Critical point symmetry in a fermion monopole and quadrupole pairing model

Joseph N. Ginocchio
MS B283, Los Alamos National Laboratory, Los Alamos, New Mexico 87545 USA

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Recent interest in symmetries at a critical point of phase transitions in nuclei prompts a revisit to the fermion monopole and quadrupole pairing model. This model has an exactly solvable symmetry limit that is transitional between spherical nuclei and γ-unstable deformed nuclei. The eigenenergies, eigenfunctions, pairing strength, and quadrupole transtion rates in this limit are derived. Comparison with empirical quadrupole transition rates suggest that the Xe isotopes may have this symmetry.

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I. INTRODUCTION

Nuclei can undergo phase transitions associated with a change of shape of their equilibrium configuration as a function of neutron number. A second-order shape phase transition between spherical and deformed γ-unstable (axially symmetric) nuclei has been suggested [1] to exist for which empirical examples have been found in ^{134}Ba [2,3] and possibly in ^{104}Ru [4], ^{102}Pd [5], ^{108}Pd [6], and ^{128}Xe [7]. This phase transition has been studied within the interacting boson model (IBM) [8].

The SO(8) fermion monopole and quadrupole pairing model [9] has an SO(7) dynamical symmetry. This dynamical symmetry limit has been shown to describe the phase transition between the spherical limit and the γ-unstable limit of the model [10,11], with the nucleon number being the control parameter. In fact, the energy surface in the SO(7) symmetry limit, obtained by the method of coherent states [10,11], is independent of the deformation angle γ and exhibits a flat-bottomed behavior in the deformation value β, particularly at midshell, which resembles the infinite-square-well potential used in the geometric approach to this phase transition [1]. An example of this energy surface versus the deformation for different values of the number of pairs of nulceons, N, for $\Omega = 16$, where 2Ω is the number of single-nucleon valence states, is given in Fig. 1.

The SO(8) fermion monopole and quadrupole pairing model has two other dynamical symmetry limits, one relating to monopole pairing [SU(2) ⊗ SO(5)] and one to a γ-unstable rotor [SO(6)]. The detailed eigenfunctions, eigenfunctions, and quadrupole transition rates have been published for these limits [9]. In this paper we derive the detailed eigenfunctions, eigenfunctions, and quadrupole transition rates for the SO(7) limit in light of the recent interest in phase transitions in nuclei.

II. THE SO(8) FERMION MONOPOLE AND QUADRUPOLE PAIRING MODEL

In the SO(8) fermion monopole and quadrupole pairing model, fermions interact in a shell model space with orbitals that have angular momentum $j = k+\frac{3}{2}, k+\frac{1}{2}, k-\frac{1}{2}, k-\frac{3}{2}$, where k is the integer pseudo-angular momentum. The total number of single-particle states is $2\Omega = 4(2k+1)$. Within the entire shell model space, there is a subspace composed solely of collective monopole and quadrupole pairs, and there exists shell model Hamiltonians that do not connect this subspace with the rest of the shell model space. The collective pairs are given by [9]

$$S^\dagger = \frac{1}{2}\Sigma_{j,m}(-1)^{j-m}\, a^\dagger_{j,m}a^\dagger_{j,-m}, \tag{1}$$

$$D^\dagger_\mu=\Sigma_{j,j'}(-1)^{j+k+\frac{3}{2}}\sqrt{(2j+1)(2j'+1)}\begin{Bmatrix} j & j' & 2\\ \frac{3}{2} & \frac{3}{2} & k\end{Bmatrix}[a^\dagger_j a^\dagger_{j'}]^{(2)}_\mu, \tag{2}$$

where $a^\dagger_{j,m}$ creates a nucleon in the orbital angular momentum j and projection m, $[a^\dagger_j a^\dagger_{j'}]^{(J)}_\mu$ creates a pair of nucleons coupled to angular momentum J and projection μ, and

$$\begin{Bmatrix} \ell & \ell' & L\\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}$$

is the 6-j symbol. The multipole operators of rank $\ell = 0, 1, 2, 3$ are

$$P^{(\ell)}_\mu = \Sigma_{j,j'}(-1)^{\ell+j+k+\frac{3}{2}}\sqrt{(2j+1)(2j'+1)} \times \begin{Bmatrix} j & j' & \ell\\ \frac{3}{2} & \frac{3}{2} & k\end{Bmatrix}[a^\dagger_j\tilde{a}_{j'}]^{(\ell)}_\mu, \tag{3}$$

where $\tilde{a}_{j,m} = (-1)^{j+m}a_{j,-m}$ [12].

The pair creation and destruction operators and multipole operators form an SO(8) algebra:

$$[S, S^\dagger] = \Omega - 2\hat{N}, \tag{4}$$

$$[\tilde{D}, D^\dagger]^{(L)}_\mu = (\Omega - 2\hat{N})\sqrt{5}\delta_{L,0} + 2\begin{Bmatrix} 2 & 2 & L\\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}P^{(L)}_\mu, \tag{5}$$

$$[D^\dagger_\mu, S] = 2P^{(2)}_\mu, \tag{6}$$

$$\left[P^{(\ell)}_\mu, S^\dagger\right] = \delta_{\ell,2}D^\dagger_\mu + \delta_{\ell,0}\delta_{\mu,0}S^\dagger, \tag{7}$$

$$[P^{(\ell)}, D^\dagger]^{(L)}_\mu = -2\sqrt{5(2\ell+1)}\delta_{L,2}\begin{Bmatrix} 2 & 2 & \ell\\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}D^\dagger_\mu + \sqrt{5}\delta_{\ell,2}\,\delta_{L,0}S^\dagger, \tag{8}$$

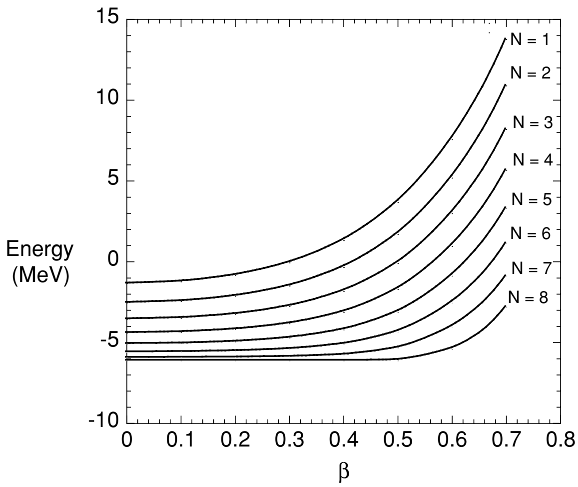

FIG. 1. The SO(7) energy surface vs the deformation for different values of the number of pairs of nucleons, N, for $\Omega = 16$.

$$[P^{(\ell)}, P^{(\ell')}]^{(L)}_{\mu} = (-1)^{\ell+\ell'}[1-(-1)^{\ell+\ell'+L}]\sqrt{(2\ell+1)(2\ell'+1)} \times \left\{ \begin{matrix} \ell & \ell' & L \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{matrix} \right\} P^{(L)}_{\mu}, \quad (9)$$

where $[P^{(\ell)}, P^{(\ell')}]^{(L)}_{\mu}$ is the commutator coupled to angular momemtum rank L,

$$[P^{(\ell)}, P^{(\ell')}]^{(L)}_{\mu} = [P^{(\ell)}P^{(\ell')}]^{(L)}_{\mu} - (-1)^{\ell+\ell'+L}[P^{(\ell')}P^{(\ell)}]^{(L)}_{\mu}, \quad (10)$$

and $\hat{N}$ counts the number of pairs and is related to the monopole multipole operator, $\hat{N} = \frac{1}{2}\Sigma_{j,m}a^{\dagger}_{jm}a_{jm} = P^{(0)}_0$. There are three dynamical subgroup chains that conserve the angular momentum [SO(3)] generated by the angular momenta operators $J_{\mu} = \sqrt{5}\, P^{(1)}_{\mu}$:

$$SO(8) \supset SO(5) \otimes SU(2) \supset SO(3) \otimes SU(2) \text{ (the vibrational limit)}, \quad (11)$$

$$SO(8) \supset SO(6) \otimes U(1) \supset SO(5) \otimes U(1) \supset SO(3) \otimes U(1) \text{ (the } \gamma\text{-unstable rotor limit)}, \quad (12)$$

$$SO(8) \supset SO(7) \supset SO(5) \otimes U(1) \supset SO(3) \otimes U(1) \text{ (the transitional limit)}. \quad (13)$$

The SO(5) group is generated by the multipole operators $P^{(L)}_{\mu}$, $L = 1, 3$, and is conserved for γ-unstable nuclei. The SU(2) group is the monopole pair subgroup generated by $S, S^{\dagger}, \frac{\Omega}{2} - \hat{N}$. The SO(6) group is generated by the multipole operators $P^{(L)}_{\mu}$, $L = 1, 2, 3$, whereas the $U(1)$ group is generated by $C_1 = \frac{\Omega}{2} - \hat{N}$. Finally, the SO(7) group is generated by the quadrupole pairs $D_{\mu}, D^{\dagger}_{\mu}$ and the generators of the SO(5) and U(1) groups.

