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Boson dominance in nuclei

Fabrizio Palumbo*
INFN – Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati, Italy
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We present a new method of bosonization of fermion systems applicable when the partition function is dominated by composite bosons. By restricting the partition function to such states, we obtain a Euclidean bosonic action from which we derive the Hamiltonian. Such a procedure respects all the fermion symmetries, particularly the fermion number conservation, and provides a boson mapping of all fermion operators.

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I. INTRODUCTION

The importance of effective bosons in nuclear physics became clear after the observation that heavy deformed nuclei display some of the features of superconducting systems [1]. In these nuclei there must then be Cooper pairs of nucleons. In this line of thought, Arima and Iachello [2] introduced two different composite bosons, the s and d bosons. Their model, the interacting boson model (IBM), proved extremely successful in reproducing low energy nuclear properties, but it has not yet been derived in a fully satisfactory way from a nuclear Hamiltonian.

Many attempts were made to reformulate the nuclear Hamiltonian in terms of effective bosons before the interacting boson model was invented. Of special importance are the works of Beliaev and Zelevinsky [3], who constructed a composite boson operator requiring its commutation with the nuclear Hamiltonian, and of Marumori, Yamamura, and Togunaga [4], who developed a method based on a map of fermion into boson matrix elements.

The first important step in the derivation of the IBM respecting nucleon number conservation was completed by Otsuka, Arima, and Iachello [5]. Their work is based on a map of a single j-shell nucleon space into a boson space. The boson Hamiltonian so obtained reproduces exactly the spectrum of the pairing model. Their procedure has been somewhat extended [6] but not sufficiently generalized.

There are several recipes for bosonization [7] based on a mapping of the nucleon model space into a boson space. Such methods do not violate nucleon conservation and in principle yield an exact solution to the problem, but in practice one has to perform a truncation in the nucleon space related to a selection of degrees of freedom guided by physical insight and calculational convenience. One shortcoming of this procedure is the appearance of "intruders," namely, states which in spite of their low energy do not appear in the boson space generated by the mapping [8].

A different approach to bosonization, which does not require a preliminary truncation of the nucleon space and does not violate nucleon conservation [9,10], is based on the Hubbard-Stratonovich transformation. The latter renders quadratic the fermion interaction by introducing bosonic auxiliary fields which in the end become the physical fields. The typical resulting structure is that of chiral theories [11]. In such an approach, an energy scale emerges naturally, and only excitations of lower energy can be described by the auxiliary fields. In our opinion, this approach has not received enough attention, and its potentiality has not been fully explored.

The physical idea behind bosonization is that certain composite bosons dominate the partition function at low energy (boson dominance), an assumption certainly justified for Goldstone bosons like Cooper pairs. We present a new way to implement boson dominance. We introduce generic nucleon composites whose structure will be determined by a variational procedure, and we evaluate the partition function restricted to such composites. In this way, we obtain a Euclidean bosonic action in closed form.

In the derivation of the effective action, we need only one approximation, concerning the identity in the space of the composites, but we respect all the nucleon symmetries, in particular, nucleon number conservation. We emphasize that the closed form of the action opens the way to numerical simulations of fermion systems in terms of bosonic variables, avoiding the "sign problem" [12].

Bosonization is achieved within the path integral formalism. In this framework, the standard procedure to evaluate physical quantities is to first find the minimum of the action at constant fields. Depending on the solution, one has spherical or deformed nuclei. In the latter case, rotational excitations appear as Goldstone modes associated with the spontaneous breaking of rotational symmetry. The notion of spontaneous symmetry breaking survives in fact with a precise definition also in finite systems [10]. Next, the quantum fluctuations must be taken into account. But we anticipate a subtlety in this program which in the present case is not completely standard because of the composite nature of the bosons.

In nuclear physics, Hamiltonian formalism is of much wider application. Because the effective bosonic action, due to compositeness, is not in canonical form, it has been necessary to devise an appropriate procedure to derive the Hamiltonian. In this context, the mentioned subtlety finds a natural solution.

Only composites that have components on many nucleon states can be approximated by bosons. But the boson space cannot be arbitrarily truncated. For instance, even if the nuclear potential contains only monopole and quadrupole pairing interactions, the s and d bosons will be coupled to all the other bosons permitted by angular momentum conservation.

*E-mail: fabrizio.palumbo@lnf.infn.it

An approximate decoupling should arise dynamically; but at the moment, a clearcut mechanism is not known. We will say more about this point in our conclusions.

Bosonization appears in several many-fermion systems and relativistic field theories. The effective bosons fall into two categories, depending on their fermion number. The Cooper pairs of the BCS model of superconductivity, of the interacting boson model of nuclear physics, of the Hubbard model of high T superconductivity [13], and of the color superconductivity in QCD have fermion number 2. Similar composite bosons with fermion number zero appear as phonons, spin waves, and chiral mesons in QCD. The latter bosons can be included in the present formalism by replacing in the composites one fermion operator by an antifermion (hole) one. This becomes necessary when the interaction contains, as in nuclear physics, important particle-hole terms. This extension of the method will be presented in a future work.

A preliminary presentation of our results was given in [14].

This paper is organized as follows. In Sec. II, we outline our approach. In Sec. III, we report the bosonic effective action. In Sec. IV, we derive the bosonic Hamiltonian as the normal ordered form of a nonpolynomial function of creation-annihilation boson operators. This result includes the boson mapping of all fermion operators. For practical use, this Hamiltonian must be expanded in the inverse of the dimension of the nucleon space. In Sec. V, we report an independent derivation of the boson Hamiltonian valid for a small number of nucleons, and we specify this Hamiltonian to the case of a single j shell. In Sec. VI, we come back to the path integral formalism, introducing the Goldstone field, and in Sec. VII we end with our conclusions.

In the presentation of our results, to facilitate the understanding of the logical development, we relegate many technical details to a number of appendixes. In Appendix A, we report results concerning Berezin integrals, and in Appendix B we discuss their use in calculations with coherent states of composites. The basics of this formalism can be found in a condensed form in [15], while an exhaustive presentation is given in [16]. In Appendix C, we discuss the properties of an operator that approximates the identity in the space of the composites. In Appendix D, we show some intermediate steps in the derivation of the effective bosonic action.

II. OUTLINE OF THE APPROACH

Consider a nuclear partition function

$$Z = \mathrm{tr}\left\{\exp\left[-\frac{1}{T}(H-\mu_N\hat{n}_N)\right]\right\} \tag{1}$$

where T is the temperature, μ_N the nucleon chemical potential, and $\hat{n}_N$ the nucleon number operator. A sector of n_N nucleons can be selected by the constraint

$$T\frac{\partial}{\partial\mu_N}\ln Z = n_N. \tag{2}$$

Under the assumption of boson dominance, we can restrict the trace to nucleon bosonic composites. The restricted partition function can be written

$$Z = \mathrm{tr}\left\{\mathcal{P}\exp\left[-\frac{1}{T}(H-\mu_N\hat{n}_N)\right]\right\}, \tag{3}$$

where $\mathcal{P}$ is a projection operator in the subspace of the composites. We will only be able to construct an approximation to such an operator. This is the only approximation we will do (beyond the physical assumption of boson dominance). Similar to the case of elementary bosons, we will assume

$$\mathcal{P} = \int d\mu(\beta^*,\beta)|\beta\rangle\langle\beta|, \tag{4}$$

where $d\mu(\beta^*,\beta)$ is an integration measure (to be specified later) over the holomorfic variables β^*, β, and $|\beta\rangle$ are coherent states of composites

$$|\beta\rangle = \left|\exp\left(\sum_J \beta_J^*\hat{b}_J^\dagger\right)\right\rangle. \tag{5}$$

