New evaluation of the $\pi N \Sigma$ term

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A new evaluation of the $\pi N \Sigma$ term is presented that incorporates recent s-channel phase shifts and t-channel $\pi\pi$ phase shifts. We also introduce analyticity-based extrapolation techniques that, along with standard dispersion relation methods, produce a more reliable extrapolation to the Cheng-Dashen point. A recent George Washington University (GWU) phase-shift analysis leads to a Σ term of 81 ± 6 MeV.

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I. INTRODUCTION

The "experimental" $\pi N \Sigma$ term, defined by Eq. (1) in terms of the πN amplitude $D^{(+)}$, is of key importance in hadronic physics, as it is a measure of chiral symmetry breaking and, on the QCD level, the nonstrange quark masses. It is difficult to evaluate Σ in terms of experimental data because $D^{(+)}$ is required at the "Cheng-Dashen" kinematic point, which lies outside of the physical region. Dispersion relations are the preferred tools for relating unphysical amplitudes to experimental data. In a recent article [1], the Σ term was evaluated by an application of interior dispersion relations (IDRs). The present article significantly improves that analysis by incorporating t-channel analyticity and $\pi\pi$ elastic phase shifts. In addition, we employ a more recent GWU pion-nucleon phase-shift analysis [2].

We begin with a brief preview of our method for determining Σ. As will be discussed more fully in Sec. II, IDRs express πN amplitudes as the sum of a Born term (which vanishes for our particular application) and two dispersion integrals. The first dispersion integral, I_s, requires an integration of $\mathrm{Im}\, D^{(+)}$ for the process $\pi N \to \pi N$ over a path in the physical s channel. I_s may be accurately evaluated (except for a small, smooth correction arising from the truncation of the dispersion integral at high energy) by use of well-established s-channel phase-shift analyses. The second dispersion integral, I_t, requires an integration of $\mathrm{Im}\, D^{(+)}$ for the process $\pi\pi \to \overline{N}N$ evaluated over the region $t \geqslant 4$ (pion mass units). The region $4 \leqslant t < 4m^2$, where m is the proton mass, is below the physical threshold and so is not directly accessible to experiment. For small t we will evaluate I_t up to a small, smooth contribution by use of t-channel partial-wave dispersion relations. The inputs needed to evaluate these dispersion relations are t-channel partial-wave amplitudes evaluated in the region ($t \leqslant 0$), where they are projected from the same set of s-channel amplitudes as were used to evaluate I_s, and $\pi\pi$ phase shifts for $t \geqslant 4$, which are related by unitarity to the phase of the t-channel process $\pi\pi \to \overline{N}N$. Combining these pieces, we reduce the calculation to known quantities and unknown "small, smooth" corrections that may be well represented by a low-order polynomial in t and that, therefore, may be determined or finessed through the method of "discrepancy functions." This new procedure leads to a more stable extrapolation of the πN amplitude to the Cheng-Dashen point than our previous method and hence to a more reliable value of Σ.

In Sec. II, we review the IDR method of evaluating the Σ term via a discrepancy function. In Sec. III we show how the t-channel integral, I_t, may be evaluated using $\pi\pi$ phase shifts, and t-channel partial-wave amplitudes in the region $t \leqslant 0$. The method for obtaining the latter t-channel partial-wave amplitudes from s-channel phase shifts is reviewed in Sec. IV. Numerical results are found in Sec. V. Kinematics and other details are given in the Appendix.

II. REVIEW OF IDR DISCREPANCY FUNCTION METHOD

A. Interior dispersion relations

As established by Cheng and Dashen [3], the conventional "experimental" Σ term is (in mass units of $\mu = m_{\pi^+}c^2 = 1$)

$$\Sigma = \overline{D}^{(+)}(\nu = 0,\ t = 2) f_\pi^2, \tag{1}$$

where $D^{(+)} = A^{(+)} + \frac{\nu}{4m} B^{(+)}$ is the current-algebra pion-nucleon scattering amplitude, and $f_\pi \approx 0.663 \approx 92.5$ MeV is the pion decay constant. The bar implies a Born-term subtraction. The amplitude is evaluated at the unphysical, but on-shell, Cheng-Dashen (CD) point: ($\nu \equiv s - u = 0,\ t = 2$).

One way to evaluate $D^{(+)}$ at the CD point is to use a dispersion relation defined along a path in the complex (ν, t) plane that intersects this point. This may be achieved by the use of IDRs, in which the amplitude is dispersed in the variable t along a curve of fixed path variable, $a = -[su - (m^2 - 1)^2]/t$. The natural set of independent variables with which to write IDRs is $(a,\ t)$. A curve of fixed negative a passes through the interior of the s-channel physical region (with $t < 0$), through the t-channel pseudophysical region (with $4 \leqslant t \leqslant 4m^2$), and finally through the t-channel physical region (with $t \geqslant 4m^2$). The value of a for which the path intersects the CD point is $a_{\mathrm{CD}} = -m^2 + \frac{1}{2} \approx -0.871\ \mathrm{GeV}^2 = -44.7$.

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The IDR for the amplitude $D^{(+)}(a,\ t)$ is [4]

$$\begin{aligned} D^{(+)}(a,t) &= D^{(+)}(a,0) + \frac{t}{t_N} D_N^{(+)}(a,t) \\ &\quad + \frac{t}{\pi}\int_4^{\infty} \frac{\operatorname{Im} D^{(+)}(a,t')}{t'(t'-t)}dt' \\ &\quad + \frac{t}{\pi}\int_{-\infty}^{0} \frac{\operatorname{Im} D^{(+)}(a,t')}{t'(t'-t)}dt'. \end{aligned} \tag{2}$$

The subtracted form ensures convergence of the integrals. The unsubtracted Born term, $D_N^{(+)}(a,t)$, is defined by

$$D_N^{(+)}(a,t) = \frac{g^2}{4m}\frac{(t_N-2)^2}{(m^2-a)(t-t_N)}, \tag{3}$$

where $t_N \equiv t(s=m^2,a) = (4m^2-1)/(m^2-a)$, and $g^2/4\pi = 13.73$ [5] is the pseudoscalar pion-nucleon coupling constant. It is easily verified that the Born term vanishes if $a = a_{\rm CD}$.