The SO(5) subgroup is common to all three of these dynamical symmetries of the SO(8) model and therefore the nuclei described by these dynamical symmetries will be γ unstable [11].

The eigenstates, energy spectrum, and transition matrix elements for the SO(5) ⊗ SU(2) and the SO(6) ⊗ U(1) dynamical symmetry limits have been derived [9]. In this paper we derive the eigenstates, energy spectrum, and transition matrix elements for the SO(7) dynamical symmetry limit.

III. SO(7) DYNAMICAL SYMMETRY

A. Quantum numbers

The space of monopole and quadrupole pairs is in one representation of SO(8), the symmetric irreducible representation (IR), $(\frac{\Omega}{2}, 0, 0, 0)$. The allowed irreducible representations of SO(7) in this space are also symmetric, $(\lambda, 0, 0)$, where $\lambda = 0, 1, \ldots, \frac{\Omega}{2}$. The allowed values of N, the total number of pairs, for a given λ are $N = \frac{\Omega}{2} - \lambda, \frac{\Omega}{2} - \lambda + 1, \ldots, \frac{\Omega}{2} + \lambda$. This means that the IR $\lambda = \frac{\Omega}{2}$ occurs for all the number of pairs $N = 0, 1, \ldots, \Omega$, $\lambda = \frac{\Omega}{2} - 1$ occurs for the number of pairs $N = 1, 2, \ldots, \Omega - 1$, etc., and thus $\lambda = 0$ occurs only at midshell, $N = \frac{\Omega}{2}$.

The additional quantum numbers of the dynamical symmetry chain given in Eq. (13) are the SO(5) quantum numbers τ and n_{Δ} and the SO(3) quantum numbers J and M, the total angular momentum and its projection. The quantum number τ is the number of quadrupole pairs not coupled to zero. The allowed values of τ are $\tau = N - (\frac{\Omega}{2} - \lambda), N - (\frac{\Omega}{2} - \lambda) - 2, \ldots, 0$, or 1 for $N \leqslant \frac{\Omega}{2}$ and $\frac{\Omega}{2} - N + \lambda, \frac{\Omega}{2} - N + \lambda - 2, \ldots, 0$ or 1 for $N \geqslant \frac{\Omega}{2}$. The quantum number n_{Δ} is the number of quadrupole triplets not coupled to zero. The allowed values of n_{Δ} are $n_{\Delta} = 0, 1, \ldots, [\frac{\tau}{3}]$, where [x] is the largest integer less than or equal to x. Finally, the allowed values of the total angular momentum and its projection are $J = \tau - 3n_{\Delta}, \tau - 3n_{\Delta} + 1, \ldots, 2(\tau - 3n_{\Delta}) - 2, 2(\tau - 3n_{\Delta})$ and $M = -J, -J + 1, \ldots, J$.

A schematic view of the quantum numbers as a function of the number of pairs, N, is given in Fig. 2. On the bottom row of Fig. 2 the number of pairs is plotted. The states belonging to the same IR of SO(7) and lableled by λ are connected by a straight line. On the vertical axis is plotted the allowed value of τ, so that, in general, each horizontal line is a multiplet of states. Since monopole pairing is attractive, the lowest state for a given N is the one with the lowest λ as we shall see. As N increases, the λ of the lowest state decreases from its maximum value, $\lambda = \frac{\Omega}{2}$, at $N = 0$ to $\lambda = 0$ at the half-filled shell, $N = \frac{\Omega}{2}$. Beyond the half-filled shell the value of the lowest λ begins increasing until it becomes the highest value allowed, $\lambda = \frac{\Omega}{2}$, for the filled shell, $N = \Omega$. Hence there is a symmetry in the pattern about the half–filled shell.

The fact that the ground state belongs to a different IR of SO(7) differs from the monopole pairing limit [SO(5) ⊗ SU(2)] for which the ground state belongs to the same IR for all N but is similar to the γ-unstable rotor limit [SO(6)] for which the ground state belongs to a different IR for each N.

B. Casimir operators

The Casimir operator for the SO(8) group is

$$C_8 = S^{\dagger}S + P^{(2)} \cdot P^{(2)} + C_7 + C_1, \quad (14)$$

where the scalar product is $P^{(2)} \cdot P^{(2)} = \Sigma_{\mu}(-1)^{\mu} P^{2}_{\mu} P^{2}{}_{-\mu}$.

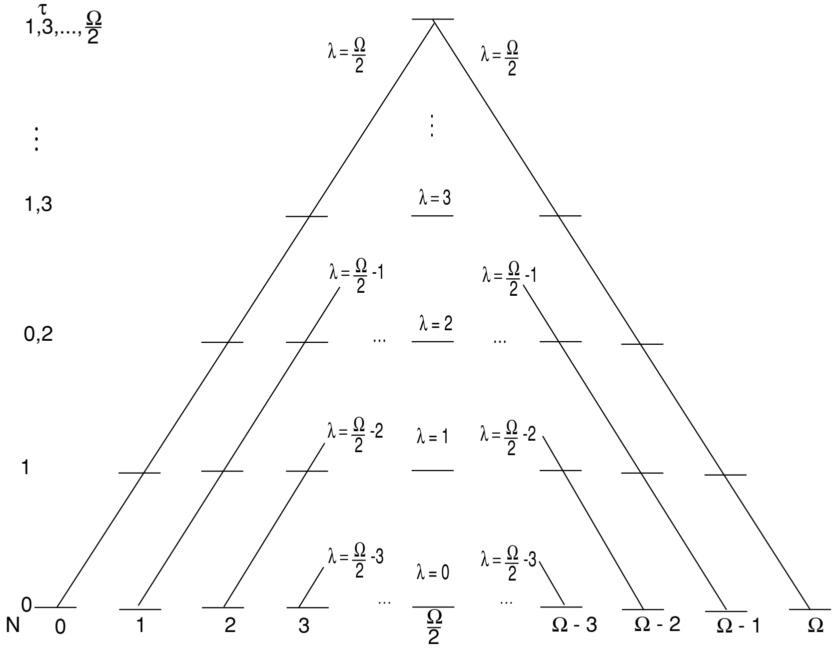

FIG. 2. A schematic view of the allowed quantum numbers as a function of the number of pairs of nucleons, N. See text for a detailed explanation.

The Casimir operator for the SO(7) group is

$$C_7 = D^\dagger \cdot \tilde{D} + C_5 + \left(\frac{\Omega}{2} - \hat{N}\right)\left(\frac{\Omega}{2} - \hat{N} + 5\right), \quad (15)$$

where the SO(5) Casimir operator is

$$C_5 = \Sigma_{\ell=1,3} P^{(\ell)} \cdot P^{(\ell)}. \quad (16)$$

The Casimir operator for SO(3) is $C_3 = 5\ P^{(1)} \cdot P^{(1)} = \hat{J} \cdot \hat{J}$ and the Casimir operator for U(1) is $C_1 = \frac{\Omega}{2} - \hat{N}$.