Definition and properties of the operator $\mathcal{P}$ are discussed in Appendix C. The $|\beta\rangle$ are defined in terms of composite creation operators

$$\hat{b}_J^\dagger = \frac{1}{2\sqrt{\Omega_J}}c^\dagger B_J^\dagger c^\dagger = \frac{1}{2\sqrt{\Omega_J}}\sum_{m_1,m_2} c_{m_1}^\dagger (B_J^\dagger)_{m_1,m_2}c_{m_2}^\dagger. \tag{6}$$

The $c^\dagger$ are nucleon creation operators, m represents all the nucleon quantum numbers, the matrices B_J are the form factors of the composites with quantum numbers J. Ω_J is the index of nilpotency of the J composite, which is defined as the largest integer such that

$$(\hat{b}_J)^{\Omega_J} \neq 0. \tag{7}$$

In the present paper, we will assume for simplicity the index of nilpotency independent of the quantum numbers of the composites and equal to half the dimension of the nucleon space, but we will mention a possible consequence of this simplification.

It is obvious that a necessary condition for a composite to resemble an elementary boson is that its index of nilpotency be large. But such condition is in general not sufficient. Consider for instance the case

$$(B^\dagger B)_{m_1m_2} = \delta_{m_1m_2}d_{m_1}, \tag{8}$$

where $d_1 = 1$, $d_m \ll 1$, $m \neq 1$. Such a composite, irrespective of its index of nilpotency, consists essentially of a unique state of a nucleon pair. We must instead require that the composites actually live in a large part of the nucleon space. This can be ensured by the further requirement

$$\det(B_J^\dagger B_J) \sim 1. \tag{9}$$

Solutions to the equations for the B matrices that do not satisfy the above condition must be discarded.

Evaluation of the trace (reported in Appendix D) gives

$$Z = \int\left[\frac{d\beta^* d\beta}{2\pi i}\right]\exp[-S_{\mathrm{eff}}(\beta^*,\beta)]. \tag{10}$$

Bosonization is thus achieved, and the nuclear dynamics can be studied by functional or numerical methods. The last possibility appears interesting because it avoids the "sign"

problem [12] which affects the Monte Carlo approach to the study of many-fermion systems.

In nuclear physics, a Hamiltonian formalism is generally used. The Hamiltonian of the effective bosons, H_B, cannot be read directly from the effective action, because $S_{\rm eff}(\beta^*, \beta)$ does not have the form of an action of elementary bosons. Indeed it contains anomalous time derivative terms, anomalous couplings of the chemical potential, and nonpolynomial interactions, which are all features of compositeness. Therefore, it has been necessary to devise an appropriate procedure to derive H_B, which also has been given in closed form, in terms of boson operators $b^\dagger, b$, satisfying canonical commutation relations. In conclusion,

$$Z = \mathrm{tr}\left[-\frac{1}{T}(H_B - \mu_B \hat{n}_B)\right], \tag{11}$$

where μ_B is the boson chemical potential and $\hat{n}_B$ the boson number operator.

For a practical use, however, it is necessary to perform an expansion of H_B. The expansion parameter is the inverse of Ω.

III. THE EFFECTIVE BOSONIC ACTION

We write the nucleon-nucleon potential as a sum of multipole pairing terms, so that the Hamiltonian has the form

$$H = c^\dagger h_0 c - \sum_K g_K \frac{1}{2} c^\dagger F_K^\dagger c^\dagger \frac{1}{2} c F_K c. \tag{12}$$

The one-body term includes the single-particle energy with matrix e, the nucleon chemical potential μ_N, and any single-particle interaction with external fields included in the matrix $\mathcal{M}$ and is written as

$$h_0 = e - \mu_N + \mathcal{M}. \tag{13}$$

The matrices F_K are the form factors of the potential, normalized according to

$$\mathrm{tr}\left(F_{K_1}^\dagger F_{K_2}\right) = 2\Omega \delta_{K_1 K_2}. \tag{14}$$

Any potential can be written in the above form [17]. But it is well known that this form is not convenient is when particle-hole terms are important. To properly account for such terms in the present scheme, it is necessary to introduce phonons, which will be done in a separate work.

To evaluate Z, we divide the inverse temperature in N_0 intervals of size τ

$$T = \frac{1}{N_0 \tau}. \tag{15}$$

Then as shown in Appendix D, the Euclidean effective action has the form

$$\begin{aligned} S_{\rm eff}(\beta^*, \beta) = N_0 \ln \mathcal{J} + \tau \sum_t \mathrm{tr} \Bigg\{ & \frac{1}{2\tau} \ln[\mathbb{1} + \tau \Gamma \Phi^\dagger \nabla_t \Phi] \\ & - \frac{1}{4} \sum_K g_K [(\Gamma \Phi^\dagger F_K^\dagger)\, \mathrm{tr}(\Gamma F_K \Phi) \\ & + 2[(\Gamma - 1) F_K^\dagger F_K] - [\Gamma \Phi^\dagger F_K^\dagger, \Gamma F_K \Phi]_+] \\ & + \frac{1}{2}[\Gamma \Phi^\dagger (\Phi h^T + h\, \Phi)] \Bigg\}, \end{aligned} \tag{16}$$

where $\mathcal{J}$ is a function appearing in the measure defining the operator $\mathcal{P}$,

$$h = h_0 - \sum_K g K F_K^\dagger F_K, \tag{17}$$

$$\nabla_t f = \frac{1}{\tau}(f_t - f_{t-1}), \tag{18}$$

$$\Phi_t = \frac{1}{\sqrt{\Omega}} \sum_J (\beta_J)_t B_J^\dagger = \frac{1}{\sqrt{\Omega}} \beta_t \cdot B^\dagger, \tag{19}$$

$$\Gamma_t = (\mathbb{1} + \Phi_t^\dagger \Phi_{t-1})^{-1}, \tag{20}$$

and $[.., ..]_+$ is an anticommutator. Notice in the second line a trace inside the trace. The variables β^*, β are always understood at times $t, t-1$ respectively. $S_{\rm eff}$ has a global $U(1)$ symmetry which implies boson conservation.

The fermion interactions with external fields are expressed in terms of the bosonic terms which involve the matrix $\mathcal{M}$ (appearing in h).

The dynamical problem of the interacting (composite) bosons can be solved within the path integral formalism. Part of the dynamical problem is the determination of the structure matrices B_J. This can be done by expressing the energies in terms of the B_J and applying a variational procedure which gives rise to an eigenvalue equation.

$S_{\rm eff}$ must be compared to the action of elementary bosons. If $H_B(b^\dagger, b)$ is the Hamiltonian of these bosons in normal form, the corresponding action is [15]

$$S_B = \tau \sum_t \{\beta^* \nabla_t \beta - H(\beta^*, \beta) + \mu_B \beta^* \beta\}, \tag{21}$$

where again the variables β^*, β are understood at times $t, t-1$, respectively. We notice that $S_{\rm eff}$ differs from S_B in many respects:

(i) There is no canonical time derivative term,
(ii) The coupling of the chemical potential (appearing in h is also noncanonical), and
(iii) There are nonpolynomial interactions because of the Γ function. This function becomes singular, as it will become clear in the sequel, when the number of bosons is of order Ω, reflecting the Pauli principle.

