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Nuclear friction and quantum mechanical diffusion

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We study dissipative properties of the motion of a slow nuclear collective variable weakly coupled to a complex quantum environment formed by the fast nucleonic degrees of freedom. The fast quantum mechanical subsystem is treated within the random matrix approach, where the complexity of the nucleonic degrees of freedom's motion can be changed by a parameter from regular to fully chaotic. Classical dynamics is assumed for the slow variable, and the equation of motion is determined from conservation of the total energy of the nuclear many-body system. We show that the macroscopic equation of motion for the collective variable is subject to a memory-dependent friction force, with a retardation defined by the chaoticity of the fast nucleonic environment.

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I. INTRODUCTION

The nuclear many-body problem involves hundreds of nucleonic degrees of freedom. However, many nuclear processes can be theoretically studied in terms of the dynamics of only a few macroscopic collective variables. Examples are fission, giant multipole resonances, and the fusion of heavy ions, where the collective variables are related to the shape of the nucleus. Usually the choice of collective variables is dictated by our intuition or a model of the physical problem. Once such a choice is made we are led immediately to the concept of dissipation, or the energy flow between the collective and nucleonic modes. Dissipation of the collective energy can be taken into account by introducing friction into the equations of motion for the collective variables.

A natural question that appears is how the properties of the macroscopic friction depend on the degree of chaoticity of the nucleonic motion. A system of noninteracting nucleons in a deformed mean field is almost regular at low excitations because only the few lowest many-body states of the system are excited. With the growth of excitation energy, the number of excited many-body states increases exponentially, and at high excitations many-body states are very close lying in energy. In this case, any residual interaction may lead to extremely complex, chaotic nucleonic dynamics. One may ask, what is the effect of this chaotic dynamics on the collective motion?

The effect of classical chaos on collective dynamics has been studied in some previous works. Burgio *et al.* [1] considered the motion of classical noninteracting particles confined in a two-dimensional "nuclear" billiard whose walls undergo periodic and slow shape oscillations. They found that the chaotic one-particle dynamics generates dissipation of the collective energy, while for the regular intrinsic dynamics there is no damping of collective motion. This conclusion was confirmed by the semiclassical study of the question by Bauer *et al.* [2,3]. They studied the damping of collective motion in nuclei within the Vlasov equation, which is the semiclassical approximation to the time-dependent Hartree-Fock equation.

If the nuclear many-body system is considered quantum mechanically, it is assumed that we can select a few slowly varying (collective) degrees of freedom, while all remaining degrees of freedom are treated as a fast quantum environment. To model such a nucleonic bath, the random matrix approach may be utilized [4–8]. The random matrices are usually taken in the Gaussian orthogonal ensemble (GOE) limit, corresponding to the fully chaotic dynamics of the nucleonic degrees of freedom. In contrast to that, Refs. [9,10] showed that the chaoticity of the energy spectrum of the fast quantum environment can be changed by the strength of the residual interaction introduced so that it acts between all eigenstates of the system. In this case, the residual interaction leads to Landau-Zener transitions from occupied to unoccupied energy levels, giving rise to a diffusion of the energy. It is interesting to note that such quantum mechanical diffusion of the energy strongly depends on the complexity of the fast quantum system. Thus, for the mixed dynamics, the energy evolves quadratically in time, while normal energy diffusion (linear time dependence) appears for the fully chaotic dynamics of the fast quantum environment [10].

We address the question of what is the macroscopic manifestation of the quantum mechanical diffusion: How does the complexity of the quantum environment reveal itself in the time evolution of the slow collective variable? We expect that the slow but finite perturbation of the complicated motion of the nucleonic degrees of freedom may lead to a time delay in the response of the nucleonic bath. And since the motions of the collective and nucleonic degrees of freedom are coupled due to the conservation of the total energy of the nuclear many-body system, this will imply the presence of memory effects in the dynamics of the slow collective variable. The memory effects in the nuclear collective motion were found earlier within different approaches. For example, in the nuclear Fermi-liquid model [11,12] the non-Markovian features of the collective dynamics arise because of the Fermi-surface distortions and depend on the relaxation time of the collective excitations. In this respect, we also mention the linear response theory

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[13,14] and dissipative diabatic dynamics model [15,16], with which the macroscopic collective equations of motion with the retarded friction force have been investigated.

The plan of the paper is as follows. In Sec. II, we present a model for the nuclear many-body system which is assumed to have one fast (nucleonic) and one slow (collective) part. The microscopical derivation of the quantum mechanical response of the fast nucleonic subsystem on the slow variations of the collective variable is contained in Sec. III. Section IV is devoted to the discussion of an influence of the macroscopic retardation of the nucleonic response on the dynamics of the slow collective degree of freedom. Summary and conclusions are given in Sec. V.

II. NUCLEAR MANY-BODY SYSTEM

Our basic assumption is that the nuclear many-body system can be separated into one fast (nucleonic) and one slow (collective) part,

$$H_{\rm tot} = H_{\rm fast}(Q; p_i) + H_{\rm slow}(Q, \dot{Q}), \tag{1}$$

where Q represents the collective degree of freedom and p_i the fast nucleonic degrees of freedom. The model for $H_{\rm fast}$ is given in Sec. II A, while the dynamics of the slow degree of freedom Q is described in Sec. II B.