The most general Hamiltonian in this dynamical symmetry chain is

$$H_7 = c_8 C_8 + c_7 C_7 + c_5 C_5 + c_3 C_3 + c_1 C_1, \quad (17)$$

which can be rewritten as

$$H_7 = G_0(S^\dagger S + P^{(2)} \cdot P^{(2)}) + G_2\left[D^\dagger \cdot \tilde{D} + \left(\frac{\Omega}{2} - \hat{N}\right)^2\right] + b_3 P^{(3)} \cdot P^{(3)} + b_1 \hat{J} \cdot \hat{J} + b_0\left(\frac{\Omega}{2} - \hat{N}\right), \quad (18)$$

where $G_0 = c_8$, $G_2 = c_8 + c_7$, $b_3 = c_8 + c_7 + c_5$, $b_1 = (c_8 + c_7 + c_5)/5 + c_3$, and $b_0 = 5c_7 + 6c_8 + c_1$. Thus the SO(7) limit corresponds to equal strength for the monopole pairing and quadrupole interaction and arbitrary strength for the quadrupole pairing. For realistic Hamiltonians the monopole pairing is attractive so we expect $G_0 < 0$.

C. Energy eigenvalues

The eigenvalues of the Casimir operators are

$$\langle N, \lambda, \tau, n_\Delta, J, M | C_8 | N, \lambda, \tau, n_\Delta, J, M\rangle = \frac{\Omega}{2}\left(\frac{\Omega}{2} + 6\right), \quad (19)$$

$$\langle N, \lambda, \tau, n_\Delta, J, M | C_7 | N, \lambda, \tau, n_\Delta, J, M\rangle = \lambda(\lambda + 5), \quad (20)$$

$$\langle N, \lambda, \tau, n_\Delta, J, M | C_5 | N, \lambda, \tau, n_\Delta, J, M\rangle = \tau(\tau + 3), \quad (21)$$

$$\langle N, \lambda, \tau, n_\Delta, J, M | C_3 | N, \lambda, \tau, n_\Delta, J, M\rangle = J(J + 1), \quad (22)$$

$$\langle N, \lambda, \tau, n_\Delta, J, M | C_1 | N, \lambda, \tau, n_\Delta, J, M\rangle = \frac{\Omega}{2} - N, \quad (23)$$

which gives for the energy eigenvalues of the Hamiltonian,

$$E_7(N, \lambda, \tau, n_\Delta, J) = c_8 \frac{\Omega}{2}\left(\frac{\Omega}{2} + 6\right) + c_7\lambda(\lambda + 5) + c_5\tau(\tau + 3) + c_3 J(J + 1) + c_1\left(\frac{\Omega}{2} - N\right). \quad (24)$$

For $N = 0$ the allowed value of λ is $\lambda = \frac{\Omega}{2}$ whereas for $N = 1$ the allowed values of λ are $\lambda = \frac{\Omega}{2}, \frac{\Omega}{2} - 1$. The $\lambda = \frac{\Omega}{2}$ state with $N = 1$ is created by operating on the $N = 0$ state with the generator of SO(7), $D^\dagger_\mu|0\rangle$, where $|0\rangle$ is the vacuum state with no valence nucleons. Thus the $\lambda = \frac{\Omega}{2} - 1$ state is $S^\dagger|0\rangle$, which should be the lowest state to agree with experiment. This state has $\tau = J = 0$. Therefore, for it to be the lowest in energy, c_7 must be positive. This means that, in general, the states with the smallest λ will be the lowest in energy. For $N \leqslant \frac{\Omega}{2}$, the smallest λ is $\lambda_0 = \frac{\Omega}{2} - N$ whereas for $N \geqslant \frac{\Omega}{2}$, $\lambda_0 = N - \frac{\Omega}{2}$; that is,

$$\lambda_0 = \frac{\Omega}{2} - N, \quad N \leqslant \frac{\Omega}{2} \quad (25)$$

$$\lambda_0 = N - \frac{\Omega}{2}, \quad N \geqslant \frac{\Omega}{2}, \quad (26)$$

as illustrated in Fig. 2.

The excitation energy in a given nucleus is $E^*_7(N, \lambda, \tau, n_\Delta, J) = E_7(N, \lambda, \tau, n_\Delta, J) - E_7(N, \lambda_0, \tau = 0,$

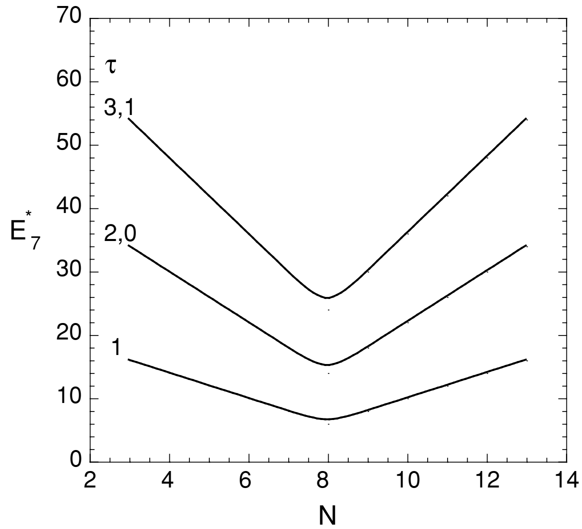

FIG. 3. The excitation energy $E_7^*(N, \lambda, \tau, n_\Delta, J)$ for $\tau = 0, 1, 2, 3$ as a function of N with $\Omega = 16$.

$n_\Delta = 0, J = 0$), and so

$$E_7^*(N, \lambda, \tau, n_\Delta, J) = c_7(\lambda - \lambda_0)(\lambda + \lambda_0 + 5) + c_5\tau(\tau + 3) + c_3 J(J + 1). \quad (27)$$

For $N \geqslant \frac{\Omega}{2}$, $\lambda_0 = N - \frac{\Omega}{2} = \frac{\Omega}{2} - \bar{N}$, where $\bar{N} = \Omega - N$ is the number of hole pairs. Thus, the energy becomes

$$E_7^*(N, \lambda, \tau, n_\Delta, J) = c_7\left[\lambda - \left(\frac{\Omega}{2} - N\right)\right]\left(\lambda + \frac{\Omega}{2} - N + 5\right) + c_5\tau(\tau + 3) + c_3 J(J + 1), \qquad N \leqslant \frac{\Omega}{2}, \quad (28)$$

$$E_7^*(N, \lambda, \tau, n_\Delta, J) = c_7\left[\lambda - \left(\frac{\Omega}{2} - \bar{N}\right)\right]\left(\lambda + \frac{\Omega}{2} - \bar{N} + 5\right) + c_5\tau(\tau + 3) + c_3 J(J + 1), \qquad N \geqslant \frac{\Omega}{2}; \quad (29)$$

that is, above midshell, the excitation energy is obtained by replacing N by $\bar{N}$ and, thus, the excitation energy will be symmetrical about midshell.

In Fig. 3 we plot the excitation energy as a function of N with $c_7 = 1$, $c_5 = 0$, and $c_3 = 0$ for $\Omega = 16$ to illustrate the N dependence, ignoring the splittings within the multiplets. The energy splitting among the multiplets decreases linearly until midshell ($N = 8$ in this case) and then increases again to be completely symmetrical about midshell.