IV. THE BOSONIC HAMILTONIAN

Let us start by examining the features of compositeness when the number of bosons is much smaller than Ω. Since the expectation value of $\beta^* \cdot \beta$ is of the order of the number of bosons, in this case we can perform an expansion of logarithm and Γ function in inverse powers of Ω. Expanding the logarithm, we have

$$\begin{aligned} & \frac{1}{2\tau} \mathrm{tr} \ln[\mathbb{1} + \tau \Gamma \Phi^\dagger \nabla \Phi] \\ & \quad = \frac{1}{2} \mathrm{tr}(\Phi^\dagger \nabla_t \Phi) - \frac{1}{4} \mathrm{tr}[\Phi^\dagger \Phi \Phi^\dagger \nabla_t \Phi] + \cdots \end{aligned} \tag{22}$$

The first term can be made canonical by normalizing the boson form factors as the potential form factors

$$\mathrm{tr}(B_J^\dagger B_K) = 2\Omega \delta_{J,K}. \tag{23}$$

The other terms are then of order Ω^{-1}. Notice that the diagonal condition is only a matter of normalization, but the off-diagonal one must be compatible with the dynamics. If this is not the case, a redefinition of β is necessary.

Expanding the Γ function, we get the following couplings of the nucleon chemical potential:

$$\mu_N \mathrm{tr}\big[\Phi_t^\dagger \Phi_{t-1} - \tfrac{1}{2}(\Phi_t^\dagger \Phi_{t-1})^2 + \cdots\big]. \tag{24}$$

Only the first term is canonical, and μ_N is not half the boson chemical potential as one might expect. As we will see in Sec. V, for $n \ll \Omega$ these anomalous couplings can be eliminated by a redefinition of the chemical potential, so that in the case of a small number of bosons, the Hamiltonian can be derived without difficulty.

But when the number of bosons is of order Ω, the expansion of logarithm and Γ function can be performed only after an appropriate subtraction, which can be performed by the change of variables

$$\beta_K = \left(1 - \frac{1}{\Omega} r^2\right)^{-\frac{1}{2}} \beta'_K, \tag{25}$$

where r is a parameter that will be fixed later. *This subtraction corresponds to the Bogoliubov transformation in other approaches, but does not violate nucleon number conservation*. We can then rewrite the Γ function in the form

$$\Gamma = \left(1 - \frac{1}{\Omega} r^2\right) \Gamma', \tag{26}$$

where

$$\Gamma' = \left[\mathbb{1} + \frac{1}{\Omega}(\beta'^* \cdot B\ \beta' \cdot B^\dagger - r^2)\right]^{-1}. \tag{27}$$

For a suitable choice of r, Γ' admits an expansion in Ω^{-1}. Now we take the function $\mathcal{J}$ appearing in S_{eff} equal to the Jacobian of the transformation (25)

$$\mathcal{J} = \left(1 - \frac{1}{\Omega} r^2\right)^{-N_B}, \tag{28}$$

where N_B is the number of bosonic degrees of freedom.

Therefore, the partition function becomes

$$Z = \int \left[\frac{d\beta'^* d\beta'}{2\pi i}\right] \exp(-S'), \tag{29}$$

where

$$\begin{aligned} S'(\beta'^*, \beta') = \tau \sum_t \mathrm{tr} \Bigg\{ & \frac{1}{2\tau} \ln(\mathbb{1} + \tau \Gamma' \Phi'^* \nabla_t \Phi') \\ & - \frac{1}{4} \sum_K g_K \left(1 - \frac{r^2}{\Omega}\right) [\Gamma' \Phi'^* F_K^\dagger \mathrm{tr}(\Gamma' F_K \Phi') \\ & + 2(\Gamma' F_K^\dagger F_K) - [\Gamma' \Phi'^* F_K^\dagger, \Gamma' F_K \Phi']_+] \\ & + \frac{1}{2}[\Gamma' \Phi'^* (\Phi' h^T + h\, \Phi')] + \frac{1}{2} \sum_K g_K (F_K^\dagger F_K) \Bigg\}, \end{aligned} \tag{30}$$

with

$$\Phi'_t = \frac{1}{\sqrt{\Omega}} \beta'_t \cdot B^\dagger. \tag{31}$$

We assume, and we will verify *a posteriori*, that the parameter r can be chosen in such a way that the anomalous time derivative terms be of order Ω^{-1}. Then to this order, Z can be written as a trace in a boson space

$$Z = \mathrm{tr} \exp\left(-\frac{1}{T} H'\right). \tag{32}$$

The Hamiltonian H' is obtained [15] by omitting time derivative and chemical potential terms and replacing the variables β'^*, β' by corresponding creation-annihilation operators $b^\dagger$, b. These satisfy canonical commutation relations and should not be confused with the corresponding operators for the composites, which are distinguished by a hat,

$$\begin{aligned} & H'(r, \mu_N) \\ & =: \mathrm{tr} \Bigg\{ -\frac{1}{4} \sum_K g_K \left(1 - \frac{r^2}{\Omega}\right) [\Gamma_b \Phi_b^\dagger F_K^\dagger \\ & \quad \times \mathrm{tr}(\Gamma_b F_K \Phi_b) + 2\Gamma_b F_K^\dagger F_K - [\Gamma_b \Phi_b^\dagger F_K^\dagger, \Gamma_b F_K \Phi_b]_+] \\ & \quad + \frac{1}{2}[\Gamma_b \Phi_b^\dagger (\Phi_b h^T + h\, \Phi_b)] + \frac{1}{2} \sum_K g_K F_K^\dagger F_K \Bigg\} :. \end{aligned} \tag{33}$$

The colons denote normal ordering, and

$$\begin{aligned} \Phi_b &= \frac{1}{\sqrt{\Omega}} b \cdot B^\dagger, \\ \Gamma_b &= \left[\mathbb{1} + \frac{1}{\Omega}(b^\dagger \cdot B\ b \cdot B^\dagger - r^2)\right]^{-1}. \end{aligned} \tag{34}$$

Here we meet with a subtlety. H' commutes with the boson number operator, so we can select sectors with a given number of bosons. But we are not guaranteed that these bosons carry nucleon number 2, because of the noncanonical coupling of the chemical potential. We can enforce this fundamental property in the following way. Let us denote by $E'_0(n)$ the lowest eigenvalue of H' in the sector of n bosons. We require that $E'_0(n)$ be the lowest eigenvalue for $n = \frac{1}{2} n_N$

$$\frac{\partial}{\partial n} E'_0| = 0, \quad \text{for} \quad n = \frac{1}{2} n_N. \tag{35}$$

This determines r as a function of the number of bosons and the nucleon chemical potential: $\overline{r} = \overline{r}(n, \mu_N)$, ensuring that the bosons carry nucleon number 2. Condition (2) then determines the nucleon chemical potential as a function of n: $\overline{\mu}_N = \overline{\mu}_N(n)$. The boson Hamiltonian in the sector of n bosons is finally

$$H_B(n) = H'(\overline{r}, \overline{\mu}_N) + 2\overline{\mu}_N\, n. \tag{36}$$

It depends on n explicitly and through the dependence on n of $\overline{r}, \overline{\mu}_N$. Therefore also the matrices B_J will depend on n; that is, the form factors of the bosons depend on the number of the nucleons.

