_\alpha^2 \rangle$ and $\langle N_\alpha \rangle^2$ separately with an accuracy corresponding to the effects of the canonical suppression. Nevertheless, it does allow us to calculate their difference $\langle (\Delta N_\alpha)^2 \rangle$ with the effects of the CE correctly included. This means that canonical suppression effects in the occupation numbers $\langle n_p^\alpha \rangle$ lead to changes of the order of $\langle N_\alpha \rangle$ in both $\langle N_\alpha^2 \rangle$ and $\langle N_\alpha \rangle^2$, but these changes are the same and the correction terms cancel in the calculation of $\langle (\Delta N_\alpha)^2 \rangle$. Therefore, the macroscopic fluctuations of multiplicities are not affected by the CE corrections to the average particle numbers. The scaled variances of the CE in the thermodynamic limit $V \to \infty$ feel the consequences of exact charge conservation owing to the suppression of the single mode fluctuations $\langle (\Delta n_p^\alpha)^2 \rangle$ and owing to the (anticorrelations) correlations between different modes $p \neq k$ with the (same) different charge states α, β. All these effects are absent in the GCE.

VII. A SYSTEM WITH TWO CONSERVED CHARGES

In the previous sections we have considered a system with one conserved charge. In high-energy collisions the measurements of fluctuations for the particle numbers and (transverse) energies are mainly done for electrically charged hadrons. Therefore, in applying the CE results to an analysis of the data on fluctuations it would be reasonable to start with the case when the charge Q is assumed to be an electric charge. Nonetheless, other conserved charges are also present in the system created in high-energy collisions. In this section we consider a system with two exactly conserved charges—electric charge Q and baryonic number B. As an example we study an ideal pion-nucleon gas and neglect quantum statistics effects. This is the simplest realistic case where we can study the influence of an exact B conservation to the CE fluctuations of electrically charged particles. The partition function of this system in the CE is

$$
\begin{aligned}
Z_{\text{c.e.}}(V,T,Q,B) &= \sum_{N_p, N_{\bar p}=0}^{\infty} \sum_{N_n, N_{\bar n}=0}^{\infty} \sum_{N_\pi, N_{\bar \pi}=0}^{\infty} \frac{(\lambda_p z_p)^{N_p}}{N_p!} \frac{(\lambda_{\bar p} z_{\bar p})^{N_{\bar p}}}{N_{\bar p}!} \frac{(\lambda_n z_n)^{N_n}}{N_n!} \frac{(\lambda_{\bar n} z_{\bar n})^{N_{\bar n}}}{N_{\bar n}!} \frac{(\lambda_{\pi^+} z_\pi)^{N_{\pi^+}}}{N_{\pi^+}!} \frac{(\lambda_{\pi^-} z_\pi)^{N_{\pi^-}}}{N_{\pi^-}!} \\
&\quad \times \delta[(N_p - N_{\bar p} + N_{\pi^+} - N_{\pi^-}) - Q]\,\delta[(N_p - N_{\bar p} + N_n - N_{\bar n}) - B] \\
&= \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp(-iQ\varphi - iB\phi) \times \exp[z_p(\lambda_p e^{i(\varphi+\phi)} + \lambda_{\bar p} e^{-i(\varphi+\phi)})] \\
&\quad \times \exp[z_n(\lambda_n e^{i\phi} + \lambda_{\bar n} e^{-i\phi})] \times \exp[z_\pi(\lambda_{\pi^+} e^{i\varphi} + \lambda_{\pi^-} e^{-i\varphi})] \\
&= \sum_{k=-\infty}^{\infty} I_{k-Q}(2z_p) I_{k+B-Q}(2z_n) I_k(2z_\pi),
\end{aligned} \tag{70}
$$

where we have used $\exp[x(t+1/t)] = \sum_{k=0}^{\infty} t^k I_k(2x)$. From Eq. (70) it follows that