The first integral in Eq. (2) is integrated over the cut in the t-channel physical/pseudo-physical region, and the second integral is integrated over the cut in the s-channel physical region. Changing the integration variable in the second integral to $s' = s(a,\ t')$, we may rewrite the IDR as [6]

$$\begin{aligned} D^{(+)}(a,t) &= D^{(+)}(a,0) + \frac{t}{t_N} D_N^{(+)}(a,t) \\ &\quad + \frac{t}{\pi}\int_4^{\infty} \frac{\operatorname{Im} D^{(+)}(a,t')}{t'(t'-t)}dt' \\ &\quad + \frac{t}{\pi}\int_{s_{\rm th}}^{\infty} \frac{\operatorname{Im} D^{(+)}[a,t(s',a)]}{t(s',a)} \\ &\quad \times \left(\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right) ds', \end{aligned} \tag{4}$$

where $s_{\rm th} = (1+m)^2$ is the s-channel threshold, and where it is understood that s and u are evaluated at the point corresponding to $(a,\ t)$; see the Appendix. It is also assumed throughout this paper that both t' and s' in the kernels of the integrals have small negative imaginary parts, $-i\epsilon$. Consequently, Re $D^{(+)}$ results from use of the principal-value prescription. It also follows that $D^{(+)}$ is real in the interval, $0 \leqslant t \leqslant 4$, between the s- and t-channel branch points.

From the dispersion relation, Eq. (4), it is seen that if the pion-nucleon coupling constant, the imaginary part of the amplitude on the cuts, and the isospin-even scattering length $a^{(+)}$ [$\equiv \frac{1}{3}(2a_3 + a_1) = \frac{1}{4\pi(1+1/m)} D^{(+)}(a,0)$] are known, then the dispersion relation gives the real part of the amplitude everywhere on the path defined by the chosen value of a. However, the imaginary parts are not known for the entire integration regions, making it necessary to use an approximation scheme, to which we now turn.

B. Discrepancy function

The discrepancy-function method is based on the assumption that it is possible to find known quantities that, when subtracted from the amplitude, leave a smooth remainder that may be fit by a low-order polynomial in (and presumably somewhat beyond) the region in which the amplitude is known. The quantities to be subtracted may be Born terms, dispersion integrals, etc. that account for the amplitude's other than smooth behavior. The amplitude is then reconstituted from the low-order polynomial and the other quantities and provides a good approximation in all regions in which the quantities are known and accounts for all the amplitude's nonsmooth behavior.

To make this idea more precise, assume that the imaginary part of the amplitude is known accurately for $s_{\rm th} < s < s_m$, and define the discrepancy function as

$$\begin{aligned} d(a,\ t) &= D^{(+)}(a,\ t) - \frac{t}{t_N} D_N^{(+)}(a,\ t) \\ &\quad - \frac{t}{\pi}\int_4^{\infty} \frac{\operatorname{Im} D^{(+)}(a,t')}{t'(t'-t)}dt' \\ &\quad - \frac{t}{\pi}\int_{s_{\rm th}}^{s_m} \frac{\operatorname{Im} D^{(+)}[a,t(s',a)]}{t(s',a)} \\ &\quad \times \left(\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right) ds' \\ &\equiv D^{(+)}(a,\ t) - \frac{t}{t_N} D_N^{(+)}(a,\ t) - I_t(a,\ t) - I_s(a,\ t). \end{aligned} \tag{5}$$

By defining the discrepancy function in this manner, the property of IDRs that all a curves pass through the s-channel threshold point ($s = s_{\rm th}$, $t = 0$) ensures that $d(a,0) = D^{(+)}(a,0) = 4\pi(1+1/m)\,a^{(+)}$ and is independent of the path parameter a. The s-channel integral, I_s, may be evaluated directly from extant πN phase-shift analyses. The t-channel integral, I_t, will be evaluated, up to a low-order polynomial, by an indirect method in Sec. IV. Under the assumption that both integrals may be evaluated, the procedure follows three steps: (1) For fixed path parameter a (equal to $a_{\rm CD}$ in this application), the discrepancy function $d(a,\ t)$ is evaluated over an interval in t in the physical s-channel region for which $D^{(+)}(a,t)$ is accurately known from the same partial-wave analysis. (2) The discrepancy function is then fit to a low-order polynomial in t whose range of validity includes the desired kinematic point. (3) Equation (5) is rearranged to read

$$D^{(+)}(a,\ t) = d(a,\ t) + \frac{t}{t_N} D_N^{(+)}(a,\ t) + I_t(a,\ t) + I_s(a,\ t), \tag{6}$$

and the variable t is set equal to the desired value. Assuming the path parameter were chosen to be $a_{\rm CD}$, we simply set $t = 2$ to obtain $D^{(+)}(a_{\rm CD},\ 2)$. The Σ term is then obtained from Eq. (1).

In our previous article, the term $-I_t(a,\ t)$ was omitted in the definition of the discrepancy function [Eq. (5)], so that the branch cut at $t = 4$ was not removed. The discrepancy function was then fit not with a polynomial in t, which lacked the branch point, but with a polynomial in the the t-channel $\pi\pi$ momentum $q = \sqrt{t/4 - 1}$. The present method is superior because it, to a large extent, removes the t-channel threshold branch cut, leaving a smoother discrepancy. We now turn to the evaluation of $I_t(a,\ t)$.

III. THE t-CHANNEL INTEGRAL

In this and in the next section, we use dispersion relations for the t-channel partial-wave amplitudes to write $I_t(a,t)$ in terms of known quantities (s-channel πN amplitudes and $\pi\pi$ partial waves) plus a rapidly convergent series in t that can be incorporated into the discrepancy function for $D^{(+)}$.