Notice that H' provides the mapping of the nucleon interactions with external fields

$$c^\dagger \mathcal{M} c \to :\tfrac{1}{2} \mathrm{tr}[\Gamma_b \Phi_b^\dagger (\Phi_b\, \mathcal{M}^T + \mathcal{M}\, \Phi_b)] :. \tag{37}$$

A. *s*-boson condensates

This procedure becomes particularly simple if the ground state contains only one species of bosons, which, having in mind the interacting boson model, we call *s* bosons. We will refer to such a ground state as an *s*-boson condensate. In such a case, assuming for simplicity

$$F_0^\dagger F_0 = B_0^\dagger B_0 = \mathbb{1}, \tag{38}$$

in the evaluation of the ground state energy we can set

$$\begin{aligned} \Phi_b &= \frac{1}{\sqrt{\Omega}} b_0 B_0^\dagger, \\ \Gamma_b &= \left[1 + \frac{1}{\Omega}(b_0^\dagger b_0 - r^2)\right]^{-1}. \end{aligned} \tag{39}$$

It is then convenient to adopt the following definitions

$$\begin{aligned} \hat{g}_K &= \frac{1}{4\Omega^2} |\mathrm{tr}(B_0 F_K^\dagger)|^2 g_K, \\ \hat{G} &= \sum_K \hat{g}_K, \\ G &= \sum_K g_K. \end{aligned} \tag{40}$$

Let us disregard normal ordering for a moment. This is equivalent to the semiclassical approximation in the path integral formalism, namely, to neglect quantum fluctuations. We then get for the lowest eigenvalue of H' in the n boson sector

$$\begin{aligned} E_0' = &-\left[1 + \frac{1}{\Omega}(n - r^2)\right]^{-1} \left[\Omega G \left(1 - \frac{r^2}{\Omega}\right) - 2\overline{h} n\right] \\ &+ \Omega G - \left[1 + \frac{1}{\Omega}(n - r^2)\right]^{-2} \left(1 - \frac{r^2}{\Omega}\right) \\ &\times n(\Omega \hat{G} - G), \end{aligned} \tag{41}$$

where

$$\overline{h} = \frac{1}{2\Omega} \mathrm{tr}\, h. \tag{42}$$

Condition (35) determines r as a function of n_N, μ_N

$$\overline{r}^2 = \frac{-2(\Omega + n)\overline{h} + \Omega(\Omega - 2n)\hat{G} - 2\Omega G}{\Omega(-2\overline{h} - 2G + \Omega \hat{G})}. \tag{43}$$

Inserting this value in E_0', we get

$$E_0' = -\frac{\Omega}{\Omega \hat{G} - G} \left[\overline{h} - \frac{1}{2}(\Omega \hat{G} - 2G)\right]^2. \tag{44}$$

Condition (2) then determines μ_N

$$\overline{\mu}_N = \overline{e} - \frac{1}{2}\Omega \hat{G} + n\hat{G} - \frac{n}{\Omega} G, \tag{45}$$

where

$$\overline{e} = \frac{1}{2\Omega} \mathrm{tr}\, e. \tag{46}$$

It is not surprising that with this value of $\overline{\mu}_N$,

$$\overline{r}^2 = n. \tag{47}$$

We then also get the lowest eigenvalue of H_B (neglecting normal ordering)

$$E_0 = E_0' + 2\overline{\mu}_N n = 2n\overline{e} - n\Omega \hat{G} + n^2 \left(\hat{G} - \frac{1}{\Omega} G\right). \tag{48}$$

In the case of a monopole pairing interaction, $\hat{G} = G = g_0$, comparing E_0 to the exact spectrum (65), we see that the coefficients of the powers of n are affected by errors of order Ω^{-1}. The form factor of the s boson is determined by the minimizing E_0

$$B_0 = F_0. \tag{49}$$

To get an expression of H_B of practical use, we must perform an expansion in Ω^{-1}. Since the energy scale is set by the single-particle energies, we must make an assumption concerning the magnitude of the coupling constants g_K with respect to $\overline{e}$. For a system with infinitely many degrees of freedom, $\Omega \to \infty$, in order to get finite energies we must assume $g_K \sim \Omega^{-1}$, in which case μ_N is of order Ω^0. Such a behavior is also acceptable for many nuclei.

To take into account quantum effects, we must put H' in normal order. This corresponds to including quantum fluctuations in the path integral formalism and requires an expansion with respect to Ω^{-1}. In the case where the ground state is an *s*-boson condensate, we only need the following equation:

$$\begin{aligned} \frac{1}{n!} \left\langle b_0^n \sum_{s=0}^{\infty} c_s : (b_0^\dagger b_0)^s : (b_0^\dagger)^n \right\rangle = &\sum_{s=0}^{n} c_s n(n-1) \times \cdots \\ &\times (n - s + 1). \end{aligned} \tag{50}$$

Notice that the expectation value of any normal ordered function in a state of n bosons is a polynomial of degree not greater than n.

A further simplification occurs if the number of the other bosons, which we denote by the label $\overline{K}$, is much smaller than Ω. In such a case we can obviously classify the terms appearing in the function Γ_b, Eq. (34), according to

$$\begin{aligned} &b_0^\dagger b_0 - r^2 + \sum_{\overline{K}_1, \overline{K}_2} b_{\overline{K}_1}^\dagger b_{\overline{K}_2} B_{\overline{K}_1} B_{\overline{K}_2}^\dagger \sim 1, \\ &b_0^\dagger B_0 \sum_{\overline{K}} b_{\overline{K}} B_{\overline{K}}^\dagger \sim \sqrt{\Omega}. \end{aligned} \tag{51}$$

It is then easy to see that neglecting terms of order $\Omega^{-\frac{1}{2}}$ or smaller, H_B is at most quartic in the $\overline{K}$ boson operators.

V. AN ALTERNATIVE DERIVATION OF THE BOSON HAMILTONIAN FOR $n \ll \Omega$

In this section, we restrict ourselves to the case of a small number of bosons. Then the subtraction is not necessary, we can set $\overline{r} = 0$, and we can put the effective action in canonical form by a shift of the chemical potential. It is to be emphasized that no other quantum corrections are necessary after such a shift.