IV. THE SO(7) EIGENFUNCTIONS

A. The ground state

We have just determined that the lowest state in the spectrum for a given N belongs to the λ_0 IR of SO(7) as given in Eqs. (25) and (26). Only $\tau = 0$ is allowed for $\lambda = \lambda_0$. Therefore, using the Casimir operator C_7 in Eqs. (15) and (20), we get

$$C_7|N, \lambda, \tau = 0, n_\Delta = 0, J = 0\rangle = \lambda(\lambda + 5)|N, \lambda, \tau = 0, n_\Delta = 0, J = 0\rangle, \quad (30)$$

which, from Eq. (15), implies

$$D^\dagger \cdot \tilde{D}|N, \lambda, \tau = 0, n_\Delta = 0, J = 0\rangle = \left(\lambda - \frac{\Omega}{2} + N\right)\left(\lambda + \frac{\Omega}{2} - N + 5\right) \times |N, \lambda, \tau = 0, n_\Delta = 0, J = 0\rangle. \quad (31)$$

1. *The ground state for $N \leqslant \Omega/2$*

To determine the eigenfunctions in the representation, we start first with the state with $\lambda = \lambda_0 = \frac{\Omega}{2} - N$. The condition in Eq. (31) becomes

$$D^\dagger \cdot \tilde{D}|N, \lambda_0, \tau = 0, n_\Delta = 0, J = 0\rangle = 0. \quad (32)$$

This eigenfunction will be constructed from monopole pairs and quadrupole pairs coupled to angular momentum zero since they are SO(5) scalars:

$$\left|N = \frac{\Omega}{2} - \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0, M = 0\right\rangle = \mathcal{N}_{N,\lambda,0}\Sigma_{p=0}\alpha_p^\lambda S^{\dagger\frac{\Omega}{2}-\lambda-2p}(D^\dagger \cdot D^\dagger)^p|0\rangle, \quad (33)$$

where $P^{(\ell)}|0\rangle = S|0\rangle = D_\mu|0\rangle = 0$ and $\mathcal{N}_{N,\lambda,0}$ is the normalization. The α_p^λ are determined by the condition in Eq. (32) using the commutation relations in Eqs. (4)–(9) and the double commutation relations [9]:

$$[[S, S^\dagger], S^\dagger] = -2S^\dagger, \quad (34)$$

$$[[S, S^\dagger], D_\mu^\dagger] = [[S, D_\mu^\dagger], S^\dagger] = -2D_\mu^\dagger, \quad (35)$$

$$[[S, D_{\mu'}^\dagger], D_\mu^\dagger] = -2(-1)^\mu \delta_{\mu',-\mu} S^\dagger, \quad (36)$$

$$[[D_\mu, S^\dagger], S^\dagger] = -2(-1)^\mu D_{-\mu}^\dagger, \quad (37)$$

$$[[D_{\mu'}, S^\dagger], D_\mu^\dagger] = [[D_{\mu'}, D_\mu^\dagger], S^\dagger] = -2\delta_{\mu',\mu} S^\dagger, \quad (38)$$

$$[[D_{\mu_1}, D_{\mu_2}^\dagger], D_{\mu_3}^\dagger] = 2(\delta_{\mu_2,-\mu_3}(-1)^{\mu_1+\mu_2} D_{-\mu_1}^\dagger - \delta_{\mu_2,\mu_1} D_{\mu_3}^\dagger - \delta_{\mu_3,\mu_1} D_{\mu_2}^\dagger). \quad (39)$$

$$[[D_\mu, S^\dagger], D^\dagger \cdot D^\dagger] = -4(-1)^\mu S^\dagger D_{-\mu}^\dagger, \quad (40)$$

$$[[D_\mu, D^\dagger \cdot D^\dagger], D^\dagger \cdot D^\dagger] = -8(-1)^\mu D^\dagger \cdot D^\dagger D_{-\mu}^\dagger. \quad (41)$$

The condition in Eq. (32) gives

$$
\begin{aligned}
&D_\mu \left| N = \frac{\Omega}{2} - \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0, M = 0 \right\rangle \\
&= \mathcal{N}_{N,\lambda,0} \Sigma_{p=0} \alpha_p^\lambda \left[\left(\frac{\Omega}{2} - \lambda - 2p \right) S^{\dagger \frac{\Omega}{2}} - \lambda - 2p - 1 (D^\dagger \cdot D^\dagger)^p [D_\mu, S^\dagger] |0\rangle \right. \\
&\quad + \left(\frac{\Omega}{2} - \lambda - 2p \right) \left(\frac{\Omega}{2} - \lambda - 2p - 1 \right) 2 S^{\dagger \frac{\Omega}{2}} - \lambda - 2p - 2 (D^\dagger \cdot D^\dagger)^p [[D_\mu, S^\dagger], S^\dagger] |0\rangle \\
&\quad + p \left(\frac{\Omega}{2} - \lambda - 2p \right) S^{\dagger \frac{\Omega}{2}} - \lambda - 2p - 1 (D^\dagger \cdot D^\dagger)^{p-1} [[D_\mu, S^\dagger], D^\dagger \cdot D^\dagger] |0\rangle \\
&\quad + p S^{\dagger \frac{\Omega}{2} - \lambda - 2p} (D^\dagger \cdot D^\dagger)^{p-1} [D_\mu, D^\dagger \cdot D^\dagger] |0\rangle \\
&\quad \left. + p \frac{(p-1)}{2} S^{\dagger \frac{\Omega}{2}} - \lambda - 2p (D^\dagger \cdot D^\dagger)^{p-2} [[D_\mu, D^\dagger \cdot D^\dagger], D^\dagger \cdot D^\dagger] |0\rangle \right] = 0.[-2pt]
\end{aligned} \tag{42}
$$

By using Eqs. (4)–(9) and (34)–(41) and

$$[D_\mu, S^\dagger]|0\rangle = 0, \tag{43}$$

$$[D_\mu, D^\dagger \cdot D^\dagger]|0\rangle = (2\Omega + 6)(-1)^\mu D^\dagger_{-\mu}|0\rangle \tag{44}$$

in Eq. (42), the condition in Eq. (32) then implies that

$$
\begin{aligned}
&\left(\frac{\Omega}{2} - \lambda - 2p \right) \left(\frac{\Omega}{2} - \lambda - 2p - 1 \right) \alpha_p^\lambda \\
&\quad = (4\lambda + 4p + 14) \alpha_{p+1}^\lambda.
\end{aligned} \tag{45}
$$

This recursion relation has the solution

$$\alpha_p^\lambda = \frac{\left(\frac{\Omega}{2} - \lambda\right)!(2\lambda + 5)!!}{(2p)!!\left(\frac{\Omega}{2} - \lambda - 2p\right)!(2\lambda + 5 + 2p)!!}. \tag{46}$$

The coefficient α_0^λ has been set equal to unity ($\alpha_0^\lambda = 1$).

This state has no quadrupole pairs in the sense that it is annihilated by the quadrupole destruction operator. However, we note that we need quadrupole pairs to construct the state to satisfy the condition in Eq. (32) because the pair operators are a composite pair of fermions and are not bosons.

2. The ground state for $N \geqslant \frac{\Omega}{2}$

To determine the eigenfunctions in the representation, we start first with the state with $\lambda = \lambda_0 = N - \frac{\Omega}{2}$. The condition in Eq. (31) becomes

$$
\begin{aligned}
&D^\dagger \cdot \tilde{D} \left| N = \frac{\Omega}{2} + \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0 \right\rangle \\
&\quad = 10\lambda |N, \lambda, \tau = 0, n_\Delta = 0, J = 0\rangle.
\end{aligned} \tag{47}
$$

Using similar manipulations as in the last subsection this conditions leads to the ground state

$$
\begin{aligned}
&\left| N = \frac{\Omega}{2} + \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0 \right\rangle = \frac{\mathcal{N}_{N,\lambda,0}}{\mathcal{N}_{\frac{\Omega}{2}} - \lambda, \lambda, 0} \\
&\quad \times (D^\dagger \cdot D^\dagger)^\lambda \left| \frac{\Omega}{2} - \lambda, \lambda, \tau = 0, n_\Delta = 0, J, M = 0 \right\rangle.
\end{aligned} \tag{48}
$$

Thus the ground state for $N \geqslant \frac{\Omega}{2}$ is then the ground state for $N \leqslant \frac{\Omega}{2}$ with the same λ operated on by $(D^\dagger \cdot D^\dagger)^\lambda$.

Hence, for $\lambda = 0$, of course the ground state is the same. For $\lambda = 1$ it is proportional to $D^\dagger \cdot D^\dagger|\frac{\Omega}{2} - 1, \lambda = 1, \tau = 0, n_\Delta = 0, J, M = 0\rangle$, etc., so that, for $\lambda = \frac{\Omega}{2}$, $N = \Omega$, the ground state is proportional to $(D^\dagger \cdot D^\dagger)^{\frac{\Omega}{2}}|0\rangle$, the closed shell.