The t-channel partial-wave expansion for the amplitude $D^{(+)}$ is [7,8]

$$
\begin{aligned}
D^{(+)} &= -\frac{4\pi}{p^2}\sum_{j\ \mathrm{even}}(2j+1)(pq)^j \\
&\quad\times\left[f_+^j(t)P_j(z_t) - \frac{t}{4m}\frac{z_t}{[j(j+1)]^{\frac{1}{2}}}f_-^j(t)P_j'(z_t)\right] \\
&\equiv \sum_{j\ \mathrm{even}} f_j(a,t), \qquad (7)
\end{aligned}
$$

where $p \equiv \sqrt{t/4-m^2}$, $q=\sqrt{t/4-1}$, and $z_t = \cos\theta_t$. In this expression $f_\pm^j(t)$ are t-channel partial-wave amplitudes, and we define $f_-^0(t) \equiv 0$. The corresponding Born terms, $f_{\pm N}^j(t)$, from which $f_{j,N}$ may be constructed, are given in Ref. [9]. The Born-term free partial-wave amplitudes are defined by $\tilde{f}_j \equiv f_j - f_{j,N}$. We call $f_j(a,t)$ the jth partial-wave contribution to $D^{(+)}$ [not to be confused with the πN t-channel partial-wave amplitude, $f_\pm^j(t)$] and note that, although $f_j(a,t)$ is a function of both a and t, a is held constant, and its dependence will be formally suppressed in the rest of this paper. We will also suppress the a dependence of the other related functions such as $D^{(+)}(a,t)$ and $I_t(a,t)$.

Recalling its definition, Eq. (5), we have

$$
I_t(t) = \sum_{j\ \mathrm{even}}\frac{t}{\pi}\int_4^\infty \frac{\mathrm{Im}\, f_j(t')}{t'(t'-t)}dt'. \qquad (8)
$$

It does not matter whether f_j or $\tilde{f}_j$ is used in the integrand because $\mathrm{Im}\, f_{j,N}(t \geqslant 4) = 0$. Partial-wave dispersion relations will now be used to evaluate this integral.

The amplitudes f_j for all j are finite at $p=0$ [10]. Owing to the asymptotic behavior of f_+^j and f_-^j [11], the f_j satisfy the dispersion relation [12]

$$
\begin{aligned}
\tilde{f}_j(t) &= \tilde{f}_j(0) + \frac{t}{\pi}\int_4^\infty \frac{\mathrm{Im}\,\tilde{f}_j(t')}{t'(t'-t)}dt' \\
&\quad + \frac{t}{\pi}\int_{t_M}^0 \frac{\mathrm{Im}\,\tilde{f}_j(t')}{t'(t'-t)}dt' + \mathcal{P}_j(t), \qquad (9)
\end{aligned}
$$

where

$$
\mathcal{P}_j(t) = \frac{t}{\pi}\int_{-\infty}^{t_M}\frac{\mathrm{Im}\,\tilde{f}_j(t')}{t'(t'-t)}dt'
$$

is expected to be a rapidly convergent series in t for sufficiently small $|t/t_M|$. We have used the fact that $f_{j,N}$ satisfies the same dispersion relation as does f_j with $\mathrm{Im}\, f_{j,N}(t \geqslant 4) = 0$.

Because higher partial-waves are thought to be less important in the low-t region, the partial-wave expansion used to approximate $D^{(+)}$ will be truncated at $j=J$, and, in practice, $J=0$ is adequate to give a smooth, nearly linear discrepancy function, $d(t)$. Since $I_t(t)$ will be used in a discrepancy function calculation, it need only be defined up to a rapidly convergent series in t. Thus, without loss of generality, we redefine I_t as

$$
I_t(t) = \sum_{j=0,2,\ldots}^{J}\left[\tilde{f}_j(t) - \tilde{f}_j(0) - \frac{t}{\pi}\int_{t_M}^0 \frac{\mathrm{Im}\,\tilde{f}_j(t')}{t'(t'-t)}dt'\right], \qquad (10)
$$

as can be seen by comparison with Eqs. (5), (7), and (9). The amplitudes $\tilde{f}_j(t)$ will be evaluated in the next section for the range $t_M < t \leqslant 0$. This is sufficient information to evaluate $I_t(t)$ for negative values of t.

In reconstructing the amplitude at the CD point, however, we will also need $I_t(2)$. Although we may easily evaluate the integral in Eq. (10) for any value of t including $t=2$, $\tilde{f}_j(2)$ is not available. In the spirit of the discrepancy method, one might be tempted to think that for each j the expression in the square brackets could be evaluated at negative values of t and analytically continued to positive values. However, because of the importance and closeness of the $t \geqslant 4$ cut [see Eq. (8)] such continuations are not well represented by a low-order polynomial.

Fortunately, owing to extended unitarity, the phase of f_j is equal to that of elastic $\pi\pi$ scattering up to $t=16$ [13]. This phase can be used to define a related amplitude, $O^j(t)f_j(t)$, whose continuation is more reliable because its cut does not begin until $t=16$. The Omnes function, O^j, is defined as [14]

$$
O^j(t) = \exp\left[-\frac{t}{\pi}\int_4^\infty \frac{\delta^j(t')}{t'(t'-t-i\varepsilon)}dt'\right], \qquad (11)
$$

where we set δ^j equal to the $\pi\pi$ elastic phase shift [15] from $t=4$ to $t_c = 1\ \mathrm{GeV}^2 \approx 50.3$. For $t > t_c$, we take $\delta^j = \delta^j(t_c)\cdot t_c/t$ so that $O^j(t)$ is finite at asymptotic values of t. The final results are insensitive to this choice of asymptotic behavior. For example, if t_c is increased to $1.4\ \mathrm{GeV}^2$, the calculated value of the Σ term is altered by only a small fraction of an MeV.