$$
\langle N_{j,\alpha} \rangle_{\text{c.e.}} = c_{1,\alpha}^j z_j, \quad \left\langle N_{j,\alpha}^2 \right\rangle_{\text{c.e.}} = c_{1,\alpha}^j z_j + c_{2,\alpha}^j z_j^2, \tag{71}
$$

where j numerates pion, neutron, and proton; $\alpha = 1$ corresponds to particles π^+, n, and p and $\alpha = -1$ to antiparticles $\pi^-, \overline{n}$, and $\overline{p}$; and $(m = 1, 2)$

$$
c_{m,\alpha}^p = \sum_{k=-\infty}^{\infty} I_{k+\alpha\cdot m-Q}(2z_p) I_{k+B-Q}(2z_n) I_k(2z_\pi) \times [Z_{\text{c.e.}}(V,T,Q,B)]^{-1}, \tag{72}
$$

$$
c_{m,\alpha}^n = \sum_{k=-\infty}^{\infty} I_{k-Q}(2z_p) I_{k+\alpha\cdot m+B-Q}(2z_n) I_k(2z_\pi) \times [Z_{\text{c.e.}}(V,T,Q,B)]^{-1}, \tag{73}
$$

$$
c_{m,\alpha}^\pi = \sum_{k=-\infty}^{\infty} I_{k+\alpha\cdot m-Q}(2z_p) I_{k+\alpha\cdot m+B-Q}(2z_n) I_k(2z_\pi) \times [Z_{\text{c.e.}}(V,T,Q,B)]^{-1}. \tag{74}
$$

Formulas for the cross-averages $\langle N_i N_j \rangle$ can be obtained in a similar manner. The calculations with Eqs. (72)–(74) are effective for small systems. In this case the k sums in these equations converge rapidly and a small number of terms leads to accurate results. In the limit of large system volume we can use another technique, similar to that developed in the previous section. This leads to simple analytical results. Using this method one can obtain, for example, the scaled variances for (negatively) positively charged particles in the thermodynamic limit. The same pictures can be obtained directly from Eqs. (72)–(74) by numerical calculations at $z_p, z_n, z_\pi \gg 1$.

First, we consider the case when the electric charge Q is exactly conserved and the baryonic number B conservation is treated within the GCE. This results in

$$
\omega_Q^\pm = 1 - \frac{z_p^\pm + z_\pi^\pm}{z_p x_p + z_\pi x_\pi}, \tag{75}
$$

where

$$z_j^{\pm} = z_j \exp\left(\pm\frac{\mu_j}{T}\right), \quad x_j = \exp\left(\frac{\mu_j}{T}\right) + \exp\left(-\frac{\mu_j}{T}\right), \tag{76}$$

and the chemical potentials μ_j are equal to $\mu_p = \mu_Q + \mu_B$ for protons and $\mu_\pi = \mu_Q$ for π^+mesons. When both Q and B are exactly conserved, the CE scaled variances of (negatively) positively charged particles are equal to

$$\omega^{\pm}_{Q,B} = 1 - \frac{z_p^{\pm 2}(z_n x_n + z_\pi x_\pi) + z_\pi^{\pm 2}(z_p x_p + z_n x_n) + 2 z_p^{\pm} z_\pi^{\pm} z_n x_n}{\left(z_p^{\pm} + z_\pi^{\pm}\right)(z_p x_p z_n x_n + z_p x_p z_\pi x_\pi + z_n x_n z_\pi x_\pi)}, \tag{77}$$

where $x_n = \exp(\mu_B/T) + \exp(-\mu_B/T)$. Let us repeat that both $\omega_Q^{\pm}$ (75) and $\omega^{\pm}_{Q,B}$ (77) are obtained in the thermodynamic limit $V \to \infty$. $\omega_Q^{\pm}$ (75) corresponds to the CE for electric charge and the GCE for baryonic number. The $\omega^{\pm}_{Q,B}$ (77) corresponds to the CE for both conserved charges. It is easy to prove that $\omega^{\pm}_{Q,B}, \leqslant \omega^{\pm}_Q$; that is, an additional exact conservation law reduces the fluctuations. However, the additional CE suppression of the scaled variances of (negatively) positively charged particles resulting from exact baryonic number conservation is rather small. We have plotted (75) and (77) in Fig. 6 for $\mu_Q = 0$ to study the influence of baryon charge conservation on the fluctuations of electrically charged particles.