A. Fast (nucleonic) subsystem

To model quantum chaoticity of the nucleonic degrees of freedom, we use a time-dependent random matrix model developed in [9] and write the corresponding Hamiltonian in the form

$$H_{\rm fast}[Q(t)] = H_{\rm fast}^{(0)} + \mathcal{A}\cdot[Q(t) - Q_0], \tag{2}$$

with $Q(t=0) = Q_0$,

$$H_{\rm fast}^{(0)} = \sum_{n=-N'}^{N'} \varepsilon_n c_n^\dagger c_n + \sum_{n>k} W_{nk}(c_n^\dagger c_k + c_k^\dagger c_n), \tag{3}$$

and

$$\mathcal{A} = \sum_{n=-N'}^{N'} A_n c_n^\dagger c_n. \tag{4}$$

The creation and annihilation operators in Eqs. (3) and (4) refer to the time-dependent basis, $|n\rangle$, which may be thought of as describing single-particle states or many-body configurations by Slater determinants, where the latter is assumed here. W_{nk} in (3) are the matrix elements of the residual interaction acting between all eigenstates of $H_{\rm fast}^{(0)}$. The time-dependent term $\mathcal{A}\cdot[Q(t) - Q_0]$ in the right-hand side (rhs) of Eq. (2) represents the coupling to the slow subsystem. In contrast to [17], we take the matrix elements of a coupling operator $\mathcal{A}$ (4) to be diagonal with respect to the instantanious states $|n\rangle$ of the Hamiltonian $H_{\rm fast}[Q(t)]$. This can always be done by choosing a basis where $\mathcal{A}$ has diagonal form.

All matrix elements in Eqs. (3) and (4) are defined as random numbers,

$$\varepsilon_n \in G\left(0, \sqrt{\tfrac{2}{N}}\right), \qquad W_{nk} \in (1-\delta_{n,k})G\left(0, \Delta\sqrt{\tfrac{1}{N}}\right),$$
$$A_n \in G\left(0, \sigma_A\sqrt{\tfrac{2}{N}}\right), \tag{5}$$

i.e., $H_{\rm fast}[Q(t)]$ constitutes an ensemble of $N = (2N'+1)$-dimensional, time-dependent random matrices with Gaussian distributed matrix elements, which are all Q independent (or time independent). In the following, the ensemble averaging (denoted by a bar above a quantity) is understood as the averaging over different sets of the Hamiltonians of the fast subsystem (2).

The two parameters in Eq. (5), Δ and σ_A, determine the properties of the system (2). By varying the strength of the residual interaction, the "chaoticity parameter" Δ, between 0 and 1, the fluctuations of the energy spectrum as well as of the eigenfunctions of $H_{\rm fast}[Q(t)]$ smoothly change between Poisson (regular) and GOE (chaos) [9]. In the limit of no residual interaction ($\Delta = 0$), each eigenvalue has a set of good quantum numbers, and its classical counterpart is regular. The coupling $\mathcal{A}\cdot[Q(t) - Q_0]$ to the slow subsystem does not break these good quantum numbers, since $\mathcal{A}$ has diagonal form. At $\Delta = 0$, the eigenvalues of $H_{\rm fast}[Q(t)]$ depend linearly on the slow variable $Q(t)$, and the slopes are determined by the dispersion σ_A.

When $\Delta > 0$, a random residual interaction is introduced which is assumed not to depend on Q. This is a quite reasonable approximation that implies matrix elements of the residual interaction W_{nk} between unmixed (time-dependent) eigenstates $|n\rangle$ and $|k\rangle$ are independent of Q. In many realistic cases, as, e.g., for nuclear fission, the coupling matrix elements may, however, show a small but smooth variation with the collective variable, i.e., shape parameters.

The dynamics of the fast subsystem is determined by the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H_{\rm fast}[Q(t)]|\Psi(t)\rangle. \tag{6}$$

As the initial condition, we shall assume that the middle eigenstate in the random matrix description of $H_{\rm fast}[Q(t)]$ is occupied at $t = 0$.

We define the static basis states, $|\mu(Q)\rangle$, through the time-independent Schrödinger equation,

$$H_{\rm fast}(Q)|\mu(Q)\rangle = E_\mu(Q)|\mu(Q)\rangle, \tag{7}$$

where the orthonormal wave functions $|\mu\rangle$ and energies E_μ are calculated for each fixed Q value.

The states $|\mu(Q)\rangle$ are mixtures of unpertubed states,

$$|\mu\rangle = \sum_n b_{\mu n}|n\rangle. \tag{8}$$

The spread of $|\mu\rangle$ over the unpertubed states $|n\rangle$ can be characterized by the spreading width Γ_μ that depends on the size of the matrix elements of the residual interaction W_{nk}. For small perturbations, Γ_μ may be estimated through Fermi's

Golden Rule,

$$\Gamma_\mu = 2\pi \frac{\langle W_{nk}^2 \rangle}{D}, \quad (9)$$

where D is the mean level spacing. In the following, we assume that $\Gamma_\mu > D$.

The static energies E_μ of Eq. (7) are unfolded such that the mean level density is constant, $\rho = \text{const}$, at each value of Q. Later, we shall consider a more realistic energy dependence of the level density suitable for the considered many-body system.

To see how the energy of the fast subsystem evolves with time, one expands the time-dependent wave function $\Psi(t)$ in the static basis (7),

$$|\Psi(t)\rangle = \sum_\mu d_\mu(t)|\mu(Q)\rangle, \quad (10)$$

where $d_\mu(t=0) = \delta_{\mu,\eta}$. Because of the residual interaction, the initially occupied state η spreads over neighboring states through the jumps, in the same way as in the standard two-level Landau-Zener picture. This gives rise to a diffusion of the energy

$$\mathcal{E}_{\text{fast}}(t) = \langle \Psi(t)|H_{\text{fast}}[Q(t)]|\Psi(t)\rangle = \sum_\mu |d_\mu|^2(t) E_\mu(Q). \quad (11)$$

As shown in Ref. [10], an important peculiarity of the energy $\mathcal{E}_{\text{fast}}(t)$ of the driven fast quantal system (2)–(5) is that it saturates with time. Such saturation is a pure quantum effect caused by the discreteness of individual energy levels of quantal system. Times, over which $\mathcal{E}_{\text{fast}}(t)$ saturates, are of the order of the Heisenberg time, $t \sim \hbar/D$ (for example, $\hbar/D \approx 3 \times 10^{-5}$ s for 10 MeV excitation of the ^{236}U), which is much larger than typical times of nuclear collective motion, $\tau_{\text{coll}} \sim 10^{-22}$–$10^{-20}$ s.