For $t<4$ the Omnes function is real, whereas for $t \geqslant 4$ its phase is $-\delta^j$. Therefore in the interval $4 \leqslant t \leqslant 16$, extended unitarity implies that the phase of the Omnes function cancels that of f_j, and so in this interval $\mathrm{Im}\,(O^j f_j)$ is zero if the $\pi\pi$ phase shifts are exact, and it is expected to be small if they are only approximately correct. In practice, it is believed that $\mathrm{Im}\,(O^j f_j)$ is small up to $t \approx 50$ [13].

Since our parametrization of δ^j vanishes at large t, $O^j\tilde{f}_j$ satisfies Eq. (9), the same dispersion relation as does $\tilde{f}_j$. If in the dispersion relation for $O^j\tilde{f}_j$, we write $\mathrm{Im}\,(O^j\tilde{f}_j) = \mathrm{Im}\,[O^j(f_j - f_{j,N})]$ in the $t \geqslant 4$ integral, then the unknown contribution to the integral comes from $\mathrm{Im}\,(O^j f_j)$, which does not contribute significantly until $t \geqslant 16$ in contrast to the $t \geqslant 4$ cut contribution of $\mathrm{Im}\,\tilde{f}_j$ in Eq. (8). $\Delta^j(t)$ is defined to be the sum of the unknown portions of the (subtracted) dispersion integrals:

$$
\begin{aligned}
\Delta^j(t) &= \frac{t}{\pi}\int_4^\infty \frac{\mathrm{Im}\,(f_j O^j)}{t'(t'-t)}dt' + \frac{t}{\pi}\int_{-\infty}^{t_M}\frac{O^j \mathrm{Im}\,\tilde{f}_j}{t'(t'-t)}dt' \\
&= O^j\tilde{f}_j(t) - \tilde{f}_j(0) - \frac{t}{\pi}\int_{t_M}^0 \frac{O^j\mathrm{Im}\,\tilde{f}_j}{t'(t'-t)}dt' \\
&\quad + \frac{t}{\pi}\int_4^\infty \frac{f_{j,N}\mathrm{Im}\, O^j}{t'(t'-t)}dt'. \qquad (12)
\end{aligned}
$$

The second line, which may be interpreted as a discrepancy function, follows from an application of the dispersion relation [Eq. (9)]. As with Eq. (10) it may be evaluated in terms of known amplitudes for $t < 0$. From the first line of Eq. (12) and the fact that $\mathrm{Im}\,(O^j f_j) \approx 0$ below $t = 16$, we expect the discrepancy function to be well represented by a low-order polynomial out to $t = 4$, and probably beyond. We have chosen $t_M = -70$, below which we are not able to evaluate the partial-wave projection $\tilde{f}_j(t)$ owing to the proximity of the double spectral region.

Having obtained the $t \leqslant 0$ values for Δ^j, one then extrapolates to $t > 0$ and reconstructs the amplitude from Eq. (12):

$$\tilde{f}_j(t) = \frac{1}{O^j(t)}\left[\Delta^j_{\mathrm{extrap}}(t) + \tilde{f}_j(0) + \frac{t}{\pi}\int_{t_M}^{0}\frac{O^j \mathrm{Im}\,\tilde{f}_j}{t'(t'-t)}dt' - \frac{t}{\pi}\int_4^\infty \frac{f_{j,N}\mathrm{Im}\,O^j}{t'(t'-t)}dt'\right], \tag{13}$$

where $\Delta^j_{\mathrm{extrap}}(t)$ is obtained from a polynomial fit of the $\Delta^j(t)$ values for $t < 0$ [16]. Substituting this result into our expression for I_t gives

$$I_t(t) = \sum_{j=0,2,\ldots}^{J} \frac{1}{O^j(t)}\left[\Delta^j_{\mathrm{extrap}}(t) + \tilde{f}_j(0)[1 - O^j(t)] + \frac{t}{\pi}\int_{t_M}^{0}\frac{[O^j(t') - O^j(t)]\mathrm{Im}\,\tilde{f}_j}{t'(t'-t)}dt' - \frac{t}{\pi}\int_4^\infty \frac{f_{j,N}\mathrm{Im}\,O^j}{t'(t'-t)}dt'\right]. \tag{14}$$

This expression for the t-channel contribution will be used to generate values of the discrepancy function $d(a, t)$ for $t < 0$ which will then be extrapolated to $t > 0$, where it, the s-channel contribution, I_s, and the Born term contribution, D_N, will be combined to give the full amplitude, $D^{(+)}(a_{\mathrm{CD}}, t_{\mathrm{CD}})$, and hence Σ. The only quantity on the right-hand side of the previous equation that remains to be calculated is $\tilde{f}_j$ for $t_M < t \leqslant 0$, to which we now turn.

IV. THE *t*-CHANNEL PARTIAL-WAVE AMPLITUDES

The real and imaginary parts of $f^j_\pm(t)$ are given, respectively, by integrals over the real and imaginary parts of the invariant amplitudes. (See Refs. [7] and [8].) In particular,

$$f^j_+(t) = -\frac{1}{8\pi}\frac{p^2}{(pq)^{j-1}}C_j(t) \tag{15}$$

and

$$f^j_-(t) = \frac{1}{8\pi}\frac{[j(j+1)]^{\frac{1}{2}}}{(2j+1)}\frac{1}{(pq)^{j-1}}[B_{j-1}(t) - B_{j+1}(t)], \tag{16}$$