As one can see from Fig. 6 the exact CE baryonic charge conservation leads to little additional suppression and does not change the result $\omega^+ = \omega^- = 0.5$ for zero values of the baryonic and electric net charges. Moreover, one can prove that at zero net charge any ideal Boltzmann gas with two exactly conserved charges (i.e., for any combination of particle charges and their masses) leads to the scaled variances equal to $\omega^{\pm} = 0.5$ in the thermodynamic limit.

However, Fig. 6 demonstrates a strong dependence of the ω_Q^+ and ω_Q^- values on the net baryonic density, independent

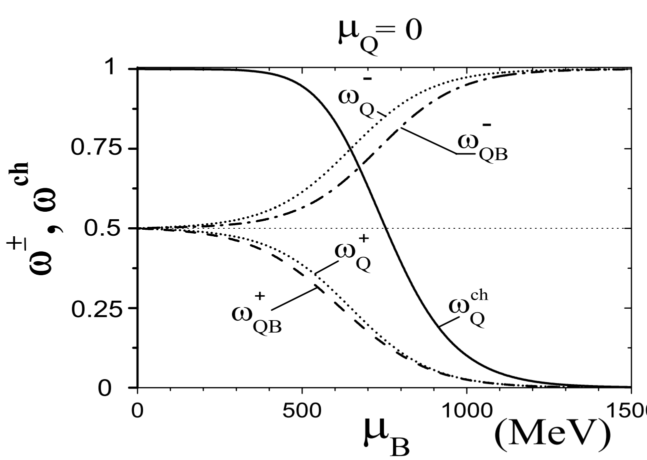

FIG. 6. The scaled variances $\omega^+_{Q,B}$ (dashed line) and $\omega^-_{Q,B}$ (dashed-dotted line) given by Eq. (77). The dotted lines show the scaled variances ω^+_Q and ω^-_Q given by Eq. (75). The solid line presents the scaled variance for all charged particles $\omega^{\rm ch}_Q$. The results correspond to $T = 120$ MeV.

of whether the baryonic number is treated within the CE or the GCE. What matters is that in the pion-nucleon gas the electric charge is $Q = N_p - N_{\bar{p}} + N_{\pi^+} - N_{\pi^-}$. At $\mu_B \simeq 0$ the electric charge of the system is close to zero. Then one finds $\omega^+_Q \simeq \omega^-_Q \simeq 0.5$ (compare to Fig. 4 at $y = 0.1$). $\mu_B > 0$ leads to $\langle N_p \rangle > \langle N_{\bar{p}} \rangle$, and this means a nonzero electric charge of the system. In this case exact electric charge conservation leads to $\omega^-_Q > \omega^+_Q$ (see Fig. 4). At $\mu_B \gg T$ the electric charge density becomes large because $\langle N_p \rangle / \langle N_{\bar{p}} \rangle \gg 1$; thus $\omega^+_Q \to 0$ and $\omega^-_Q \to 1$ (compare to Fig. 4 at $y = 2$).

VIII. SUMMARY AND CONCLUSIONS

We have considered particle number and energy fluctuations for different systems within the canonical ensemble formulation. The results are compared to those in the grand canonical ensemble. We have studied the system with an arbitrary number of different particle species and nonzero net charge in Secs. II and III.

Exact charge conservation reduces the values of N_+ and N_- fluctuations in the thermodynamic limit. At nonzero net charge Q the canonical ensemble predicts a difference for the fluctuations of N_+ and N_-; these fluctuations also differ from those of all charged particles $N_{\rm ch} = N_+ + N_-$. All these features of the canonical ensemble are in a striking contrast to those in the grand canonical ensemble. We have demonstrated in Sec. IV that the energy fluctuations of the system are mainly determined by the fluctuations of the number of particles and have the same volume dependence. Therefore, the energy fluctuations are rather different in the canonical and grand canonical ensembles. We extended our canonical ensemble results and calculated the particle number fluctuations in the system of singly and doubly charged particles in Sec. V, included the quantum statistics effects in Sec. VI, and studied systems with two conserved charges in Sec. VII.