For simplicity, we assume the coupling g_0 to be positive (attractive pairing force) and larger than the other ones, so that

at the minimum only β_0 is different from zero. Assuming B_0 to satisfy Eq. (38), $S_{\rm eff}$ at constant fields is

$$T\overline{S} = \Omega \sum_K gK - \Omega \left(g_0|\beta_0|^2 + \sum_K gK \right) \left(1 + \frac{1}{\Omega}|\beta_0|^2 \right)^{-2} + \frac{1}{\Omega}{\rm tr}(e - \mu_N)|\beta_0|^2 \left(1 + \frac{1}{\Omega}|\beta_0|^2 \right)^{-1}. \quad (52)$$

Its minimum with respect to $|\beta_0|^2$ can be determined exactly, but since we will perform the $1/\Omega$ expansion we put ourselves in this framework from the beginning. We will retain only the first-order corrections, which are of order Ω^0, with the exception of the coupling with external fields which are of order Ω^{-1}. We are reminded that the difficulties in the derivation of the boson Hamiltonian are due to anomalous time derivative terms and couplings of the chemical potential. In this approximation, the first difficulty is overcome because, as already noted, noncanonical time derivatives are of order $1/\Omega$. To get rid of the noncanonical couplings of the chemical potential, we set

$$\mu_N = \Omega\mu_1 + \tfrac{1}{2}\mu_B \quad (53)$$

and expand with respect to $1/\Omega$

$$T\overline{S} = -\Omega(2\mu_1 + g_0)|\beta_0|^2 + (2\overline{e} - \mu_B)|\beta_0|^2 + 2(\mu_1 + g_0)|\beta_0|^4. \quad (54)$$

Since $|\beta_0|^2 \ll \Omega$, the first term at the minimum must vanish separately from the others and we get

$$\mu_1 = -\frac{1}{2}g_0, \qquad |\beta_0|^2 = \frac{1}{g_0}\left(\frac{1}{2}\mu_B - \overline{e} \right). \quad (55)$$

We select a sector with a given number n of bosons by imposing the condition

$$\frac{\partial}{\partial\mu_B}(T\overline{S}) = n, \quad (56)$$

which yields

$$|\beta_0|^2 = n. \quad (57)$$

We see from Eq. (24) that to order Ω^0 the only noncanonical term is proportional to μ_1, which does not depend on n. We can then insert in the action the definition (53) and get a canonical bosonic action with canonical chemical potential μ_B.

There remains a last point. The energies are given by (minus) the logarithm of the partition function plus the chemical potential times the number of bosons. But we can subtract from the action the term $\Omega\mu_1\beta_t^* \cdot \beta_{t-1}$, and subtract in the end from the energy only μ_B times the number of bosons. In this way we get exactly the boson Hamiltonian H_B.

A. Few bosons in a single *j* shell

If the nucleons live in a single j-shell the form factors of the composites either vanish, or are equal to the form factors of the potential. But unless $I \ll j$ condition (9) is not satisfied, the composites do not have a high index of nilpotency and must be excluded.

We identify the quantum number K with the boson angular momentum, $K = (I_K, M_K)$, so that the form factors of the potential are proportional to Clebsh-Gordan coefficients

$$(F_{IM})_{m_1,m_2} = \sqrt{2\Omega}\langle jm_1 jm_2|IM\rangle, \quad \Omega = j + \tfrac{1}{2}, \quad (58)$$

with the conventions of [18].

The resulting action is

$$S(\beta^*, \beta) = \sum_t \left\{ \sum_{I_1 I_2} \beta^*_{I_1}[(\nabla_t - \mu_B) + \omega]_{I_1 I_2}\beta_{I_2} + \sum_{I_1 I_2 I_3 I_4} \sum_{IM} W^I_{I_1 I_2 I_3 I_4} \left(\beta^*_{I_1}\, \beta^*_{I_2}\right)_{IM} \left(\beta_{I_3}\beta_{I_4}\right)_{IM} \right\}, \quad (59)$$

where also for the $9j$ symbol we adopt the conventions of [18], and

$$\omega_{I_1 I_2} = \frac{1}{\Omega}{\rm tr}\left(F_{I_1} F^\dagger_{I_2} e\right) - g_{I_1}\Omega\delta_{I_1 I_2},$$
$$W^I_{I_1 I_2 I_3 I_4} = \left(-2g_0 + \sum_{i=1}^4 g_{I_i} \right) \Pi_{i=1}^4 [(2I_i + 1)]^{1/2} \times \Omega \begin{Bmatrix} j & j & I_1 \\ j & j & I_2 \\ I_3 & I_4 & I \end{Bmatrix},$$
$$\left(\beta_{I_3}\beta_{I_4}\right)_{IM} = \sum_{M_3, M_4} \langle I_3, M_3, I_4, M_4|I, M\rangle\, \beta_{I_3 M_3}\beta_{I_4 M_4}. \quad (60)$$

The Hamiltonian is obtained [15] by omitting the time derivative and chemical potential terms and replacing the variables β^*, β by corresponding creation-annihilation operators $b^\dagger$, b. These satisfy canonical commutation relations and should not be confused with the corresponding operators for the composites, which are distinguished by a hat,

$$H_B = \sum_{I_1 M_1 I_2 M_2} \omega_{I_1 M_1 I_2 M_2} b^\dagger_{I_1 M_1} b_{I_2 M_2} + \sum_{I_1 I_2 I_3 I_4} \sum_{IM} \left\{ W^I_{I_1 I_2 I_3 I_4} \left(b^\dagger_{I_1} b^\dagger_{I_2}\right)_{IM} \left(b_{I_3} b_{I_4}\right)_{IM} \right\}. \quad (61)$$

It is easy to check that due to the symmetries of the $9j$ symbols, H_B is Hermitian.

From the interaction with external fields, we get the fermion-boson mapping of other operators

$$c^\dagger \mathcal{M} c \to \sum_{I_1 M_1 I_2 M_2} \frac{2}{\Omega}{\rm tr}\left(F_{I_1 M_1} \mathcal{M} F^\dagger_{I_2 M_2}\right) b^\dagger_{I_1 M_1} b_{I_2 M_2} + \sum_{\text{all}\, I, M} \left(\frac{2}{\Omega} \right)^2 {\rm tr}\left(F_{I_1 M_1} \mathcal{M} F^\dagger_{I_4 M_4} F_{I_2 M_2} F^\dagger_{I_3 M_3}\right) \times b^\dagger_{I_1 M_1} b^\dagger_{I_2 M_2} b_{I_3 M_3} b_{I_4 M_4}. \quad (62)$$

Since the above Hamiltonian has been derived under the restriction $n \ll \Omega$ in a single j shell, we can assume

$$e_{m_1 m_2} = \overline{e}\delta_{m_1 m_2}, \quad (63)$$

so that the single boson energy matrix is diagonal

$$\omega_{I_1 I_2} = (2\overline{e} - g_{I_1}\Omega)\delta_{I_1 I_2}. \tag{64}$$

However, the bosonic interactions couple all the bosons with angular momenta for which the $9j$ symbols do not vanish, even if the corresponding potentials do vanish. It is important to remember, though, that the coefficients of the terms involving high angular momentum can change after the dependence of the index of nilpotency on the angular momentum is taken into account.