B. The remaining states

The remaining states in the same IR are produced by acting on the ground state with the quadrupole pairs, $D^\dagger_\mu|\frac{\Omega}{2} - \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0, M = 0\rangle$, which has $N = \frac{\Omega}{2} - \lambda + 1$, $D^\dagger_{\mu_1} D^\dagger_{\mu_2}|\frac{\Omega}{2} - \lambda, \lambda, \tau = 0, n_\Delta = 0, J = 0, M = 0\rangle$, which has $N = \frac{\Omega}{2} - \lambda + 2$, etc. These states have the same λ, but have different N, because the quadrupole pair operators $D^\dagger_\mu$ are generators of SO(7). The most general state in the IR will then be given by

$$
\begin{aligned}
|N, \lambda, \tau, n_\Delta, J, M\rangle &= \mathcal{N}_{N,\lambda,\tau} (D^\dagger \cdot D^\dagger)^{N + \lambda - \frac{\Omega}{2} - \frac{\tau}{2}} [D^\dagger]^{\tau, n_\Delta, J, M} \\
&\quad \times \Sigma_{p=0}\, \alpha_p^\lambda S^{\dagger \frac{\Omega}{2} - \lambda - 2p} (D^\dagger \cdot D^\dagger)^p |0\rangle \\
&= \mathcal{N}_{N,\lambda,\tau} (D^\dagger \cdot D^\dagger)^{N + \lambda - \frac{\Omega}{2} - \frac{\tau}{2}} \Sigma_{p=0} \alpha_p^\lambda \\
&\quad \times S^{\dagger \frac{\Omega}{2} - \lambda - 2p} (D^\dagger \cdot D^\dagger)^p \\
&\quad \times |\tau, \Omega/2, \tau, n_\Delta, J, M\rangle,
\end{aligned} \tag{49}
$$

with $\lambda = \lambda_0, \lambda_0 + 1, \ldots, \frac{\Omega}{2}$. The scalar products of a pair of quadrupole pairs, $D^\dagger \cdot D^\dagger$, are SO(5) and SO(3) scalars so they do not contribute to τ but do increase the number of pairs by two units. $[D^\dagger]^{\tau, n_\Delta, J, M}$ represents τ quadrupole pairs coupled to the SO(5) quantum numbers, and $|\tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M\rangle = [D^\dagger]^{\tau, n_\Delta, J, M}|0\rangle$, which is assumed to be normalized to unity. This state has $\lambda = \frac{\Omega}{2}$ since it consists of quadrupole pairs operating on the vacuum. $\mathcal{N}_{N,\lambda,\tau}$ is the normalization determined in the next section. Now,

$$\tilde{D} \cdot \tilde{D} \left| \tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M \right\rangle = S \left| \tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M \right\rangle = 0, \tag{50}$$

because both $D \cdot D$ and S are SO(5) are SO(3) scalars and therefore the resulting state is one with the number of pairs less than τ but that belongs to the τ IR of SO(5), which is impossible. From Eq. (30) and the expression for the coefficients α_p^λ, we can also show that, for the ground states with $N \leqslant \frac{\Omega}{2}$,

$$\tilde{D}\cdot\tilde{D}\left|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right\rangle = 0. \tag{51}$$

The states in Eq. (49) can be rewritten

$$|N,\lambda,\tau,n_\Delta,J,M\rangle = \frac{\mathcal{N}_{N,\lambda,\tau}}{\mathcal{N}_{\frac{\Omega}{2}-\lambda+\tau,\lambda,\tau}}(D^\dagger\cdot D^\dagger)^{N+\lambda-\frac{\Omega}{2}-\frac{\tau}{2}} \times\left|\frac{\Omega}{2}-\lambda+\tau,\lambda,\tau,n_\Delta,J,M\right\rangle, \quad \text{with } \lambda=\lambda_0,\lambda_0+1,\ldots,\frac{\Omega}{2}. \tag{52}$$

This set of states include the ground states and excited states for all N as long as λ is limited to $\lambda=\lambda_0,\lambda_0+1,\ldots,\frac{\Omega}{2}$. Note that for $N=\frac{\Omega}{2}+\lambda$ and $\tau=0$ the states in Eq. (52) reduce to the ground states in Eq. (48) for $N \geqslant \frac{\Omega}{2}$.

C. The normalization

First we calculate the normalization of the state with no quadrupole pairs coupled to zero, $|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\rangle$,

$$\left\langle\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right. \times\left|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right\rangle = 1 = \mathcal{N}_{\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau}\langle\tau,\tau,\tau,n_\Delta,J,M|S^{\frac{\Omega}{2}-\lambda} \times\left|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right\rangle, \tag{53}$$

which follows from Eq. (51). We use the overlaps from Ref. [9]:

$$\left\langle\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right|S^pS^{\dagger p}\left|\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right\rangle = \frac{p!(\Omega-2\tau)!}{(\Omega-p-2\tau)!}, \tag{54}$$

$$\left\langle\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right|(\tilde{D}\cdot\tilde{D})^{p(D^\dagger}\cdot D^\dagger)^p\left|\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right\rangle = \frac{2^pp!(2\tau+3+2p)!!(\Omega-2\tau)!!(\Omega+3)!!}{(2\tau+3)!!(\Omega-2\tau-2p)!!(\Omega+3-2p)!!}, \tag{55}$$

$$\left\langle\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right|S^NS^{\dagger N-2p}(D^\dagger\cdot D^\dagger)^p\left|\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right\rangle = \frac{(-1)^pN!(2\tau+3+2p)!!(\Omega-2\tau)!!(\Omega-2\tau-2p-1)!!}{(2\tau+3)!!(\Omega-2\tau-N)!}. \tag{56}$$

These overlaps plus the expression for α_p^λ given in Eq. (46) leads to

$$\left\langle\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right. \times\left|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right\rangle = 1 = \frac{\mathcal{N}^2_{\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau}\left(\frac{\Omega}{2}-\lambda\right)!(\Omega-2\tau)!}{\left(\frac{\Omega}{2}+\lambda-2\tau\right)!} \times{}_3F_2\left[\tau+\frac{5}{2},-\frac{\left(\frac{\Omega}{2}-\lambda+1\right)}{2},-\frac{\left(\frac{\Omega}{2}-\lambda\right)}{2};\lambda+\frac{7}{2}, -\left(\frac{\Omega-1}{2}-\tau\right);1\right], \tag{57}$$

where ${}_3F_2$ is the generalized hypergeometric function that originates from the summation in Eq. (33). Using the fact that this hypergeometric function with unit argument is known [13] and the fact that the normalization on the left-hand side of Eq. (57) is unity, we can solve for $\mathcal{N}_{\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau}$:

$$\mathcal{N}_{\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau} = \sqrt{\frac{(\Omega+5+\lambda)!(2\lambda-2\tau)!!}{\left(\frac{\Omega}{2}-\lambda\right)!(\Omega-2\tau)!!(\Omega+4)!!(2\lambda+5)!!}}. \tag{58}$$

Now we calculate the total normalization of the state in Eq. (52). We use the overlap

$$\left\langle\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right|(\tilde{D}\cdot\tilde{D})^p(D^\dagger\cdot D^\dagger)^p \times\left|\frac{\Omega}{2}+\tau-\lambda,\lambda,\tau,n_\Delta,J,M\right\rangle = \frac{2^pp!(2\tau+3+2p)!!(2\lambda-2\tau)!!(2\lambda+3)!!}{(2\tau+3)!!(2\lambda-2\tau-2p)!!(2\lambda-2p+3)!!} \tag{59}$$

to get

$$\mathcal{N}_{N,\lambda,\tau} = \sqrt{\frac{\left(\frac{\Omega}{2}+\lambda+5\right)!(2\tau+3)!!\left(\frac{\Omega}{2}-N+\lambda-\tau\right)!!\left(\frac{\Omega}{2}-N+\lambda+\tau+3\right)!!}{\left(\frac{\Omega}{2}-\lambda\right)!\left(N+\lambda-\tau-\frac{\Omega}{2}\right)!!(2\lambda+5)!!(\Omega-2\tau)!!(\Omega+4)!!\left(N+\lambda-\frac{\Omega}{2}+\tau+3\right)!!(2\lambda+3)!!}}. \tag{60}$$