B. Slow (collective) subsystem

The slow collective subsystem is considered clasically with the Hamilton function

$$H_{\text{slow}}(Q, \dot{Q}) = \mathcal{E}_{\text{slow}}(t) = \tfrac{1}{2} M \dot{Q}^2 + \tfrac{1}{2} C Q^2, \quad (12)$$

where M and C are collective mass and stiffness coefficients, respectively. These two parameters may be microscopically determined from the cranking approach and are here treated as constant parameters. The stiffness coefficient C is allowed to be positive (nuclear giant resonances situation) as well as negative (the case of nuclear descent from the fission barrier).

Different dynamical paths $Q(t)$ of the slow collective variable are attributed to different realizations of the random matrices $H_{\text{fast}}[Q(t)]$, modeling the nucleonic bath. These paths are found by the condition that the total energy of the nuclear many-body system (1) is conserved,

$$\mathcal{E}_{\text{tot}}(t) = \mathcal{E}_{\text{fast}}(t) + \mathcal{E}_{\text{slow}}(t) = \mathcal{E}_{\text{tot}}(0), \quad (13)$$

where $\mathcal{E}_{\text{fast}}(t)$ is given by Eq. (11). Since the fast nucleonic subsystem, described in terms of the random matrix approach (2)–(5), exhibits energy diffusion, the coupling Eq. (13) implies dissipative effects in the slow collective dynamics. The random matrix model contains no scales and can therefore be used only to measure the fluctuation properties of the nucleonic bath. Indeed, assuming a constant mean level density ρ of the fast subsystem's eigenstates, the ensemble averaged energy is constant in time, $\overline{\mathcal{E}}_{\text{fast}}(t) = \overline{E}_\eta$. This is so because of symmetry of the energy states implying equal transition probabilities from the initially occupied state η to higher and lower-lying unoccupied levels. In the absence of coupling between the slow and fast subsystems, the energy $\mathcal{E}_{\text{slow}}(t)$, associated with the nuclear shape parameter Q, is conserved and defined by the mass M and stiffness parameter C, obtained in the limit of the infinitely slow deformations of the nucleus. In our model, the damping of the collective motion may arise only from the time fluctuations of the energy $\mathcal{E}_{\text{fast}}$ of the nucleonic bath generated by the set of random matricies $H_{\text{fast}}[Q(t)]$. This feature causes the present model to differ from a concept of the cranking approach [18] or the linear response theory [14], where the dynamics of Q is derived from the constancy of the energy of the nucleonic subsystem.

Taking the time derivative of Eq. (13) and utilizing Eq. (12), one can obtain an equation of motion for the collective degree of freedom in the form

$$M\ddot{Q} = -C(Q - Q_0) + F_{\text{fast}}, \quad (14)$$

with the initial conditions $Q(t=0) = Q_0$ and $\dot{Q}(t=0) = \dot{Q}_0$. The force

$$F_{\text{fast}} = -\frac{1}{\dot{Q}} \frac{d\mathcal{E}_{\text{fast}}(t)}{dt} \quad (15)$$

measures the response from the fast nucleonic bath.

Equation (14) defines the complicated dynamics of the slow collective variable that arise because of the dependence of $\mathcal{E}_{\text{fast}}(t)$ (11) on Q and $\dot{Q}$. Indeed, from Eqs. (10) and (6), one can see that the occupation probabilities $|d_\mu|^2(t)$ in (11) depend on the collective coordinate and velocity through the set of equations

$$\dot{d}_\mu(t) = -\dot{Q}(t) \sum_\nu \langle \mu|\partial_Q|\nu\rangle d_\nu(t) - \frac{i}{\hbar} E_\mu[Q(t)] d_\mu(t). \quad (16)$$

Thus, the dynamical trajectory $Q(t)$ of the slow variable is obtained by solving Eqs. (14) and (16). This solution corresponds to a given set of the random matrices (2)–(5) that model the nucleonic subsystem. The average behavior of the collective degree of freedom $\overline{Q}(t)$ is obtained by ensemble averaging over all realizations of the nucleonic subsystem. In this way, we may define not only an equation of motion of the average collective variable $\overline{Q}$, but also an equation of motion of Q itself that includes the fluctuations. The latter implies the introduction of a stochastic force, resulting in the Langevin description of the nuclear collective dynamics, see, e.g. [11].

III. RESPONSE OF THE FAST SUBSYSTEM

To get a general understanding of the influence of the chaoticity of the nucleonic bath, treated within the approach (2)–(5), on the dissipative properties of the collective dynamics (12), we shall study the ensemble average of the response force F_{fast} (15). In general, the calculation of (15) is a very complicated problem. The problem simplifies significantly if one considers the response of the quantum mechanical system (2) on small variations $Q - Q_0$ of the slow parameter Q around its initial value Q_0. In this case, $F_{\rm fast}$ can be treated perturbatively, i.e., in powers of $Q - Q_0$.

Such perturbative derivation of the ensemble averaged response force is presented in Sec. III A. In Sec. III B, we show that the linearized response contains a memory-dependent friction term, with a retardation determined by the chaoticity of the fast quantum mechanical subsystem.

A. Response force

The response force (15) can be calculated from Eq. (11)

$$F_{\rm fast} = -\sum_\mu |d_\mu|^2(t)\frac{\partial E_\mu(Q)}{\partial Q} - \sum_\mu \frac{1}{\dot Q}\frac{d(|d_\mu|^2)}{dt}E_\mu(Q). \tag{17}$$

The explicit time dependence of the occupation probabilities $|d_\mu|^2$ in (17) can be obtained from Eq. (16). Since for each fixed Q value the static eigenstates $|\mu\rangle$ are orthonormal, the matrix $\langle\mu|\partial_Q|\nu\rangle$ in (16) is anti-Hermitian, implying

$$\langle\mu|\partial_Q|\nu\rangle = i\cdot \mathrm{Im}(\langle\mu|\partial_Q|\nu\rangle) \equiv i\cdot\mathcal{D}_{\mu\nu}(Q), \tag{18}$$

where the introduced functions $\mathcal{D}_{\mu\nu}$ are real numbers.