where the partial-wave projections are defined by

$$B^{(\pm)}_j(t) = \int_{-1}^{+1} B^{(\pm)}(t,\ z_t)P_j(z_t)\,dz_t \tag{17}$$

with a similar expression for $C^{(\pm)}_j(t)$. The integrals are over the range $-1 \leqslant z_t \leqslant 1$, which for negative t lies in the unphysical region between the physical s and u channels where z_t is, respectively, -1 and $+1$. To evaluate $f^j_\pm(t)$ for even values of j, which are the only ones needed in this paper, the appropriate integrands are $B^{(+)}/\nu$ and $C^{(+)}$ multiplied by even functions of ν. For even j, the symmetry of the invariant amplitudes and $P_j(z_t)$ allows us to replace these integrals by twice the same integral taken from $z_t = -1$ to $z_t = 0$, that is, from $\nu = 0$ to $\nu_s(t) = -4pq = \sqrt{(4m^2 - t)(4 - t)}$, where the physical s channel begins. Referring to $B^{(+)}/\nu$ and $C^{(+)}$ by the generic amplitude F, we need to evaluate $\mathrm{Re}\,F(\nu, t)$ and $\mathrm{Im}\,F(\nu, t)$ from $\nu = 0$ to $\nu_s(t)$. Concentrating first on $\mathrm{Im}\,F(\nu, t)$, we recognize that both u and s cuts may contribute, that is,

$$\mathrm{Im}\,F(\nu, t) = \mathrm{disc}_\nu F(\nu, t) = \mathrm{disc}_s F(\nu, t) + \mathrm{disc}_u F(\nu, t), \tag{18}$$

where $\mathrm{disc}_x F$ is the discontinuity across the x cut. We have at our disposal the s-channel partial-wave expansion for F, and within the Lehmann z_s ellipse, $\mathrm{disc}_s F$ is given by the imaginary part of the s-channel partial-wave expansion for F. We therefore use the s-u crossing properties of $B^{(+)}/\nu$ and $C^{(+)}$ to write everything in terms of $\mathrm{disc}_s F$:

$$\mathrm{Im}\,F(\nu, t) = \mathrm{disc}_\nu F(\nu, t) = \mathrm{disc}_s F(\nu, t) + \mathrm{disc}_s F(-\nu, t). \tag{19}$$

This means that if we want to evaluate $\mathrm{Im}\,F(\nu, t)$ in the projection integral over the interval $-1 \leqslant z_t \leqslant 0$ by means of the imaginary part of an s-channel partial-wave expansion, then it must be evaluated at $s = (2m^2 + 2 - t + \nu)/2$ and at $s = (2m^2 + 2 - t - \nu)/2$. Consequently, the s-channel partial-wave expansion is evaluated at every point along the s cut from threshold to where $z_t = -1$, that is, $s_{-1} = [2m^2 + 2 - t + \sqrt{(4m^2 - t)(4 - t)}]/2$, and the $-1 \leqslant z_t \leqslant 0$ projection integral is done by integrating the imaginary part of an s-channel partial-wave expansion with the appropriate factors from threshold to s_{-1}.

Since $\mathrm{Re}\,F$ cannot be evaluated using partial-wave expansions over the whole region $0 \leqslant \nu \leqslant \nu_s$, we take advantage of its fixed-t dispersion relation. Again because the region over which partial-wave expansions are known is finite, a discrepancy method is used to accommodate the high-energy contribution. We define a discrepancy function

$$d_\nu(\nu, t) = F(\nu,\ t) - F_N(\nu,\ t) - \frac{2}{\pi}\int_{\nu_0}^{\nu_m}\frac{\nu'\mathrm{Im}\,F(\nu',\ t)}{\nu'^2 - \nu^2}\,d\nu', \tag{20}$$

where $F_N(\nu,\ t)$ is the fixed-t Born term, the lower limit of integration is the minimum of 0 and $\nu(t, s_{\mathrm{th}}) = t + 4m$, and the upper limit of the integration $\nu_m = \nu(t, s_m)$, where s_m is the largest value of s where partial-wave expansions are available. If we change the integration over ν to an integration over s and use this expression for $\mathrm{Im}\,F(t, \nu)$ in terms of the imaginary parts of s-channel partial-wave expansions, d_ν can be written as

$$d_\nu(\nu, t) = F(\nu,\ t) - F_N(\nu,\ t) - \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_m} \mathrm{Im}\,F[s',\ z_s(t, s')] \times \left[\frac{1}{s' - s(\nu,\ t)} + \frac{1}{s' - u(\nu,\ t)}\right]ds', \tag{21}$$

where $\mathrm{Im}\, F(s, z_s)$ is the imaginary part of the s-channel partial-wave expansion of F. When the s and u cuts overlap *both* terms in the square brackets contribute according to the $i\varepsilon$ prescription.

In practice, the values for the discrepancy functions in the physical region are very flat and can be easily fit to a low-order polynomial in ν^2 and extrapolated to the region $0 \leqslant \nu \leqslant \nu_s$. Then Re F can be obtained from

$$\begin{aligned} F(\nu, t) = {} & d_{\nu,\mathrm{extrap}}(\nu, t) + F_N(\nu, t) \\ & + \frac{1}{\pi} \int_{s_{\mathrm{th}}}^{s_m} \mathrm{Im}\, F[s', z_s(t, s')] \\ & \times \left[\frac{1}{s' - s(\nu, t)} + \frac{1}{s' - u(\nu, t)} \right] ds'. \quad (22) \end{aligned}$$

It appears to be possible to use the projection expressions to evaluate $f^j_\pm(t)$ for values of t as low as -70.

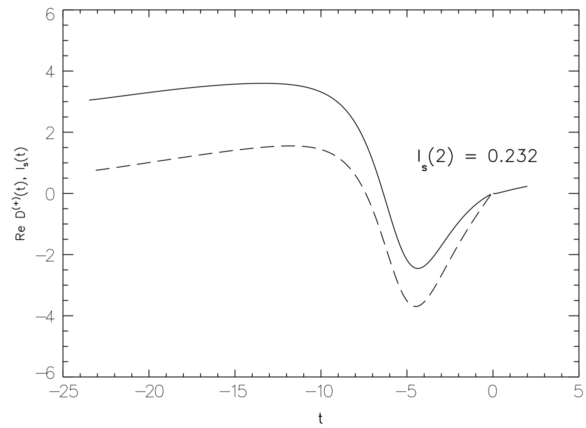

FIG. 1. Dispersion integral $I_s(t)$ as defined in Eq. (5), evaluated by use of GWU phase-shift analysis FA02. The amplitude Re $D^{(+)}(t)$ (dashed line) is also shown for comparison. All figures are displayed in pion mass units. Alternatively, t, D, I, and $\tilde{f}^0_+$ may be replaced by the dimensionless quantities t/μ^2, μD, μI, and $\tilde{f}^0_+/\mu$.