The canonical ensemble suppression effects for the charged particle multiplicities are well known, and they are successfully applied to the statistical description of hadron production in high-energy collisions [3–8]. The canonical ensemble formulation explains, for example, the suppression in the production of strange hadrons [6] and antibaryons [7] in small systems (i.e., when the total numbers of strange particles or antibaryons are small). This consideration demonstrates a difference between the canonical and grand canonical ensembles—the statistical ensembles are not equivalent for small systems. When the size of the system increases and moves to the thermodynamic limit $V \to \infty$, all average quantities in both ensembles become equal. This means that in the thermodynamic limit the canonical ensemble and grand canonical ensemble are equivalent insofar as calculating the averaged quantities of the statistical system is concerned.

Results of Ref. [23] and considerations presented here demonstrate that there are modifications of the fluctuations caused by canonical ensemble effects. In contrast to the canonical suppression of average multiplicities, the canonical effects for the multiplicity fluctuations do survive at $V \to \infty$ and they are even most clearly seen in the thermodynamic limit. Changes in the scaled variances resulting from exact charge conservation of the canonical ensemble are not small (on the order of 50%) and they are in general different for all charged particles.

To observe these new canonical ensemble effects in an analysis of the data on multiparticle production, several points should be clarified. To use the condition of exact charge conservation, one has to apply it to the system of all secondary hadrons formed in high-energy collisions. This should be done on an event-by-event basis as we are interested in the system fluctuations. In the experimental study of $A + A$ collisions at high energies only a fraction of all produced particles with conserved charges is detected. In experiments on $A + A$ collisions at SPS CERN's: Super Proton Synchrotron (SPS) and at the Relativistic Heavy Ion Collider at the (RHIC) at Brookhaven National Laboratory (BNL) the secondary hadrons are detected in a small rapidity interval with additional restrictions on particle transverse energies and azimuthal angles. By introducing a probability $0 \leqslant q \leqslant 1$ that a single final particle with the conserved charge of interest (e.g., negatively charged hadron) is accepted in the detector, and assuming that secondary particles are uncorrelated in momentum space, a simple relation between the scaled variance of the accepted particles, $\omega^-_{\rm acc}$, and the genuine scaled variance of all particles in the statistical ensemble, ω^-, was obtained [23]: $\omega^-_{\rm acc} = q \cdot \omega^- + (1 - q)$. The limiting behavior of the scaled variance of the accepted particles is rather evident. At $q \simeq 1$ most of the particles are registered and $\omega^-_{\rm acc} \simeq \omega^-$. For a very small acceptance, $q \simeq 0$, the measured particle number distribution approaches the Poisson one, and this does not depend on the shape of the genuine distribution in full phase space. Therefore, to observe real event-by-event fluctuations ω^- one needs $q \simeq 1$; otherwise, if $q \ll 1$, one always obtains $\omega_{\rm acc} \simeq 1$ and draws the wrong conclusion that the fluctuations correspond to the Poisson distribution. This fact is of a very general origin, and because of relatively small experimental acceptance a large part of the event-by-event fluctuations in high-energy multiparticle production is lost. However, to observe many interesting event-by-event fluctuations in future experimental program at the SPS and at the RHIC, for example, to search for the QCD critical point [18], or to extract the so-called dynamical fluctuations [19] from the data, one should accept an essential part of all secondary particles (i.e., q should go to 1). In this case the role of exact charge conservation as discussed in our paper should increase. In fact, the multiplicities measured on an event-by-event basis vary not only as a result of the statistical fluctuations at freeze-out but also because of the dynamical fluctuations at an early stage. For example, for the number of negatively charged particles the total fluctuations ω^- are equal to the sum of the dynamical (early stage) fluctuations and the dynamically averaged statistical fluctuations at freeze-out: $\omega^- = \omega^-_{\rm dyn} + \omega^-_{\rm stat}$. The simplest behaviors assumed in Ref. [19] correspond to $\omega^-_{\rm stat} \simeq 1$ and $\omega^-_{\rm dyn} \simeq q\alpha\langle N^-\rangle$, where $\alpha = \text{constant} \ll 1$ and $\langle N^-\rangle$ is the total average number of negatively charged particles. At $q \ll 1$ the fluctuations of experimentally accepted particles always yield $\omega^-_{\rm acc} \simeq 1$, as previously discussed. Moreover, the average of accepted negatively charged hadrons $\langle N^-_{\rm acc}\rangle = q\langle N^-\rangle$ is not large enough and consequently $\omega^-_{\rm dyn} \ll \omega^-_{\rm stat}$, because $\alpha \ll 1$. When the acceptance increases and $q \to 1$ the experimental average $\langle N^-_{\rm acc}\rangle$ also increases, so that $\omega^-_{\rm dyn} \simeq \omega^-_{\rm stat}$ or even $\omega^-_{\rm dyn} > \omega^-_{\rm stat}$. If the statistical fluctuations do not depend on q, so that $\omega^-_{\rm stat} = \text{constant} \simeq 1$, an increase of the dynamical fluctuations will be clearly observed by an increase of the total fluctuations ω^- and, therefore, by an increase of the measured fluctuations $\omega^-_{\rm acc}$ too. At $q \simeq 1$ one finds $\omega^-_{\rm acc} \simeq \omega^-$, and the dynamical fluctuations can be then easily extracted from the data: $\omega^-_{\rm dyn} \simeq \omega^-_{\rm acc} - 1$. However, the real picture can be rather different. When the acceptance q increases from 0 to 1, the statistical fluctuations $\omega^-_{\rm stat}$ decrease from 1 to 0.5 because of CE suppression effects resulting from exact charge conservation. As a result, the expected behavior of ω^- is more complicated and extracting the dynamical fluctuations $\omega^-_{\rm dyn}$ from the data requires additional special analysis. We hope that these questions will become the subjects of future experimental and theoretical studies.