So finally the eigenstates are

$$|N,\lambda,\tau,n_\Delta,J,M\rangle = \frac{\mathcal{N}_{N,\lambda,\tau}\Sigma_{p=0}\left(\frac{\Omega}{2}-\lambda\right)!(2\lambda+5)!!S^{\dagger\frac{\Omega}{2}}-\lambda-2p(D^\dagger\cdot D^\dagger)^{p+\frac{N+\lambda-\frac{\Omega}{2}-\tau}{2}}}{(2p)!!\left(\frac{\Omega}{2}-\lambda-2p\right)!(2\lambda+5+2p)!!\left|\tau,\frac{\Omega}{2},\tau,n_\Delta,J,M\right\rangle}. \tag{61}$$

V. PAIRING STRENGTH

We can calculate the pairing strength in these states and we find

$$\langle N, \lambda, \tau, n_\Delta, J, M|S^\dagger S|N, \lambda, \tau, n_\Delta, J, M\rangle = \frac{\left(\frac{\Omega}{2} - \lambda\right)(\lambda - \tau + 1)(\Omega + 2\lambda + 12)}{2\lambda + 7}. \quad (62)$$

The pairing strength in the monopole pairing limit is [9]

$$\langle N, \kappa, \tau, n_\Delta, J, M|S^\dagger S|N, \kappa, \tau, n_\Delta, J, M\rangle_M = (N - \kappa)(\Omega + 1 - N - \kappa), \quad (63)$$

where κ is the number of pairs not coupled to angular momentum zero and $\tau = \kappa, \kappa - 2, \ldots, 0$, or 1. The ground state has $\kappa = 0$, the first excited state has $\kappa = 1$, etc. In contrast to the pairing strength in the SO(7) limit, the pairing strength in the monopole pairing limit does not depend on τ.

We can ask what the pairing strength is in the SO(7) limit compared to the monopole pairing limit. In Fig. 4 we plot the ratio of the pairing strength in the SO(7) limit to the pairing strength in the monopole pairing limit for four examples: the pairing strength in the SO(7) ground state ($\lambda = \lambda_0, \tau = 0$) divided by the pairing strength in the monopole pairing ground state ($\kappa = 0$) (solid line); the pairing strength in the SO(7) $J = 2_1$ excited state ($\lambda = \lambda_0 + 1, \tau = 1$) divided by the pairing strength in the monopole pairing first excited state ($\kappa = 1$), (long dashed line); the pairing strength in the SO(7) $J = 2_2, 4_1$ excited states ($\lambda = \lambda_0 + 2, \tau = 2$) divided by the pairing strength in the monopole pairing second excited state ($\kappa = 2$) (short dashed line); and the pairing strength in the SO(7) $J = 0_2$ excited state ($\lambda = \lambda_0 + 2, \tau = 0$) divided by the pairing strength in the monopole pairing second excited state ($\kappa = 2$) (medium dashed line).
For $N = \tau = \kappa$, the SO(7) eigenfunctions are the same as the monopole pairing eigenfunctions; therefore the ratio for $\tau = \kappa$ starts out at unity, decreases until midshell, rises after midshell, and then drops again at full shell. The SO(7) pairing strength is symmetrical about midshell but the monopole pairing strength is not, so the ratio is not. The ratio for $\tau = \kappa$ has similar behavior for all κ, but the ratio for $\tau \leqslant \kappa$ has a different N dependence (medium dashed line). In fact that ratio is larger than unity for most N.

A study of two-nucleon pairing strength as a function of mass number would be a test for the SO(7) limit.

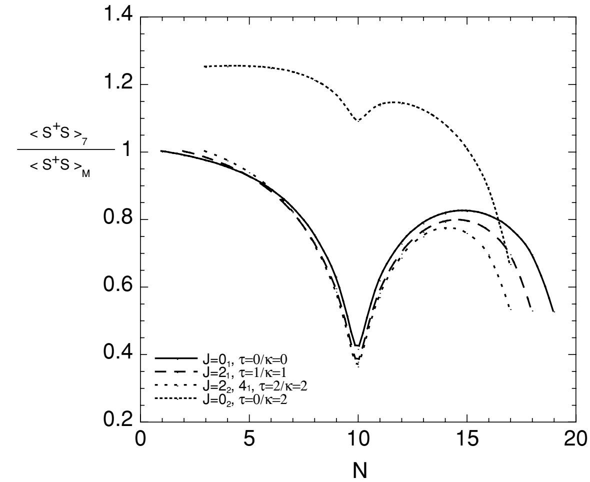

FIG. 4. The ratio of the SO(7) pairing strength, $\langle S^\dagger S\rangle_7$, to the monopole pairing strength, $\langle S^\dagger S\rangle_M$, as a function of the number of pairs of nucleons, N. See text for details.

VI. QUADRUPOLE TRANSITIONS

Operating with the quadrupole operator $P^{(2)}_\mu$ on the eigenstates in Eq. (61) and using the commutation relations in Eqs. (4)–(9), we have

$$P^{(2)}_\mu|N, \lambda, \tau, n_\Delta, J, M\rangle = \mathcal{N}_{N,\lambda,\tau}\Sigma_{p=0}\alpha^\lambda_p\left\{\left[2\left(p + \frac{N + \lambda - \frac{\Omega}{2} - \tau}{2}\right)S^{\dagger\frac{\Omega}{2}-\lambda-2p+1}(D^\dagger \cdot D^\dagger)^{p+\frac{N+\lambda-\frac{\Omega}{2}-\tau}{2}-1}\right.\right.$$
$$\left. + \left(\frac{\Omega}{2} - \lambda - 2p\right)S^{\dagger\frac{\Omega}{2}-\lambda-2p-1}(D^\dagger \cdot D^\dagger)^{p+\frac{N+\lambda-\frac{\Omega}{2}-\tau}{2}}\right]D^\dagger_\mu\left|\tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M\right\rangle$$
$$\left. + S^{\dagger\frac{\Omega}{2}-\lambda-2p}(D^\dagger \cdot D^\dagger)^{p+\frac{N+\lambda-\frac{\Omega}{2}-\tau}{2}}P^{(2)}_\mu\left|\tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M\right\rangle\right\}. \quad (64)$$

The quadrupole operator changes τ and λ each by one unit. The last term cannot increase τ because $P^{(2)}_\mu$ does not change the total number of nucleons and it is operating on a state with τ nucleons coupled to maximal SO(5). The first two terms can be rewritten as

$$\mathcal{N}_{N,,\lambda,\tau}\Sigma_{p=0}\left[\alpha^\lambda_{p+1}2\left(p + \frac{N + \lambda - \frac{\Omega}{2} - \tau}{2} + 1\right) + \alpha^\lambda_p\left(\frac{\Omega}{2} - \lambda - 2p\right)\right],$$

$$S^{\dagger\frac{\Omega}{2}-\lambda-2p-1}(D^\dagger \cdot D^\dagger)^{p+\frac{N+\lambda-\frac{\Omega}{2}-\tau}{2}}D^\dagger_\mu\left|\tau, \frac{\Omega}{2}, \tau, n_\Delta, J, M\right\rangle. \quad (65)$$

For the final state with τ increased by one unit the coefficients must be a linear combination of the normalization $\mathcal{N}$ and amplitudes α for the two states with $\lambda \pm 1$:

$$\mathcal{N}_{N,\lambda,\tau}\left[\alpha^\lambda_{p+1}2\left(p + \frac{N + \lambda - \frac{\Omega}{2} - \tau}{2} + 1\right) + \alpha^\lambda_p\left(\frac{\Omega}{2} - \lambda - 2p\right)\right]$$

$$= a\mathcal{N}_{N,\lambda+1,\tau+1}\alpha_p^{\lambda+1} + b\mathcal{N}_{N,\lambda-1,\tau+1}\alpha_{p+1}^{\lambda-1}. \quad (66)$$