It is convenient to introduce

$$f_\mu(t) = \exp\left\{\frac{i}{\hbar}\int_0^t E_\mu[Q(t')]dt'\right\} d_\mu(t), \tag{19}$$

and rewrite Eq. (16) in integral form for the new dynamical variables f_μ

$$f_\mu(t) = -i\int_0^t \dot Q(t')\sum_\nu \exp\left(-\frac{i}{\hbar}\int_0^{t'}[E_\mu - E_\nu]dt''\right) \times \mathcal{D}_{\mu\nu}(t')f_\nu(t')dt', \tag{20}$$

where $f_\mu(0) = \delta_{\mu,\eta}$.

The resulting system of integral equations is of Volterra type and can be solved by iterations

$$\begin{aligned} f_\mu(t) = \delta_{\mu,\eta} - i\int_0^t \dot Q(t_1)\mathcal{D}_{\mu\eta}(t_1) &\times \exp\left(\int_0^{t_1}\frac{i}{\hbar}[E_\mu - E_\eta]dt'\right)dt_1 \\ &- \int_0^t \dot Q(t_1)\sum_\nu \mathcal{D}_{\nu\eta}(t_1)\exp\left(\int_0^{t_1}\frac{i}{\hbar}[E_\nu - E_\eta]dt'\right) \\ &\times \int_0^{t_1}\dot Q(t_2)\mathcal{D}_{\mu\nu}(t_2) \\ &\times \exp\left(\int_0^{t_2}\frac{i}{\hbar}[E_\mu - E_\nu]dt'\right)dt_2dt_1 + \cdots \end{aligned} \tag{21}$$

The iterations are defined by the parameter

$$\alpha = -i\int_0^t \dot Q(t_1)\mathcal{D}_{\mu\nu}(t_1)\exp\left(\int_0^{t_1}\frac{i}{\hbar}[E_\mu - E_\eta]dt'\right)dt_1, \tag{22}$$

which can be estimated as follows. From Eq. (18) we have

$$i\mathcal{D}_{\mu\nu} = \frac{\langle\mu|\partial H_{\rm fast}/\partial Q|\nu\rangle}{E_\mu - E_\nu} = \frac{\langle\mu|\mathcal{A}|\nu\rangle}{E_\mu - E_\nu}, \qquad \mu\neq\nu, \tag{23}$$

where the second step is obtained from Eqs. (2) and (7). By assuming the functions $\mathcal{D}_{\mu\nu}$ to weakly depend on time, we may estimate the size of α,

$$|\alpha| \approx \left|\left[\frac{\langle\mu|\mathcal{A}|\nu\rangle}{E_\mu - E_\nu}\right]_{Q=Q_0}\cdot(Q - Q_0)\right|. \tag{24}$$

To truncate the series (21), α should be relatively small. As seen by Eq. (24), this is fulfilled if either the displacements $Q - Q_0$ of the slow variable are small, or the coupling $\langle\mu|\mathcal{A}|\nu\rangle$ between the slow and fast subsystems is weak.

Using Eqs. (19) and (21), we get up to quadratic in α terms,

$$\begin{aligned} |d_\mu(t)|^2 = |f_\mu(t)|^2 = \delta_{\mu,\eta} &+ \left[\int_0^t \dot Q(t')\mathcal{D}_{\mu\eta}(t')\cos\left(\frac{1}{\hbar}\int_0^{t'}[E_\mu - E_\eta]dt''\right)dt'\right]^2 \\ &+ \left[\int_0^t \dot Q(t')\mathcal{D}_{\mu\eta}(t')\sin\left(\frac{1}{\hbar}\int_0^{t'}[E_\mu - E_\eta]dt''\right)dt'\right]^2 \\ &- 2\delta_{\mu,\eta}\int_0^t \dot Q(t_1)\sum_\nu \mathcal{D}_{\nu\eta}(t_1)\cos\left(\int_0^{t_1}\frac{1}{\hbar}[E_\nu - E_\eta]dt'\right) \\ &\times \int_0^{t_1}\dot Q(t_2)\mathcal{D}_{\mu\nu}(t_2)\cos\left(\int_0^{t_2}\frac{1}{\hbar}[E_\mu - E_\nu]dt'\right)dt_2dt_1 \\ &- 2\delta_{\mu,\eta}\int_0^t \dot Q(t_1)\sum_\nu \mathcal{D}_{\nu\eta}(t_1)\sin\left(\int_0^{t_1}\frac{1}{\hbar}[E_\nu - E_\eta]dt'\right) \\ &\times \int_0^{t_1}\dot Q(t_2)\mathcal{D}_{\mu\nu}(t_2)\sin\left(\int_0^{t_2}\frac{1}{\hbar}[E_\mu - E_\nu]dt'\right)dt_2dt_1. \end{aligned} \tag{25}$$

One can neglect the time dependence of the functions $\mathcal{D}$ and static energies E [appearing through its parametric dependence on $Q(t)$]. Otherwise, this corresponds to including higher-order terms in α (or in $Q - Q_0$). We put the variables $\mathcal{D}$ and E equal to the corresponding values evaluated at $Q = Q_0$.

Linearizing Eq. (17) with respect to $Q - Q_0$ and utilizing Eq. (25), we obtain an expression for the response force $F_{\rm fast}$,

$$F_{\rm fast} = -\frac{\partial E_\eta}{\partial Q} - \frac{\partial^2 E_\eta}{\partial Q^2}\cdot(Q - Q_0) - \int_0^t \gamma(t - t')\dot Q(t')dt', \tag{26}$$

with

$$\gamma(t - t') = 2\sum_{\mu,\mu\neq\eta}|\langle\mu|\mathcal{A}|\eta\rangle|^2\frac{\cos([E_\mu - E_\eta][t - t']/\hbar)}{E_\mu - E_\eta}, \tag{27}$$

where Eq. (23) was used.