B. Monopole pairing

Let us consider the case of a pure monopole pairing interaction, namely, $g_I = 0$, $I \neq 0$. The exact spectrum is

$$E_{n,v} = 2n\overline{e} - g_0(\Omega+1)(n-v) + g_0(n-v)^2 + g_0(n-v)v. \tag{65}$$

We adopted definitions slightly different from the usual ones but more convenient, we think, in the present context. Here n is the total number of pairs, i.e., half the number of nucleons; and v is half the standard seniority, i.e., the number of pairs not coupled to zero angular momentum. It is then natural to relate the number of s bosons n_s to n and v according to

$$n_s = n - v. \tag{66}$$

In terms of n_s and v, the eigenvalues can be written

$$E_{n_s+v,v} = 2n\overline{e} - g_0(\Omega+1)n_s + g_0 n_s^2 + g_0 n_s(n-n_s). \tag{67}$$

Setting $b_0 = s$ and denoting by $\overline{K}$ angular momenta greater than zero, we get from Eq.(61) the bosonized pairing Hamiltonian

$$\begin{aligned} H_{\text{pairing}} = {} & [2\overline{e} - g_0(\Omega+1)]\, s^\dagger s + g_0 s^\dagger\, s\, s^\dagger s \\ & + g_0 \sum_{\overline{K}_1,\overline{K}_2,\overline{K}_3} (2\overline{K}_1+1)^{-\frac{1}{2}}[(2\overline{K}_2+1)(2\overline{K}_3+1)]^{\frac{1}{2}} \\ & \times \langle \overline{K}_2 0 \overline{K}_3 0 | \overline{K}_1 0\rangle \left[\left(s^\dagger b^\dagger_{\overline{K}_1}\right)_{\overline{K}_1} \cdot (b_{\overline{K}_2} b_{\overline{K}_3})_{\overline{K}_1} + \text{h.c.}\right] \\ & + \sum_{IM} v_2^I (d^\dagger d^\dagger)_{IM}(dd)_{IM}. \end{aligned} \tag{68}$$

The expression of v_2^I is given in the next section. The sector of zero seniority, $n = n_s$, decouples and has the exact spectrum. The study of the seniority spectrum, in which the dependence of the index of nilpotency on the angular momentum might be important, is left for a future work.

C. Monopole plus quadrupole pairing

We consider now the case in which the nuclear Hamiltonian contains a monopole pairing plus quadrupole pairing interaction. One might hope in a repetition of the pattern found in the bosonization of the pairing model, namely that the s-d bosons should decouple from the others, but this does not happen. We can write the Hamiltonian in the form

$$H = H_{s\text{-}d} + H_Q, \tag{69}$$

where H_Q contains at least one boson with angular momentum greater than 2 and H_s-d is the Hamiltonian truncated to the s-d subspace

$$\begin{aligned} H_{s\text{-}d} = {} & \omega_0 s^\dagger s + \omega_2 d^\dagger \cdot d + v_0 s^\dagger s^\dagger s s + w s^\dagger s\, d^\dagger \cdot d \\ & + \sum_I v_2^I (d^\dagger d^\dagger)_I \cdot (dd)_I + v_{02}[(d^\dagger d^\dagger)_{0,0} s s + \text{h.c.}] \\ & + \tilde{v}_{0,2}[(d^\dagger s^\dagger)_2 \cdot (dd)_2 + \text{h.c.}]. \end{aligned} \tag{70}$$

The parameters in Eq. (70) are

$$\begin{aligned} \omega_0 &= 2\overline{e} - g_0\Omega, \\ \omega_2 &= 2\overline{e} - g_2\Omega, \\ v_0 &= 2\Omega \begin{Bmatrix} j & j & 0 \\ j & j & 0 \\ 0 & 0 & 0 \end{Bmatrix} g_0 = g_0, \\ v_2^I &= 50\Omega \begin{Bmatrix} j & j & 2 \\ j & j & 2 \\ 2 & 2 & I \end{Bmatrix} (2g_2 - g_0), \\ w &= 40\Omega \begin{Bmatrix} j & j & 0 \\ j & j & 2 \\ 0 & 2 & 2 \end{Bmatrix} g_2 = 4g_2, \\ v_{02} &= 10\Omega \begin{Bmatrix} j & j & 2 \\ j & j & 2 \\ 0 & 0 & 0 \end{Bmatrix} g_2 = \sqrt{5} g_2, \\ \tilde{v}_{02} &= 10\sqrt{5}\Omega \begin{Bmatrix} j & j & 2 \\ j & j & 0 \\ 2 & 2 & 2 \end{Bmatrix} (3g_2 - g_0), \\ &\sim \sqrt{\frac{10}{7}}(3g_2 - g_0). \end{aligned} \tag{71}$$

The last approximate equality holds for $j \gg 1$. Now in order to see if we have at least an approximate decoupling of the s-d sector, we might proceed in the following way. After introducing the dependence of the index of nilpotency on the angular momentum, we should evaluate the energy of the s and d bosons first by assuming that the form factors of all the others vanish, then assuming as nonvanishing only the form factor of the boson with angular momentum 4 and so on. If these energies reach their minimum when all the other form factors vanish, we have an exact decoupling. Otherwise we can have an approximate decoupling if these energies depend little on the inclusion of higher momentum bosons. We leave this investigation for a future work [17] in which the determination of the form factors of the composites will be studied in a nucleon space including many shells.

VI. THE PATH INTEGRAL FORMALISM: THE GOLDSTONE FIELD

In this section we outline the treatment of the effective action in the standard path integral formalism. This can be helpful in performing a numerical simulation.

We must first determine the classical value of the s-boson field without breaking nucleon conservation. This can be done

by adopting the polar representation for this field

$$\beta_0 = r \exp\left(i \frac{\theta}{\overline{r}}\right), \tag{72}$$

$$r^2 = \overline{r}^2 \left(1 + \frac{\sigma}{\overline{r}}\right). \tag{73}$$

Boson number conservation is a consequence of the invariance of the action under the transformation

$$\beta_0 \to \beta_0 \exp(i\alpha). \tag{74}$$

It leaves r, and therefore σ, invariant, while

$$\theta \to \theta + \alpha. \tag{75}$$

We call [10] the Goldstone field by analogy with the Goldstone model. $\overline{r}$ is the classical field. The integration over σ in the partition function extends from $-\overline{r}$ to ∞, and one has to devise different approximations depending on the value of $\overline{r}$. All the terms in the action must be expressed by means of the polar fields. For the time derivative ones, for

$$\sum_t \beta^* \cdot \nabla_t \beta = \sum_t \left\{ +\frac{1}{8}\tau^2 (\nabla_t^{(-)}\sigma)^2 + \frac{1}{2}\tau^2 (\nabla_t^{(-)}\theta)^2 \right.$$
$$\left. + i\tau\sigma \nabla_t^{(s)}\theta + \sum_{\overline{K}} \beta_{\overline{K}}^* \cdot \nabla_t \beta_{\overline{K}} \right\}, \tag{76}$$

where $\overline{K}$ refers to all the other bosons and $\nabla_t^{(s)}$ is the symmetric time derivative. The path integral must then be evaluated as a function of $\overline{r}$ and μ_N, and these parameters are fixed by minimizing the free energy and by imposing condition (2), respectively. The evaluation of the path integral is performed by first determining the minimum of the action at constant fields, and then including the contribution of quantum fluctuations. But at the end, we must check that $2n = n_N$.

VII. SUMMARY AND OUTLOOK

We have presented a new method of bosonization in which we restrict the partition function of the nucleus to nucleon composites. We obtain in this way the Euclidean action of the effective bosons in closed form respecting all nucleon symmetries, in particular, nucleon number conservation. Indeed the presence of a large number of nucleons in the ground state is accounted for by a subtraction that does not violate nucleon number conservation. The only approximation made concerns the replacement of the identity operator in the space of the composites by the operator $\mathcal{P}$.

The nuclear dynamics can be studied by the methods of path integrals, including numerical simulations which now are not affected by the sign problem, or in the more usual Hamiltonian formalism. Since the effective boson action does not have a canonical form, it has been necessary to devise an appropriate procedure to derive the boson Hamiltonian. However, this can be put in a form useful for practical applications only by performing an expansion in the inverse of the index of nilpotency of the composites.