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[1] J. Cleymans and H. Satz, Z. Phys. C **57**, 135 (1993); J. Sollfrank, M. Gaździcki, U. Heinz, and J. Rafelski, *ibid.* **61**, 659 (1994); G. D. Yen, M. I. Gorenstein, W. Greiner, and S. N. Yang, Phys. Rev. C **56**, 2210 (1997); F. Becattini, M. Gaździcki, and J. Solfrank, Eur. Phys. J. C **5**, 143 (1998); G. D. Yen and M. I. Gorenstein, Phys. Rev. C **59**, 2788 (1999); P. Braun-Munzinger, I. Heppe, and J. Stachel, Phys. Lett. **B465**, 15 (1999); P. Braun-Munzinger, D. Magestro, K. Redlich, and J. Stachel, *ibid.* **518**, 41 (2001); F. Becattini, M. Gaździcki, A. Keränen, J. Manninen, and R. Stock, Phys. Rev. C. **69**, 024905 (2004).

[2] P. Braun-Munzinger, K. Redlich, and J. Stachel, nucl-th/0304013, review for *Quark Gluon Plasma 3*, edited by R. C. Hwa and X.-N. Wang (World Scientific, Singapore, 2003).

[3] R. Hagedorn, CERN Report 71-12 (1971); E. V. Shuryak, Phys. Lett. **B42**, 357 (1972).

[4] K. Redlich and L. Turko, Z. Phys. C **5**, 541 (1980); J. Rafelski and M. Danos, Phys. Lett. **B97**, 279 (1980); L. Turko, *ibid.* **B104**, 153 (1981); R. Hagedorn and K. Redlich, Z. Phys. C **27**, 541 (1985); L. Turko and J. Rafelsky, Eur. Phys. J. C **18**, 587 (2001).