The first two terms in the rhs of Eq. (26), determined by the variations of the energy $E_\eta(Q)$ of the initially occupied state η around $Q = Q_0$, give the static (and conservative) part of the response (17) of the fast quantum mechanical subsystem. This conservative force exists also for infinitely small parametric drivings, $\dot Q \to 0$, of the quantum subsystem $H_{\rm fast}[Q(t)]$ and, therefore, this may be included into the definition of the potential energy of the slow collective variable (12). In the sequel, we shall omit this contribution to F_{fast} and concentrate only in the second part (26) of the response force which has a dynamical nature and is given by the memory integral over velocities of the slow collective variable. One can then write for the ensemble averaged response force,

$$\overline{F}_{\mathrm{fast}} = -\int_0^t \overline{\gamma}(t-t')\dot{\overline{Q}}(t')dt'. \tag{28}$$

This retarded force is coming from the time growth of the occupations $\overline{|d_\mu|^2}$ of the initially unoccupied states μ. This may lead to a heating of the quantum mechanical subsystem or, because of the energy conservation condition (13), to the corresponding dissipation of the collective energy (12). From this perspective, the presence of the retarded force (28) in the response of the fast nucleonic environment may give rise to the friction in the collective dynamics (14).

B. Retardation of the response

Formally, the response of the quantum mechanical subsystem (28) has a retarded character. To estimate the retardation and see how it is defined by the chaoticity of the quantum environment (2)–(5), we shall evaluate the ensemble averaging of the memory kernel $\gamma(t-t')$ (27).

Since for different realizations of the random matrices (2)–(5), the squared matrix elements of the coupling operator $\mathcal{A}$, $|\langle\mu|\mathcal{A}|\eta\rangle|^2$, are statistically independent of the energy differences $E_\mu - E_\eta$, one can write that

$$\overline{\gamma}(t-t') = 2\sum_\mu \overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2} \cdot \overline{[E_\mu - E_\eta]^{-1}\cos([E_\mu - E_\eta][t-t']/\hbar)}. \tag{29}$$

Introducing the dimensionless spacings, $s_r \equiv (E_\mu - E_\eta)/D$, we have statistics of s_r described by the $r \equiv |\mu - \eta|$th order spacing distribution of the GOE ensembles of the energy levels. The distribution of nearest-neighbor energy levels (obtained at $r = 1$) is approximately described by the well-known Wigner distribution, $P(s_1) = (\pi/2)s_1 \exp(-\pi/4 s_1^2)$ also for quite small values of Δ [a]. Obviously, the mean value is $\langle s_1\rangle = 1$ and the variance $\sigma^2(s_1) = 4/\pi - 1$ in that case. In the general case of $r > 1$, the mean value of the rth order spacing distribution is just $\langle s_r\rangle = r$, while the variance behaves as $\sigma^2(s_r) = (2/\pi^2)[\ln(2\pi(r+1)) + 1 - \pi^2/8] - 1/6$; see, for example, [19]. The slow growing of the variance with the multiplicity of spacing is a consequence of the constancy of the unfolded mean level density used to measure fluctuations of the GOE spectrum. The size of spacing fluctuations between any pairs of energy levels are quite similar and relatively small, $\sigma(s_r)/r \ll 1$. In the following, we approximate $\sigma^2(s_r)$ by a constant smaller than 1 for any r.

An important feature of the rth order spacing distribution $P(s_r)$ is that it approaches a Gaussian distribution even at moderate values of r [20]. Therefore,

$$\overline{[E_\mu - E_\eta]^{-1}\cos([E_\mu - E_\eta][t-t']/\hbar)} \approx \int_{-\infty}^{+\infty} ds_r \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{[s_r - r]^2}{2\sigma^2}\right) \times s_r \cos([s_r][t-t']/[\hbar/D]), \tag{30}$$

where $s_r = (E_\mu - E_\eta)/D$ and $r = |\mu - \eta|$. Since $\sigma(s_r)/r \ll 1$, one can replace the term $1/s_r$ in Eq. (30) by $1/r$ and perform integration of the remaining integral function analytically. With this, we get for the ensemble averaged memory kernel (29) the following expression,

$$\overline{\gamma}(t-t') = 2\sum_\mu \overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2} \frac{\cos([\overline{E}_\mu - \overline{E}_\eta][t-t']/\hbar)}{\overline{E}_\mu - \overline{E}_\eta} \times \exp\left(-\frac{(t-t')^2}{2[\hbar/\sigma D]^2}\right), \tag{31}$$

where $\overline{E}_\mu - \overline{E}_\eta = rD$.

We see that the ensemble averaging procedure (29)–(31) leads to the decay of the ensemble averaged memory kernel with time. In fact, the decay time of the memory kernel, $\hbar/\sigma D$, even at relatively small excitations of the nucleus is much larger than characteristic time scales of nuclear collective dynamics $\tau_{\mathrm{coll}} \sim 10^{-22}$–$10^{-20}$ s; see estimation for $\hbar/D$ given at the end of Sec. II A. Consequently, the exponential term in Eq. (31) can be very well approximated by 1.

Thus, we make the next step evaluating the ensemble averaged memory kernel (31). Now we consider the level density ρ of the many-body states to be a growing function of the excitation energy E, relative the initial energy $\overline{E}_\eta, E \equiv \overline{E}_\mu - \overline{E}_\eta$ and make the replacement $\sum_\mu \to \int_{-\infty}^{+\infty} \rho(E)dE$ in Eq. (31). Using the Taylor expansion of $\rho(E)$ around $E = 0$, one can show that

$$\overline{\gamma}(t-t') = \sqrt{8\pi}\sum_{l=0}^{+\infty} \frac{(-1)^l}{(2l+1)!} \cdot \left.\frac{d^{(2l+1)}\rho}{dE^{(2l+1)}}\right|_{E=0} \times \frac{d^{(2l)}}{d(t''/\hbar)^{(2l)}} \left\{\mathrm{Re}\left(\mathcal{F}\left[\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}\right](t''/\hbar)\right)\right\}, \quad t'' = t - t', \tag{32}$$

where

$$\mathcal{F}[\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}](t''/\hbar) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}(E) e^{-i(t''/\hbar)E} dE \tag{33}$$

is the Fourier transform of the squared matrix elements $\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}$, which are assumed to be even functions of E. Odd terms in expansion disappear because of the symmetry of the energy spectrum. We shall proceed by considering only the first term in the infinite sum (32),

$$\overline{\gamma}(t-t') = \sqrt{8\pi}\frac{d\rho}{dE}\mathrm{Re}(\mathcal{F}[\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}](t''/\hbar)), \quad t'' = t - t', \tag{34}$$

which is equivalent to a local quadratic expansion of the many-body level density ρ in the vicinity of the energy of the initially occupied state $|\eta\rangle$. This general expression for the memory kernel is quite reasonable provided that the energy changes of the fast quantum environment are relatively small.