V. RESULTS

In this section, we will present the numerical details leading to the evaluation of the experimental Σ term. We begin by evaluating the discrepancy function:

$$d(t) = \mathrm{Re}\, D^{(+)}(t) - \frac{t}{t_N} D_N^{(+)}(t) - I_t(t) - I_s(t), \qquad (23)$$

where the last two terms are t- and s-channel dispersion integrals defined in Eq. (5). All calculations in this section have $a = a_{\mathrm{CD}}$. Our first goal is to determine $d(t)$ from experimental data for t in the interval $(-25, 0)$. We discuss in turn each contribution to the right-hand side of the definition.

$\mathrm{Re}\,[D^{(+)}(a_{\mathrm{CD}}, t)]$ was evaluated in the interval $-25 < t < 0$ by summing the s-channel partial-wave contributions determined from the GWU FA02 phase-shift solution. As mentioned earlier, the Born term is zero for $a = a_{\mathrm{CD}}$, and it is included in the discrepancy formula in case the reader wishes to extend our procedure to other values of a.

The principal value integral $I_s(t)$ was evaluated by numerical integration directly from its definition in Eq. (5). The imaginary part of $D^{(+)}(a_{\mathrm{CD}}, t')$, needed in the integrand, was also obtained from the FA02 phase-shift solution. The upper limit of the integral is the value of s that corresponds to the upper limit of validity of the FA02 partial-wave solution ($s_M = 4.8\ \mathrm{GeV}^2 \approx 250$ in pion units). Because t occurs only in the dispersion kernel, $I_s(t)$ may be evaluated for any desired value of t. Figure 1 displays $I_s(t)$ for t in the range $(-25, 2)$. [The oscillation in $I_s(t)$ is a result of the low-lying s-channel P33 resonance.] The value $I_s(2) \approx 0.232$ was used in Eq. (6) to aid in the evaluation of $D^{(+)}(a_{\mathrm{CD}}, 2)$ (i.e., at the Cheng-Dashen point). For comparison, Re $D^{(+)}(t)$ is also displayed in the range $(-25, 0)$.

To evaluate $I_t(t)$, we first determined $\Delta^j(t)$ as defined in Eq. (12). As discussed in Sec. IV, the t-channel partial-wave amplitudes $f_j(t)$ for $t < 0$ may be computed from the projection formulas [Eqs. (15)–(17)]. The imaginary parts of the invariant amplitudes within the required region $t < 0$ and $-1 < z_t < +1$ were obtained from the s-channel partial-wave series and the FA02 phase shifts. For the real parts of the invariant amplitudes, fixed-t dispersion relations were used. The final results for the $j = 0$ case are displayed in Fig. 2. The partial-wave amplitudes are very similar to those given in Table 2.4.6 of Ref. [7]. (There is a slight shift in the values of the $j = 0$ waves, which may be due to the different values of the pion-nucleon coupling constant as used in Ref. [7] ($f^2_{\mathrm{pv}} = 0.079$) and the current FA02 analysis ($f^2_{\mathrm{pv}} = 0.076$). The t-channel partial-wave amplitudes with higher j are almost identical to those in the table in [7].)

Using the values of $f_j(t)$ determined for $t \leqslant 0$, we used Eq. (12) to determine the partial-wave discrepancy function $\Delta^j(t)$. The resulting function and two simple fits (a polynomial and a Pade) are given for $j = 0$ in Fig. 3. Fits to cubic, quartic, and quintic polynomials gave mutually consistent results to within about 3%. The Pade fit used a quadratic in t divided by a linear function in t. The extrapolations to the point $t = 2$ are also shown on the figure. It was found that to obtain a

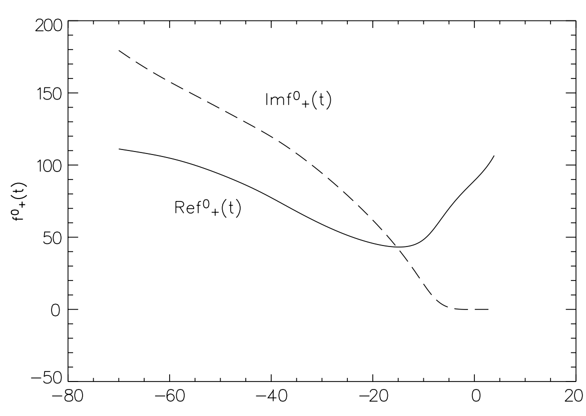

FIG. 2. t-channel partial-wave amplitudes Re $\tilde{f}^0_+(t)$ and Im $\tilde{f}^0_+(t)$ projected from invariant amplitudes evaluated by the use of the phase-shift analysis FA02. The Born term has been subtracted.

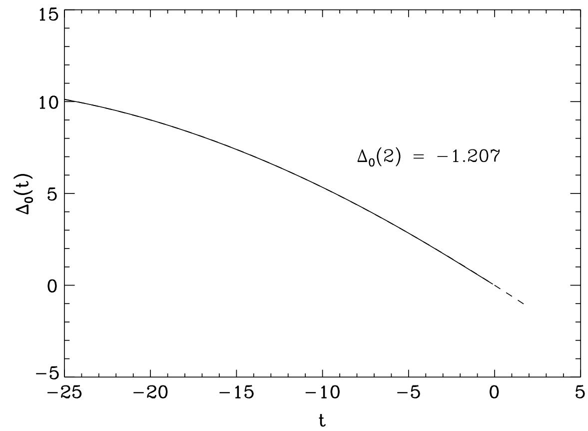

FIG. 3. $\Delta^j(t)$ as determined by Eq. (13) and the partial-wave amplitudes of Fig. 2 for the region $t < 0$. The extrapolation to $t = 2$ is performed by a quartic fit as described in the text. The dashed portion represents the extrapolation.