[5] F. Becattini, Z. Phys. C **69**, 485 (1996); F. Becattini and U. Heinz, *ibid.* **76**, 269 (1997); F. Becattini and G. Passaleva, Eur. Phys. J. C. **23**, 551 (2002).

[6] J. Cleymans, K Redlich, and E. Suhonen, Z. Phys. C **51**, 137 (1991); J. Cleymans, K Redlich, and E Suhonen, Z. Phys. C **58**, 347 (1993); J. Cleymans, A. Keranen, M. Marais, and

E. Suhonen, Phys. Rev. C **56**, 2747 (1997); J. Cleymans, H. Oeschler, and K. Redlich, *ibid.* **59**, 1663 (1999); Phys. Lett. **B485**, 27 (2000); J. S. Hamieh, K. Redlich, and A. Tounsi, *ibid.* **B486**, 61 (2000); J. Phys. G **27**, 413 (2001); P. Braun-Munzinger, J. Cleymans, H. Oeschler, and K. Redlich, Nucl. Phys. **A697**, 902 (2002); A. Tounsi, A. Mischke, and K. Redlich, Nucl. Phys. **A715**, 565 (2003).

[7] M. I. Gorenstein, M. Gaździcki, and W. Greiner, Phys. Lett. **B483**, 60 (2000).

[8] M. I. Gorenstein, A. P. Kostyuk, H. Stöcker, and W. Greiner, Phys. Lett. **B509**, 277 (2001).

[9] K. Werner and J. Aichelin, Phys. Rev. C **52**, 1584 (1995); F. M. Liu, K. Werner, and J. Aichelin, *ibid.* **68**, 024905 (2003); F. M. Liu, K. Werner, J. Aichelin, M. Bleicher, and H. Stöcker, J. Phys. G **30**, S589 (2004); F. Becattini and L. Ferroni, Eur. Phys. J. C **35**, 243 (2004); **38**, 225 (2004).

[10] S. Pratt and J. Ruppert, Phys. Rev. C **68**, 024904 (2003).

[11] F. Gulminelli and Ph. Chomaz, Phys. Rev. E **66**, 046108 (2002).

[12] H. Heiselberg, Phys. Rep. **351**, 161 (2001); S. Jeon and V. Koch, hep-ph/0304012, review for *Quark-Gluon Plasma 3*, edited by R. C. Hwa and X.-N. Wang (World Scientific, Singapore, 2003).

[13] M. Gaździcki and St. Mrówczyński, Z. Phys. C **54**, 127 (1992).

[14] L. Stodolsky, Phys. Rev. Lett. **75**, 1044 (1995).

[15] E. V. Shuryak, Phys. Lett. **B423**, 9 (1998).

[16] St. Mrówczyński, Phys. Lett. **B430**, 9 (1998).

[17] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. D **60**, 114028 (1999).

[18] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998); M. A. Stephanov, Prog. Theor. Phys. **153**, 139 (2004).

[19] M. Gaździcki, M. I. Gorenstein, and St. Mrówczyński, Phys. Lett. **B585**, 115 (2004); M. I. Gorenstein, M. Gaździcki, and O. S. Zozulya, *ibid.* **585**, 237 (2004).

[20] S. Pratt, Phys. Rev. Lett. **84**, 4255 (2000).

[21] C. B. Das, S. Das Gupta, W. G. Lynch, A. Z. Mekjian, and M. B. Tsang, Phys. Rep. **406**, 1 (2005).

[22] S. Cheng and S. Pratt, Phys. Rev. C **67**, 044904 (2003).

[23] V. V. Begun, M. Gaździcki, M. I. Gorenstein, and O. S. Zozulya, Phys. Rev. C **70**, 034901 (2004).

[24] V. V. Begun, M. I. Gorenstein, A. P. Kostyuk, and O. S. Zozulya, nucl-th/0410044 (Phys. Rev. C, in press).

[25] M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).

[26] L. D. Landau and E. M. Lifschitz, *Statistical Physics* (Fizmatlit, Moscow, 2001).