In the case (34), the time properties of the memory kernel $\overline{\gamma}(t-t')$ are given by the Fourier transform (33) of the ensemble averaged values of the squared matrix elements $\overline{|\langle\mu|\mathcal{A}|\eta|\rangle|^2}$. To analyze the E dependence of $\overline{|\langle\mu|\mathcal{A}|\eta|\rangle|^2}$ within the random matrix approach (2)–(5), we expand the perturbed states $|\mu\rangle$ in the unperturbed ones $|n\rangle$ [see Eq. (8)], yielding for the ensemble averaged values of the squared matrix elements,

$$\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2} = \sum_n \overline{|b_{\mu n}|^2} \cdot \overline{|b_{\eta n}|^2} \cdot \overline{A_n^2}. \tag{35}$$

Since the coupling operator (4) is diagonal in basis $|n\rangle$, $\overline{A_n^2} = 2\sigma_A/N$, where σ_A^2 is the spreading of the slopes of the unperturbed energies, and N is the size of the random matrices in Eqs. (2)–(5). The ensemble averaged strength functions $\overline{|b_{\mu n}|^2} = \overline{|b_{\mu n}|^2}(\overline{E}_n)$ in Eq. (35) are peaked around the perturbed average energy $\overline{E}_\mu$.

For small matrix elements of the residual interaction ($\Delta \ll 1$), $\overline{|b_{\mu n}|^2}$ are of Breit-Wigner shape [21,22],

$$\overline{|b_{\mu n}|^2} = \frac{1}{\pi\rho} \cdot \frac{\Gamma_\mu/2}{(\overline{E}_n - \overline{E}_\mu)^2 + (\Gamma_\mu/2)^2}, \tag{36}$$

with the spreading width Γ_μ obtained from Fermi's Golden Rule (9), and where the level density ρ is taken as constant. We use the expression for the spreading width derived in Ref. [23],

$$\Gamma_\mu = 0.039\left(\frac{A}{160}\right) E_{\text{exc}}^{3/2} \text{ MeV}, \tag{37}$$

where the damping of a one-quasiparticle state was assumed to be $\Gamma^{\downarrow}_{\nu=1} = E^2_{\text{exc}}/15$ MeV. Here A is mass number and E_{exc} is the excitation energy of the nucleus. Replacing the summation in (35) by the integration $\sum_n \to \rho\int_{-\infty}^{+\infty} d\overline{E}_n$, we obtain

$$\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2} = \frac{2\sigma_A^2}{\pi\rho N} \cdot \frac{\Gamma_\mu}{(\overline{E}_\mu - \overline{E}_\eta)^2 + \Gamma_\mu^2}, \tag{38}$$

which gives for the memory kernel (34) the following expression

$$\overline{\gamma}(t-t') = \frac{4\sigma_A^2}{N}\frac{1}{\rho}\frac{d\rho}{dE}\exp\left(-\frac{|t-t'|}{(\hbar/\Gamma_\mu)}\right). \tag{39}$$

The memory kernel $\overline{\gamma}(t-t')$ is mainly concentrated in the time interval $|t-t'| \leqslant \tau$, where $\tau = \hbar/\Gamma_\mu$. Taking Γ_μ from Eq. (37) and considering the nucleus as a Fermi gas with temperature T that is related to the excitation energy by $E_{\text{exc}} = (A/10)T^2$ MeV, we get

$$\tau \approx \left(\frac{A}{230}\right)^{-5/2} T^{-3} 10^{-22} \text{ s}. \tag{40}$$

For example, for low-energy ($T < 1$ MeV) fission of heavy nuclei with mass numbers $A = 200$–260, the memory kernel (39) is spread out over times $\tau \sim 10^{-22}$–10^{-21} s, which are comparable to typical saddle-to-scission times for nuclear descent from the fission barrier [24]. Such a macroscopic retardation of the response force (28) of the fast quantum mechanical subsystem indeed implies that the dynamics (14)–(15) of the slow collective variable is subject to memory effects.

Note that an exponential form $\exp(-|t-t'|/\tau)$ of the memory kernel of the retarded friction force in the macroscopic equations of motion for the nuclear collective deformations is used in the Fermi-liquid model [25] and in the linear response theory [14]. In these approaches, τ is a relaxation time of the collective excitations, which in the limit of small temperatures T of the nucleus goes as $\tau \sim 1/T^2$, and which can be compared to the temperature dependence shown in Eq. (40), $\tau \sim 1/T^3$.

For fairly large matrix elements of the residual interaction ($\Delta < 1$, when the spectrum is mixed), the ensemble averaged strength functions of the perturbed states in (35) may have a Gaussian shape [22],

$$\overline{|b_{\mu n}|^2} = \frac{1}{\sqrt{2\pi}\rho\Gamma_G} \cdot \exp\left[-\frac{(\overline{E}_n - \overline{E}_\mu)^2}{2\Gamma_G^2}\right], \tag{41}$$

and are characterized by the spreading width Γ_G that is larger than Γ_μ. This leads to

$$\overline{\gamma}(t-t') = \frac{4\sigma_A^2}{N}\frac{1}{\rho}\frac{d\rho}{dE}\exp\left[-\frac{(t-t')^2}{(\hbar/\Gamma_G)^2}\right]. \tag{42}$$

Here the memory kernel has a smaller time spread $\tau = \hbar/\Gamma_G$ than in the previous case, i.e., when $\overline{\gamma}(t-t')$ (34) was obtained with the ensemble averaged strength functions of the Breit-Wigner shape (36). In addition, the memory kernel (42) will result in a much stronger decay of correlations between the values of the slow collective velocity $\dot{Q}$ at different times t and t' in the equation of motion (14)–(15) and, therefore, will give rise to less pronounced non-Markovian features of the collective dynamics as compared to $\overline{\gamma}(t-t')$ given by Eq. (39).