The formalism is consistent only if all the composite bosons involved have a high index of nilpotency. Obviously, bosons with high angular momenta do not satisfy such a condition, as shown by the example of a boson with maximum angular momentum

$$\hat{b}_{2j-1,2j-1} = \sum_{m_1 m_2} \langle j m_1 j m_2 | 2j-1, 2j-1 \rangle c_{m_1} c_{m_2}, \tag{77}$$

which has an index of nilpotency of 1. It is then necessary to ascertain that bosons with low index of nilpotency will decouple. This problem sometimes is not explicit in some approaches, because the boson space is truncated directly or as a consequence of a truncation of the nucleon space. In this connection, it is important to observe that the restriction to a definite boson space, like in the interacting boson model, does not require an exact decoupling. An approximate decoupling is sufficient, because states weakly coupled can easily be integrated out in the path integral before deriving the Hamiltonian. This fundamental feature is left for a future investigation.

Another important point is the inclusion of particle-hole terms in the nucleon-nucleon potential. This requires the introduction of fonons, which will be done by an appropriate extension of the present technique.

It is perhaps worthwhile to emphasize that for a derivation of the interacting boson model it is not at all necessary that phonons be unimportant. It is sufficient that they can be integrated out like pions in the derivation of the nucleon-nucleon potential.

In a future paper we will discuss in detail the determination of the form factors of the composites [17]. This can be done by adopting the natural parametrization

$$(B_{J,M})_{m_1,m_2} = \sum_{j_1 j_2} p^J_{j_1 j_2} \langle j_1 m_1 j_2 m_2 | J M \rangle. \tag{78}$$

The energies of the bosons are functions of the matrices p^J which can be determined by a variational calculation.

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APPENDIX A: BASIC FORMULAS FOR BEREZIN INTEGRALS

We report for the convenience of the reader some basic formulas for Berezin integrals that we need. Their definition for a single Grassmann variable is

$$\int d\gamma (a\gamma + b) = a, \tag{A1}$$

the generalization to many variables being obvious. For a change of variables

$$\gamma_i = \gamma_i(\gamma') \tag{A2}$$

in a multiple integral we have

$$\int \prod_i (d\gamma_i) f(\gamma) = \left(\det \frac{\partial \gamma_h}{\partial \gamma'_k}\right)^{-1} \int \prod_i (d\gamma'_i) f(\gamma'). \tag{A3}$$

Notice the appearance of the inverse of the Jacobian, contrary to the case of ordinary variables.

Gaussian integrals can be evaluated exactly, as for ordinary variables. There are two types of such integrals

$$\int \prod_h (d\gamma_h^* d\gamma_h) \exp \sum_{ij} \gamma_i^* M_{ij} \gamma_j = \det M, \tag{A4}$$

$$\int \prod_h (d\gamma_h) \exp \sum_{ij} \frac{1}{2} \gamma_i A_{ij} \gamma_j = \mathrm{Pf}\, A, \tag{A5}$$

where Pf A is called the Pfaffian of A. The following algebraic identity holds:

$$(\mathrm{Pf}\, A)^2 = \det A. \tag{A6}$$

APPENDIX B: INNER PRODUCTS OF COMPOSITE STATES

Let us first consider the case of only one composite. To evaluate the inner product of coherent states, we use the identity operator in the fermion Fock space

$$\mathcal{I} = \int d\gamma^* d\gamma \langle \gamma | \gamma \rangle^{-1} |\gamma\rangle\langle\gamma|, \tag{B1}$$

where γ^*, γ are Grassmann variables and $|\gamma\rangle$ are coherent nucleon states

$$|\gamma\rangle = \exp(-\gamma c^\dagger)\rangle. \tag{B2}$$

We then have

$$\langle \beta_1 | \beta \rangle = \langle \beta_1 | \mathcal{I} | \beta \rangle = \int d\gamma^* d\gamma \exp(-\gamma^* \gamma) \langle \beta_1 | \gamma \rangle \langle \gamma | \beta \rangle. \tag{B3}$$

The matrix element $\langle \beta_1 | \gamma \rangle$ can be evaluated using the defining property of coherent states

$$c|\gamma\rangle = \gamma |\gamma\rangle, \tag{B4}$$

with the result

$$\langle \beta_1 | \gamma \rangle = \exp\left(\frac{1}{2\sqrt{\Omega}} \beta_1^* \gamma B \gamma\right). \tag{B5}$$

Therefore $\langle \beta_1 | \beta \rangle$ becomes

$$\langle \beta_1 | \beta \rangle = \int d\gamma^* d\gamma\, E(\gamma^*, \gamma, \beta_1^*, \beta), \tag{B6}$$

where the function E is

$$E(\gamma^*, \gamma, \beta^*, \beta) = \exp\left(-\gamma^* \gamma + \frac{1}{2\sqrt{\Omega}} \beta^* \gamma B \gamma + \frac{1}{2\sqrt{\Omega}} \beta \gamma^* B^\dagger \gamma^*\right). \tag{B7}$$

By the change of variables,

$$\gamma' = \gamma^* - \frac{\sqrt{\Omega}}{\beta} (B^\dagger)^{-1} \gamma, \tag{B8}$$

the integral is factorized according to

$$\langle \beta_1 | \beta \rangle = \int d\gamma' \exp\left(\frac{1}{2\sqrt{\Omega}} \gamma' \beta B^\dagger \gamma'\right) \times \int d\gamma \exp\left\{\frac{1}{2} \gamma \left[\sqrt{\Omega} (\beta B^\dagger)^{-1} + \frac{1}{\sqrt{\Omega}} \beta_1^* B\right] \gamma\right\}. \tag{B9}$$

The factors are of the form (A5), so that finally

$$\langle \beta_1 | \beta \rangle = \left[\det\left(\frac{1}{\sqrt{\Omega}} \beta B^\dagger\right)\right]^{\frac{1}{2}} \left\{\det\left[\sqrt{\Omega} (\beta B^\dagger)^{-1} + \frac{1}{\sqrt{\Omega}} \beta_1^* B\right]\right\}^{\frac{1}{2}} \det\left[\mathbb{1} + \frac{1}{\sqrt{\Omega}} \beta \beta_1^* B^\dagger B\right]^{\frac{1}{2}}. \tag{B10}$$

It is perhaps worthwhile noticing that in the limit of infinite Ω, assuming the structure function to satisfy the condition (38) we recover exactly the expressions valid for elementary bosons, in particular,

$$\langle \beta_t | \beta \rangle = \left(1 + \frac{1}{\Omega} \beta_t^* \beta\right)^\Omega \to \exp(\beta_t^* \beta), \qquad \Omega \to \infty. \tag{B11}$$

We will further need the inner product

$$\langle \beta | (\hat{b}^\dagger)^n \rangle = C_n (\beta^*)^n, \tag{B12}$$

where

$$C_n = \frac{\Omega!}{(\Omega - n)! \Omega^n} = \left(1 - \frac{1}{\Omega}\right) \times \left(1 - \frac{2}{\Omega}\right) \times \cdots \times \left(1 - \frac{n-1}{\Omega}\right). \tag{B13}$$