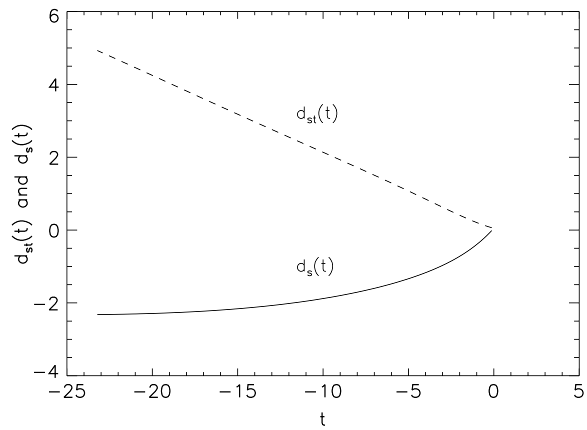

FIG. 5. Comparison of the discrepancy functions $d_s(t)$ and $d_{st}(t)$. Note that the latter is much more linear, indicating that it may be more reliably extrapolated.

smooth discrepancy function it was sufficient to include only the $j = 0$ contribution to $I_t(t)$. The resulting function $I_t(t)$, reconstructed from Eq. (14), is displayed in Fig. 4.

Figure 5 compares the discrepancy function $d_{st}(t) \equiv d(t)$ as defined in Eq. (5) with the "old fashioned" discrepancy function $d_s(t)$ for which only $I_s(t)$ has been subtracted. That is, $d_{st}(t) = d_s(t) - I_t(t)$. On comparison with Fig. 1, it is seen that the large resonance structures in $I_s(t)$ and $D^{(+)}(t)$ cancel, leading to a much smoother function. Furthermore, it is seen that the further subtraction of $I_t(t)$ yields a yet smoother function $d_{st}(t)$, one that may be expected to extrapolate more reliably than $d_s(t)$.

To illustrate this point, we have extrapolated $d_s(t)$ (see Fig. 6) by fitting it to several polynomials in t of different order. It is seen that high-order polynomials (fourth through sixth order) are required to give a respectable fit to $d_s(t)$ in the region $(-25 < t < 0)$. The extrapolations of these polynomials to $t = 2$ give a rather wide range of values, which lead to a large uncertainty in the corresponding Σ term (the values of which are shown in parentheses on the figure). This contrasts with the extrapolation of the improved discrepancy function $d_{st}(t)$ as shown in Fig 7. The resulting linear and quadratic extrapolations are much more consistent, yielding Σ terms within 1 MeV. To assign meaningful uncertainties to this value is difficult. From variations in the Σ term observed when different polynomial orders are used in extrapolating Δ_j and d_{st}, when the cutoff parameter in the Omnes function is varied, and when we vary the fitting range of the discrepancy function, we estimate $\Sigma \approx 81 \pm 6$ MeV. The "error bar" is simply our estimate of the extrapolation uncertainty and does not include any uncertainty inherent in the phase-shift analysis. Thus we again see that the recent GWU phase-shift analysis implies a significantly larger Σ term than was previously obtained (as with, for example, the KH80 phase shifts). This is in agreement with our earlier calculation [1] and also with the recent calculation of Pavan *et al.* [17].

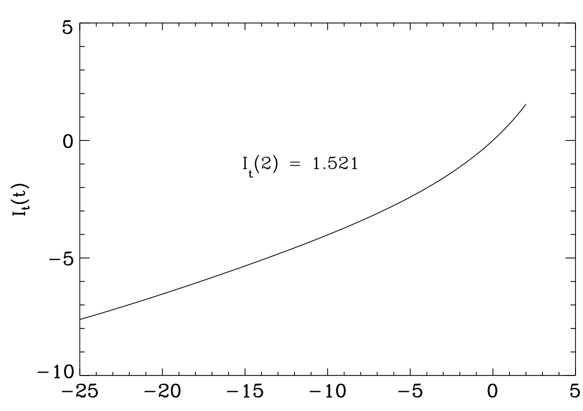

FIG. 4. The reconstructed function $I_t(t)$ using Eq. (14) and the results of the previous two figures.

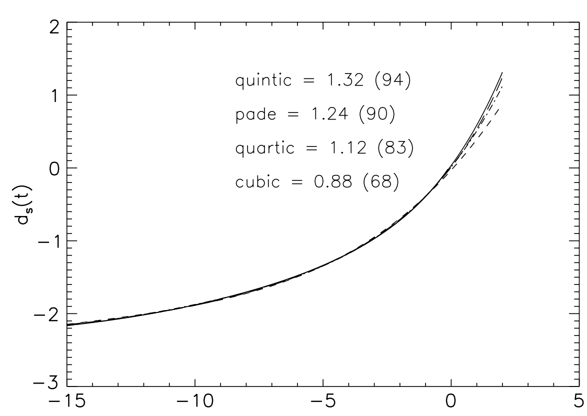

FIG. 6. Extrapolation of $d_s(t)$ to obtain the Σ term, whose value in MeV is given in parenthesis. d_s is fit to a polynomial in the region $t \leqslant 0$ and is then extrapolated to $t > 0$. Note that high-order polynomials are required to fit the function and that the extrapolated values are rather sensitive to the order of the polynomial.