In the GOE limit ($\Delta = 1$, fully chaotic energy spectrum of the quantum mechanical subsystem), the ensemble averaged strength functions are constant, $\overline{|b_{\mu n}|^2} = 1/N$. We then get

$$\overline{\gamma}(t-t') = \frac{4\pi\sigma_A^2}{N^2}\frac{1}{\rho}\frac{d\rho}{dE}\delta(t-t'), \tag{43}$$

corresponding to the pure viscous response (28)

$$\overline{F}_{\text{fast}} = -\gamma_0\dot{Q}(t), \tag{44}$$

with a friction coefficient $\gamma_0 = (4\sigma_A^2/N)(d\rho/dE)/\rho$. Note that within the Fermi gas model for sufficiently high excitations of the nucleus, when the nuclear level density ρ grows exponentially with the excitation energy E_{exc}, $\rho \sim \exp(2\sqrt{(10/A)E_{\text{exc}}})$, the friction coefficient γ_0 behaves with the temperature $T = \sqrt{(10/A)E_{\text{exc}}}$ as $\gamma_0 \sim 1/T$.

IV. MEMORY EFFECTS AND FRICTION

From the studies presented in the previous section, we conclude that

(i) Coupling of the slow collective degree of freedom Q to the fully chaotic quantum environment (2)–(5) formed by the fast nucleonic degrees of freedom gives rise to the appearence of the usual (Markovian) friction force in the macroscopic equation of motion for Q, (14) and (15),

$$M\ddot{\overline{Q}} = -C(\overline{Q} - Q_0) - \gamma_0\dot{\overline{Q}}, \quad (45)$$

while

(ii) the collective dynamics (14) and (15) becomes memory dependent if the quantum nucleonic environment has mixed (or almost regular) energy spectrum,

$$M\ddot{\overline{Q}} = -C(\overline{Q} - Q_0) - \int_0^t \overline{\gamma}(t - t')\dot{\overline{Q}}(t')dt'. \quad (46)$$

What is the role of the memory effects caused by the retarded force in the motion of the slow collective degree of freedom (46)? To answer this question, we show in the Appendix that the retarded force can be split into conservative and friction parts, such that

$$\int_0^t \overline{\gamma}(t - t')\dot{\overline{Q}}(t')dt' = \mathcal{C}(t)[\overline{Q}(t) - Q_0] + \mathcal{G}(t)\dot{\overline{Q}}(t). \quad (47)$$

The memory effects are reflected here in the time dependence of the coefficients $\mathcal{C}$ and $\mathcal{G}$ determined by the explicit form of the memory kernel $\overline{\gamma}(t - t')$, see Eqs. (A8) and (A9), correspondingly.

First of all, we see that the memory effects lead to the renormalization of the stiffness C of the slow variable's potential energy,

$$\tilde{C} = C + \mathcal{C}(t). \quad (48)$$

As is demonstrated in Refs. [11,12] for the retarded force of the form (47), the dynamical correction $\mathcal{C}(t)$ to the "adiabatic" stiffness C always stabilizes the slow collective subsystem, i.e., increases its total stiffness $\tilde{C}$. Moreover, the time dependence of $\mathcal{C}$ leads to the peculiarity in the motion of the collective deformation parameter: the nucleus will undergo characteristic shape oscillations. One can say that with the growth of the correlations between values of the collective velocity at different moments of time, the relative role of $\mathcal{C}(t)$ will increase. In other words, the stronger the memory effects, the more elastic the response of the quantum nucleonic bath. Microscopically, the dynamical correction to the adiabatic stiffness of the slow collective subsystem is coming from the time-reversible transitions of the probability from occupied to unoccupied states of the fast quantum mechanical environment giving rise to its time-reversible energy change.

The other manifestation of the memory effects is the presence of friction in the collective dynamics described by the time-dependent friction coefficient $\mathcal{G}(t)$. The friction is defined by the residual interaction acting between all eigenstates of the fast subsystem, and its relative contribution to the retarded force (47) increases with the increase of the strength of this interaction, i.e., with the increase of the chaoticity parameter Δ. The microscopic origin of the friction is the time-irreversible growth of the ensemble averaged occupations $\overline{|d_\mu|^2}$. This would lead to the heating of the fast nucleonic degrees of freedom's subsystem provided that its mean level density ρ is an increasing function of the energy. The latter is an important point because otherwise, at $\rho = \text{const}$, the energy of the fast subsystem $\overline{\mathcal{E}}_{\text{fast}}(t)$ averaged over all ensembles of the random matrices (2)–(5) will be constant in time, $\overline{\mathcal{E}}_{\text{fast}}(t) = \overline{E}_\eta$, since the occupation probability $\overline{|d_\mu|^2}$ concentrated initially at the level with the energy E_η will spread out symmetrically on higher- and lower-lying states. This fact is simply expressed in terms of the memory kernel (31) as

$$\overline{\gamma}(t - t') = 2\sum_\mu \overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}\frac{\cos([\overline{E}_\mu - \overline{E}_\eta][t - t']/\hbar)}{\overline{E}_\mu - \overline{E}_\eta} = 0, \quad (49)$$

provided that the ensemble averaged values of the squared coupling matrix elements $\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}$ are even functions of the energy difference $\overline{E}_\mu - \overline{E}_\eta$. It is obvious that the asymmetrical distribution of $\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}$ with $\overline{E}_\mu - \overline{E}_\eta$ will lead to the heating of the fast quantum mechanical subsystem, $\overline{\gamma} \neq 0$, even at $\rho = \text{const}$.