We solve for a, b and thus determine the reduced matrix elements

$$\langle N, \lambda+1, \tau+1, n'_\Delta, J'||P^{(2)}||N, \lambda, \tau, n_\Delta, J\rangle = \sqrt{\frac{\left(\frac{\Omega}{2}-\lambda\right)\left(\frac{\Omega}{2}+6+\lambda\right)\left(\frac{\Omega}{2}+5-N+\lambda+\tau\right)\left(-\frac{\Omega}{2}+5+N+\lambda+\tau\right)}{(2\lambda+7)(2\lambda+5)(\Omega-2\tau)(2\tau+5)}} \times \left\langle \tau+1, \frac{\Omega}{2}, \tau+1, n'_\Delta, J'\middle| |D^\dagger| \middle| \tau, \frac{\Omega}{2}, \tau, n_\Delta, J\right\rangle, \quad (67)$$

$$\langle N, \lambda-1, \tau+1, n'_\Delta, J'||P^{(2)}||N, \lambda, \tau, n_\Delta, J\rangle = \sqrt{\frac{\left(N+\lambda-\frac{\Omega}{2}-\tau\right)\left(\frac{\Omega}{2}-\lambda+1\right)\left(\frac{\Omega}{2}+5+\lambda\right)\left(\frac{\Omega}{2}-N+\lambda-\tau\right)}{(2\lambda+5)(2\lambda+3)(\Omega-2\tau)(2\tau+5)}} \times \left\langle \tau+1, \frac{\Omega}{2}, \tau+1, n'_\Delta, J'\middle| |D^\dagger| \middle| \tau, \frac{\Omega}{2}, \tau, n_\Delta, J\right\rangle. \quad (68)$$

The values of the matrix elements of the quadrupole pair operator with $n_\Delta = 0$ and maximal J are

$$\left\langle N=\tau+1, \lambda'=\frac{\Omega}{2}, \tau'=\tau, J'=2\tau+2\middle| |D^\dagger| \middle| N=\tau, \lambda=\frac{\Omega}{2}, \tau, J=2\tau\right\rangle = \sqrt{(\Omega-2\tau)(\tau+1)(4\tau+5)}. \quad (69)$$

The values of the matrix element of the quadrupole pair operator for the states with $\tau'=2$ and any J' are

$$\left\langle N=2, \lambda=\frac{\Omega}{2}, \tau'=2, J'\middle| |D^\dagger| \middle| N=1, \lambda=\frac{\Omega}{2}, \tau=1, J=2\right\rangle = \sqrt{2(\Omega-2)(2J'+1)}, \quad (70)$$

and for the state with $\tau'=3, n_\Delta=0$, and $J=0$ the matrix element is

$$\left\langle N=3, \lambda=\frac{\Omega}{2}, \tau'=3, n_\Delta=0, J'=0\middle| |D^\dagger| \middle| N=2, \lambda=\frac{\Omega}{2}, \tau=2, J=2\right\rangle = \sqrt{3(\Omega-4)}. \quad (71)$$

The transition strengths from the initial state i to the final state f are

$$B(E2: i \to f) = e_{\text{eff}}^2 \frac{|\langle f \| P^{(2)} \| i\rangle|^2}{2J_i+1}$$

(where e_{eff} is the effective charge), and for the states with $\tau \leqslant 3$ are then

$$B(E2: N; \lambda+1, \tau'=1, J'=2 \to \lambda, \tau=J=0) = e_{\text{eff}}^2 \frac{\left(\frac{\Omega}{2}-\lambda\right)\left(\frac{\Omega}{2}+6+\lambda\right)\left(\frac{\Omega}{2}+5-N+\lambda\right)\left(N-\frac{\Omega}{2}+5+\lambda\right)}{5(2\lambda+7)(2\lambda+5)}, \quad (72)$$

$$B(E2: N; \lambda+1, \tau'=2, J' \to \lambda, \tau=1, J=2) = e_{\text{eff}}^2 \frac{2\left(\frac{\Omega}{2}-\lambda\right)\left(\frac{\Omega}{2}+6+\lambda\right)\left(\frac{\Omega}{2}+6-N+\lambda\right)\left(N-\frac{\Omega}{2}+6+\lambda\right)}{7(2\lambda+7)(2\lambda+5)}, \quad (73)$$

$$B(E2: N; \lambda+1, \tau'=3, J'=6 \to \lambda, \tau=2, J=4) = e_{\text{eff}}^2 \frac{\left(\frac{\Omega}{2}-\lambda\right)\left(\frac{\Omega}{2}+6+\lambda\right)\left(\frac{\Omega}{2}+7-N+\lambda\right)\left(N-\frac{\Omega}{2}+7+\lambda\right)}{3(2\lambda+7)(2\lambda+5)}, \quad (74)$$

$$B(E2: N; \lambda+1, \tau'=3, J'=0 \to \lambda, \tau=2, J=2) = e_{\text{eff}}^2 \frac{\left(\frac{\Omega}{2}-\lambda\right)\left(\frac{\Omega}{2}+6+\lambda\right)\left(\frac{\Omega}{2}+7-N+\lambda\right)\left(N-\frac{\Omega}{2}+7+\lambda\right)}{3(2\lambda+7)(2\lambda+5)}, \quad (75)$$

$$B(E2: N; \lambda, \tau'=J'=0 \to \lambda-1, \tau=1, J=2) = e_{\text{eff}}^2 \frac{\left(\frac{\Omega}{2}-\lambda+1\right)\left(\frac{\Omega}{2}+5+\lambda\right)\left(\frac{\Omega}{2}-N+\lambda\right)\left(N-\frac{\Omega}{2}+\lambda\right)}{(2\lambda+5)(2\lambda+3)}, \quad (76)$$

$$B(E2: N; \lambda, \tau'=1, J'=2 \to \lambda-1, \tau=2, J) = e_{\text{eff}}^2 \frac{2(2J+1)\left(\frac{\Omega}{2}-\lambda+1\right)\left(\frac{\Omega}{2}+5+\lambda\right)\left(\frac{\Omega}{2}-N+\lambda-1\right)\left(N-\frac{\Omega}{2}+\lambda-1\right)}{35(2\lambda+5)(2\lambda+3)}. \quad (77)$$

We note that the transition $\lambda+1, \tau+1, n'_\Delta, J' \to \lambda, \tau, n_\Delta, J$ is independent of n_Δ and J for the examples given and we conjecture that this is true in general.

For the states lowest in the spectrum with $\lambda = \lambda_0+1, \lambda_0+2, \lambda_0+3$, and, for $N \leqslant \frac{\Omega}{2}, \lambda_0 = \frac{\Omega}{2} - N$, these transitions become

$$B(E2: N; \lambda_0+1, \tau'=1, J'=2 \to \lambda_0, \tau=J=0) = e_{\rm eff}^2 \frac{N(\Omega+6-N)}{\Omega+7-2N}, \tag{78}$$

$$B(E2: N; \lambda_0+2, \tau'=2, J' \to \lambda_0+1, \tau=1, J=2) = e_{\rm eff}^2 \frac{2(N-1)(\Omega+7-N)}{\Omega+9-2N}, \tag{79}$$

$$B(E2: N; \lambda_0+2, \tau'=J'=0 \to \lambda_0+1, \tau=1, J=2) = e_{\rm eff}^2 \frac{2(N-1)(\Omega+2-2N)(\Omega+7-N)}{(\Omega+9-2N)(\Omega+7-2N)}, \tag{80}$$

$$B(E2: N; \lambda_0+3, \tau'=3, J'=6 \to \lambda_0+2, \tau=2, J=4) = e_{\rm eff}^2 \frac{3(N-2)(\Omega+8-N)}{(\Omega+11-2N)}, \tag{81}$$

$$B(E2: N; \lambda_0+3, \tau'=3, J'=0 \to \lambda_0+2, \tau=2, J=2) = e_{\rm eff}^2 \frac{3(N-2)(\Omega+8-N)}{(\Omega+11-2N)}, \tag{82}$$

$$B(E2: N; \lambda_0+3, \tau'=1, J'=2 \to \lambda_0+2, \tau=2, J) = e_{\rm eff}^2 \frac{4(2J+1)(N-2)(\Omega+2-2N)(\Omega+8-N)}{35(\Omega+11-2N)(\Omega+9-2N)}. \tag{83}$$

For $N \geqslant \frac{\Omega}{2}$ the transition rates are obtained by replacing $N \to \bar{N}$ in Eqs. (78)–(83). The variation of the $B(E2)$s with respect to the number of pairs, N, will then be symmetric about midshell as are the excitation energies. In Fig. 5 we show an example of this variation for $\Omega = 16$. The transition from the first excited state, $\tau = 1$, $J_i = 2_1^+$, to the ground state (solid line) increases monotonically as a function of N to midshell and then decreases again. The transition from the $\tau = 2$, $J_i = 2_2^+, 4_1^+$ to the first excited state (dashed line) increases even faster than this transition whereas the transition from the $\tau = 0$, $J_i = 0_2^+$ to the first excited state (dotted line) increases slower than either of these. In Fig. 6 the ratio of the transition from the $\tau = 2$, $J_i = 2_2^+, 4_1^+$ to the first excited state to the transition from the first excited state to the ground state (solid line) and the ratio of the transition from the $\tau = 0$, $J_i = 0_2^+$ to the transition from the first excited state to the ground state (dashed line) are plotted as a function of the number of pairs.