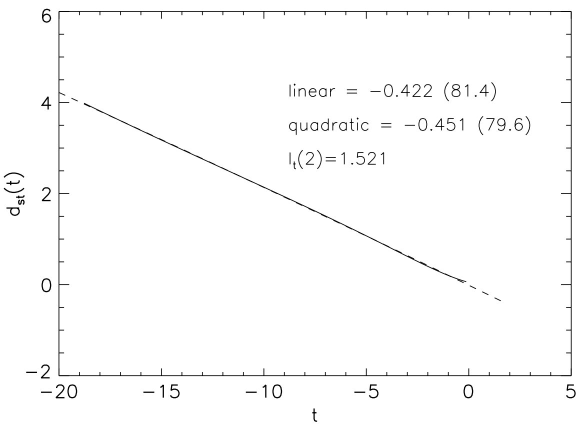

FIG. 7. Extrapolation of the improved discrepancy function $d_{st}(t)$ to obtain the Σ term, whose value in MeV is given in parenthesis. Because of the near linearity of $d_{st}(t)$ both linear and quadratic fits are adequate and give consistent results.

ACKNOWLEDGMENTS

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APPENDIX: KINEMATICS

We use the conventional Mandelstam variables (s, t, u), which are related by $s + t + u = 2m^2 + 2\mu^2$, where m and μ are the nucleon and charged pion masses, respectively. From here on and in the body of this paper we set $\mu = 1$. In addition, we define the IDR "path" variable as

$$a = -[su - (m^2 - 1)^2]/t. \tag{A.1}$$

In terms of a, the s-channel physical boundaries are $a = -\infty$ $(t = 0)$ for forward scattering and $a = 0$ for backward scattering. For fixed negative a, the interval $t \in (0, -\infty)$ defines a path of fixed laboratory angle

$$\cos\theta_L = -\frac{a + m^2 - 1}{[a^2 - 2a(m^2 + 1) + (m^2 - 1)]^{\frac{1}{2}}}. \tag{A.2}$$

The c.m. angle, given by

$$\cos\theta_C = \frac{a + s}{a - s}, \tag{A.3}$$

varies over the path of fixed a. Any pair of the four variables (s, t, u, a) may be taken as independent. Fixed-t dispersion relations are customarily written with $\nu \equiv s - u$ as the second variable. The pair ν, t is convenient because the invariant amplitudes are, at fixed t, even or odd under the exchange of s and u (i.e., $\nu \to -\nu$.) For fixed-a dispersion relations (or IDR), the natural choice of independent variables is a and t. With this choice, ν is given by

$$\nu(a, t) = \sqrt{(t - 4m^2)(t - 4) + 4at} \tag{A.4}$$

and s and u become

$$s(a, t) = m^2 + 1 - \frac{t}{2} + \frac{\nu(a, t)}{2} \tag{A.5}$$

and

$$u(a, t) = m^2 + 1 - \frac{t}{2} - \frac{\nu(a, t)}{2}. \tag{A.6}$$

Finally, in evaluating the t-channel contribution to the discrepancy function, it is natural to adopt t, the square of the t-channel c.m. energy, and z_t, the cosine of the t-channel c.m. scattering angle, as independent variables. z_t is related to the previous variables by

$$z_t = \cos\theta_t = \frac{\nu}{4pq}, \tag{A.7}$$

where $p^2 = t/4 - m^2$ and $q^2 = t/4 - 1$ are the squares of the $N\bar{N}$ and $\pi\pi$ t-channel c.m. momenta.

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[2] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky, R. L. Workman, and M. M. Pavan, Phys. Rev. C **69**, 035213 (2004). The GWU-FA02 phase shift analysis is available at the Center for Nuclear Studies Web site: gwdac.phys.gwu.edu. Choose "pion-nucleon" and "current solution."
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[4] Throughout this paper whenever a dispersion relation is written for a quantity, it will be understood that only the real part of the quantity is being evaluated and that all integrals are principal-value integrals.
[5] The pion-nucleon coupling constant, $g^2_{\pi N}/4\pi = 13.73$, is consistent with the GWU phase shift analysis [2].
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[7] G. Höhler, in *πN Scattering: Phenomenological Analysis,* edited by H. Schopper, Landolt-Börnstein, New Series, Group 1, Vol. 9b Pt. 2 (Springer, Berlin, 1983). We will make particular use of Sec. 2.4.6.3, p. 268. This reference is referred to as LB in the text. It should be noted that this reference uses $\nu \equiv (s - u)/4m$ instead of our choice $\nu = s - u$.
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[9] G. Höhler, in Ref. [7], A7.37, which defines $f^j_{\pm N}(t)$.
[10] G. Höhler, in Ref. [7], A3.75, which relates $f^j_+(4m^2)$ and $f^j_-(4m^2)$.
[11] G. Höhler, in Ref. [7], A.7.48, which gives the asymptotic form of $f^j_\pm(t)$.
[12] G. Höhler, in Ref. [7], A.7.47. In deriving Eq. (9) we note the following: (1) $f_j(t)$ has branch cuts $(-\infty, b)$ and $(4, \infty)$, where

$b = 4 - 1/m^2$, (2) $\text{Im}\, f_{j,N}(t) = \text{Im}\, f_j(t)$ if $0 < t < b$, and (3) $\text{Im}\, f_{j,N}(t) = 0$ if $t > b$.

[13] To see this, note that below the four-pion threshhold, extended unitarity reads $\text{Im}\, f^{I,J}_{\pi\pi\to N\bar{N}} = \rho^J(p_{\pi\pi}) f^{I,J}_{\pi\pi\to\pi\pi}{}^* f^{I,J}_{\pi\pi\to N\bar{N}}$ in this interval, where $\rho^J(p_{\pi\pi})$ is the two-body $\pi\pi$ phase space. Since the left side is real, so is the right, and consequently the phases of $f^{I,J}_{\pi\pi\to\pi\pi}$ and $f^{I,J}_{\pi\pi\to N\bar{N}}$ are equal. Above $t = 16$, the 4π intermediate state also contributes, and the phases of the $\pi\pi \to \pi\pi$ and $\pi\pi \to N\bar{N}$ amplitudes may become different. Nonetheless, because $2\pi \to 4\pi$ is weak, it is believed that $\delta_J(t)$ is well approximated by the $\pi\pi$ phase shift that is much higher in energy, perhaps as high as $t \approx 50$.

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