In the general case of many composites, the above equations become

$$\langle \beta_t | \beta \rangle = \left[\det\left(\mathbb{1} + \Phi_1 \Phi\right)\right]^{\frac{1}{2}}, \tag{B14}$$

$$\langle \beta_t | \left(\hat{b}_{I_0}^\dagger\right)^{n_0} \times \cdots \times \left(\hat{b}_{I_i}^\dagger\right)^{n_i} \rangle = \frac{\partial^{n_0}}{\partial x_0^{n_0}} \times \cdots \times \frac{\partial^{n_i}}{\partial x_i^{n_i}} \times \exp\left\{\frac{1}{2} \mathrm{tr} \ln\left[\mathbb{1} + \frac{1}{\sqrt{\Omega}} (x \cdot B^\dagger) \Phi_t^\dagger\right]\right\} |_{x=0}, \tag{B15}$$

where

$$x \cdot B^\dagger = \sum_J x_J B_J^\dagger. \tag{B16}$$

APPENDIX C: THE OPERATOR $\mathcal{P}$

To implement the assumption of boson dominance, we need the projection operator in the space of the composites describing the physical degrees of freedom which dominate the

partition function. Guided by the comparison with elementary bosons, we approximate this operator by $\mathcal{P}$. This comparison suggests that the measure in $\mathcal{P}$ should be $\langle\beta|\beta\rangle^{-1}$. But, not surprisingly, we find that the measure must depend on the number of nucleons, namely, of bosons

$$d\mu(\beta^*,\beta) = J^{-1}\langle\beta|\beta\rangle^{-1}, \tag{C1}$$

where the Jacobian is given in Eq. (28).

For this reason it is necessary to introduce two types of composite bosons. We distinguish by an overbar the new boson operators and their structure matrices. These new bosons are to be regarded as the "physical" ones, while those appearing in $\mathcal{P}$ are auxiliary operators.

Let us see the action of $\mathcal{P}$ on physical states. Let us first consider the case in which there is only one physical composite with structure function satisfying the equation

$$\overline{B}^\dagger\overline{B} = \overline{\gamma}^2\mathbb{1}. \tag{C2}$$

Using the results of Appendix B we find

$$\mathcal{P}|(\overline{b}^\dagger)^n\rangle = J^{-1}(\Omega\overline{\gamma}^2)^{\frac{n}{2}}\frac{\Omega^2}{(\Omega-n)(\Omega-n-1)}|(\overline{b})^n\rangle. \tag{C3}$$

Setting $r^2 = n$, $\mathcal{P}$ is the identity on states of n bosons provided

$$\overline{\gamma}^2 = \frac{1}{\Omega}\left(\frac{\Omega-n+1}{\Omega}\right)^{\frac{2}{n}}. \tag{C4}$$

In the case of many composites, using the inequality

$$\mathrm{tr}\left(\frac{1}{\Omega}B^\dagger B\right)^n \ll \left[\mathrm{tr}\left(\frac{1}{\Omega}B^\dagger B\right)\right]^n, \tag{C5}$$

which follows from condition (9) we find again that $\mathcal{P}$ approximates the identity with an error of order $1/\Omega$, that is,

$$\begin{aligned}&\mathcal{P}\left|\left(\overline{b}^\dagger_{I_0}\right)^{n_0}\times\cdots\times\left(\overline{b}^\dagger_{I_i}\right)^{n_i}\right\rangle\\ &\quad= \left|\left(\overline{b}^\dagger_{I_0}\right)^{n_0}\times\cdots\times\left(\overline{b}^\dagger_{I_i}\right)^{n_i} + O(1/\Omega)\right\rangle.\end{aligned} \tag{C6}$$

APPENDIX D: DERIVATION OF THE EFFECTIVE ACTION

For the following manipulations, we need the Hamiltonian in the antinormal form

$$H = \frac{1}{2}\mathrm{tr}(h+h_0) - ch^Tc^\dagger - \sum_K gK\frac{1}{2}cF_Kc\frac{1}{2}c^\dagger F_K^\dagger c^\dagger, \tag{D1}$$

where the T means "transposed" and h was given in Eq. (17). Now we must evaluate the matrix element $\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle$. To this end, we expand to first order in τ (which does not give any error in the final $\tau\to 0$ limit) and insert the operator $\mathcal{P}$ between annihilation and creation operators

$$\begin{aligned}\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \exp\left[-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right]\langle\beta_t|\mathcal{P}\\ - ch^T\tau\mathcal{P}c^\dagger\sum_k g_k\tau\frac{1}{2}cF_Kc\mathcal{P}\\ \times\frac{1}{2}c^\dagger F_K^\dagger c^\dagger|\beta_{t-1}\rangle.\end{aligned} \tag{D2}$$

Using the identity in the nucleon Fock space, we find

$$\begin{aligned}\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \int d\gamma^*d\gamma\,E(\gamma^*,\gamma,\beta_t^*,\beta_{t-1})\\ \times\exp\left[-\frac{1}{2}\mathrm{tr}(h+h_0)\tau - \gamma^*h\tau\gamma\right]\\ \times\exp\left(\sum_K g_K\tau\frac{1}{2}\gamma\,F_K\,\gamma\frac{1}{2}\gamma^*F_K^\dagger\gamma^*\right),\end{aligned} \tag{D3}$$

where the function $E(\gamma^*,\gamma,\beta^*,\beta)$ is defined in (B7). By means of the Hubbard-Stratonovich transformation, we can make the exponents quadratic in the Grassmann variables and evaluate the Berezin integral

$$\begin{aligned}&\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle\\ &= \int\prod_K da_K^*da_K\exp(-a^*\cdot a)\\ &\quad\times\exp\left\{\frac{1}{2}\mathrm{tr}\ln\left[\mathbb{1}+\left(\Phi_t^\dagger+\sum_{K_1}\sqrt{g_{K_1}\tau}a^*_{K_1}F_{K_1}\right)\right.\right.\\ &\quad\left.\left.\times R^{-1}\left[\Phi_{t-1}+\sum_{K_2}\sqrt{g_{K_2}\tau}a_{K_2}\left(F_{K_2}\right)^\dagger\right](R^T)^{-1}\right]\right\}\\ &\quad\times\det R\exp\left[-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right],\end{aligned} \tag{D4}$$

where

$$R = \mathbb{1} + h\tau. \tag{D5}$$

Performing the integral over the auxiliary fields a_K^*, a_K we obtain

$$\begin{aligned}&\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle\\ &= \int\prod_K da_K^*da_K\exp(-a^*\cdot a)\\ &\quad\times\exp\left\{\frac{1}{2}\mathrm{tr}\ln\left[\mathbb{1}+\left(\Phi_t^\dagger+\sum_{K_1}\sqrt{g_{K_1}\tau}a^*_{K_1}F_{K_1}\right)\right.\right.\\ &\quad\left.\left.\times R^{-1}\left[\Phi_{t-1}+\sum_{K_2}\sqrt{g_{K_2}\tau}a_{K_2}\left(F_{K_2}\right)^\dagger\right](R^T)^{-1}\right]\right\}\\ &\quad\times\det R\exp\left[-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right].\end{aligned} \tag{D6}$$

The functional form of the composites partition function is

$$Z = \int\left[\frac{d\beta^*d\beta}{2\pi i}\right]\exp[-S_{\mathrm{eff}}(\beta^*,\beta)], \tag{D7}$$

where S_{eff} is given in Eq. (16).

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