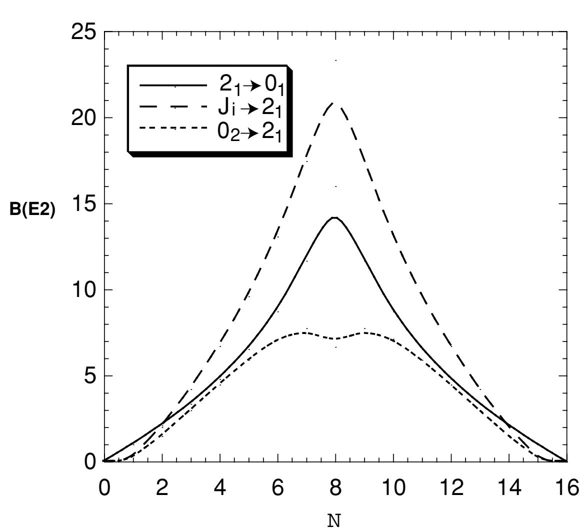

FIG. 5. $B(E2)$ values as a function of the number of pairs of nucleons, N. See text for a detailed explanation.

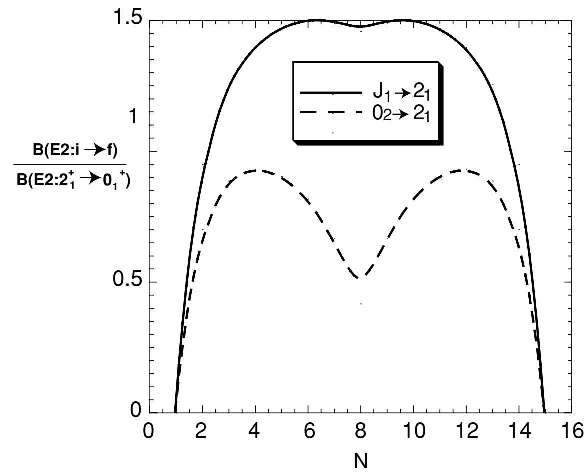

FIG. 6. The ratio of $B(E2: i \to f)/B(E2: 2_1^+ \to 0_1^+)$ as a function of the number of pairs of nucleons, N. See text for a detailed explanation.

VII. COMPARISON WITH EXPERIMENT

Experimental evidence for SO(7) symmetry has been observed in the Pd-Ru region of the periodic table [14]. The Xe isotopes may also be good examples of SO(7) symmetry in nuclei. In Fig. 7 the spectra of the Xe isotopes are plotted

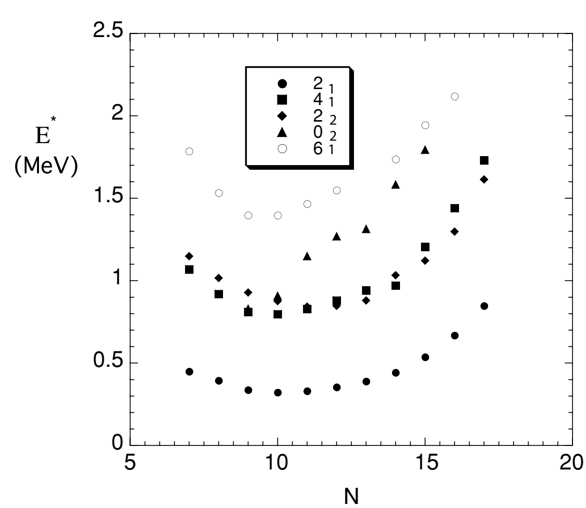

FIG. 7. The spectra of the Xe isotopes as a function of the number of pairs of nucleons, N.

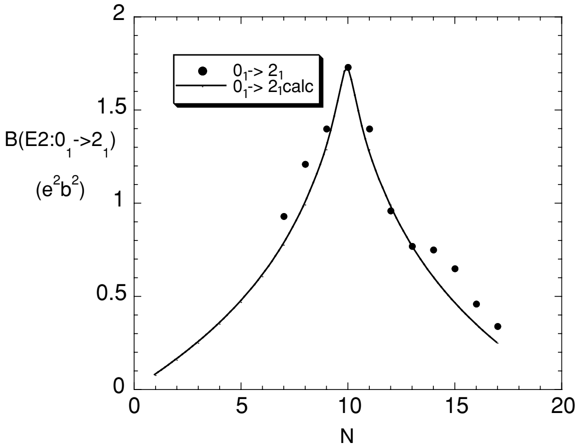

FIG. 8. The $B(E2:0_1 \to 2_1)$ of the Xe isotopes as a function of the number of pairs of nucleons, N. The solid dots are the empirical values; the solid line is a fit using Eq. (78) with e^2_{eff} determined by fitting the $B(E2:0_1 \to 2_1)$ for ^{120}Xe($N = 10$).

versus the number of pairs of nucleons. The qualitative features of the SO(7) spectrum of Fig. 3 are seen in this spectrum. The excitation energy decreases to a minimum at $N = 10$(^{120}Xe), which implies an effective $\Omega = 20$, and the SO(5) multiplets are quasi-degenerate. However, the N dependence is not simply linear as in the SO(7) example shown in Fig. 3.

The transition strengths are a better test of the eigenfunctions. In Fig. 8 we compare the empirical transition strengths [15] from the ground state to the first excited state with the SO(7) limit given in Eq. (78) with e_{eff} being fit to the empirical strength at $N = 10$. The agreement is very good.

Recently, transitional nuclei have been analyzed in terms of a Bohr Hamiltonian with a γ-independent potential that is a square well in the deformation β. Table I summarizes the characteristic features of the $B(E2)$ values of this model in the column labeled E(5). In the notation $J_{i,\tau}$, i designates the ith state with quantum numbers J, τ. The SO(7) $B(E2)$ s are in the next column for $N = 14$, which corresponds to ^{128}Xe. The empirical results are in the last column. The $B(E2)$ values are given relative to the transition from the first excited state to the ground state. Both the $E(5)$ and SO(7) predictions agree with experiment within the experimental error.

TABLE I. The $B(E2)$ values of excited states relative to the $B(E2)$ from the first excited state to the ground state for the $E(5)$ model, the SO(7) model, and the experimental value.

	$E(5)$	SO(7) $\Omega = 20, N = 14$	^{128}Xe Expt. [16]
$\frac{B(E2;4^+_{1,2}\to 2^+_{1,1})}{B(E2;2^+_{1,2}\to 0^+_{1,1})}$	1.68	1.54	1.47(15)
$\frac{B(E2;6^+_{1,3}\to 4^+_{1,2})}{B(E2;2^+_{1,2}\to 0^+_{1,1})}$	2.21	1.74	1.94 (20)
$\frac{B(E2;0^+_{2,0}\to 2^+_{1,1})}{B(E2;2^+_{1,2}\to 0^+_{1,1})}$	0.86	1.03	—

VIII. SUMMARY AND CONCLUSIONS

We have derived the energy spectrum, eigenfunctions, pairing strength, and quadrupole transition strengths in the SO(7) limit of a fermion monopole and quadrupole pairing model that includes Pauli effects. This limit corresponds to a transitional limit between spherical nuclei and γ-unstable deformed nuclei and a shell model Hamiltonian with monopole pairing and quadrupole interactions having equal strength. The agreement of the trend of the quadrupole transition from the first excited to the ground state with respect to atomic mass for the Xe isotopes suggests that this region of the periodic table may have SO(7) symmetry. More detail information on both quadrupole transitions from other excited states and two-nucleon transfer strengths as a function of mass number could determine whether SO(7) symmetry has a wide empirical validity.

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