It should be stressed that the separation of the retarded force (47) is general in the sense that the memory integral in (47) can always be separated into time-reversible (conservative) and time-irreversible (friction) parts. For the linear dynamics of the slow variable, the separation can be performed explicitly, while for nonlinear dynamics it cannot. In the latter case, the friction part of the retarded force (which can be a more complicated function of the coordinate and velocity) will be determined by the odd powers of the velocity, while the conservative one will be the function of the even powers of $\dot{\overline{Q}}$.

V. SUMMARY

We have studied dissipative motion of the slow nuclear collective variable Q in the complex quantum environment formed by the fast nucleonic degrees of freedom. The fast quantum mechanical subsystem was considered with the help of the ensembles of the time-dependent random matrices $H_{\text{fast}}[Q(t)]$ (2)–(5), which are linearly dependent on the slow coordinate $Q(t)$. The coupling operator $\mathcal{A}$ (4) between the fast and slow subsystems was taken diagonally on the basis of the many-body states of the fast Hamiltonian. The complexity of the quantum mechanical subsystem was generated by the inclusion of the residual interaction, acting between all eigenstates of $H_{\text{fast}}[Q(t)]$, and which can be controlled by the relative strength of the interaction Δ, with the two limits, $\Delta = 0$ and $\Delta = 1$, corresponding to the regular and fully chaotic energy spectrum of the system. The initial configuration of the quantum mechanical subsystem is described by the occupation of the middle eigenstate $|\eta\rangle$ of the Hamiltonian H_{fast}.

The dynamics of the slow collective variable $Q(t)$ (12) was treated as a small-amplitude motion around the initial value Q_0 of a classical particle with constant mass M in the field of the potential force $-C(Q - Q_0)$ and subject to the influence of the irregular force $-1/\dot{Q}(d\mathcal{E}_{\text{fast}}/dt)$ caused by the coupling of the collective motion to the nucleonic bath with the energy $\mathcal{E}_{\text{fast}}(t)$ (11).

We have shown that for relatively small variations of the slow variable $Q(t) - Q_0$, the ensemble averaged response of the nucleonic bath $\overline{-1/\dot{Q}(d\mathcal{E}_{\text{fast}}/dt)}$ can be represented as the memory integral $-\int_0^t \overline{\gamma}(t - t')\dot{\overline{Q}}(t')dt'$. To evaluate the memory kernel $\overline{\gamma}(t-t')$ (31), we represented it as an infinite sum (32) of terms containing a product of derivatives of the many-body level density of the fast quantum environment ρ and the time derivatives of the Fourier transform of the squared coupling matrix element $\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}$. By making a local expansion for the level density around the energy $\overline{E}_\eta$ of the initially occupied state $|\eta\rangle$, we took only the first term (34) in the sum (32), which is proportional to the Fourier transform of the matrix element itself. Our next step was to measure how the chaoticity of the fast quantum mechanical subsystem (2)–(5) is reflected in the properties of $\overline{|\langle\mu|\mathcal{A}|\eta\rangle|^2}$. For that purpose, we expanded the perturbed many-body states $|\mu\rangle$ of the fast Hamiltonian $H_{\rm fast}$ in terms of the unperturbed ones $|n\rangle$ (8) and considered its ensemble averaged strength functions $\overline{|b_{\mu n}|^2}$.

When the matrix elements of the residual interaction are relatively small (which corresponds to the almost regular energy spectrum of the quantum mechanical subsystem), the distribution of $\overline{|b_{\mu n}|^2}$ as a function of the perturbed energy $\overline{E}_\mu$ is of the Breit-Wigner shape (36), see Ref. [21]. The distribution of the strength functions (36) is characterized by the spreading width of the perturbed states Γ_μ, which can be estimated in the same way [Eq. (37)] as in [23]. In this case, we get the memory kernel $\overline{\gamma}(t-t')$ of the exponential form, $\overline{\gamma}(t-t')\sim\exp(-|t-t'|/[\hbar/\Gamma_\mu])$, concentrated on the macroscopic time range $\tau=\hbar/\Gamma_\mu$, i.e., on typical times of the collective motion in nuclei. This implies the presence of the memory effects in the dynamics (14) and (15) of the slow collective degree of freedom. With the growth of the chaoticity of the fast quantum environment, the functional form of the memory kernel (34) is changing. Thus, for sufficiently large matrix elements of the residual interaction, the distribution of the strengths $\overline{|b_{\mu n}|^2}$ becomes Gaussian (41) [22,26] with the larger spreading width Γ_G, which results in the memory kernel $\overline{\gamma}(t-t')\sim\exp\{-(t-t')^2/[\hbar/\Gamma_G]^2\}$, see Eq. (42). Such memory kernel of the retarded force (28) in the equation of motion (14) and (15) for the slow collective variable will lead to much weaker memory effects. The non-Markovian features of the collective dynamics disappear in the limit of the fully chaotic energy spectrum of the quantum mechanical subsystem, when $\overline{|b_{\mu n}|^2}$ are constant. Here the response of the fast quantum environment (28) is given by the usual (Markovian) friction force $-\gamma_0\dot{\overline{Q}}(t)$.

To measure the influence of the memory effects on the dynamics of the slow collective variable, we have split analytically the retarded force $-\int_0^t\overline{\gamma}(t-t')\dot{\overline{Q}}(t')dt'$ into the conservative $\mathcal{C}(t)[\overline{Q}(t)-Q_0]$ and friction $\mathcal{G}(t)\dot{\overline{Q}}(t)$ forces with the time-dependent stiffness $\mathcal{C}$ (A8) and friction $\mathcal{G}$ (A9) coefficients determined by the explicit form of the memory kernel $\overline{\gamma}(t-t')$. The conservative and friction parts of the retarded response force are defined, correspondingly, by the time-reversible and time-irreversible probability transitions from occupied to unoccupied energy levels of the fast nucleonic bath.

In the paper, we have studied how the memory effects for the linear average collective dynamics (46) are determined by the complexity of the nucleonic quantum mechanical subsystem (2)–(5). One can expect that the relative role of the memory effects does not depend exclusively on the structure of the fast quantum environment itself, but also on the initial excitation of the slow collective subsystem (12), measured by the initial velocity $\dot{Q}_0$, and the characteristic time of the changes of the slow variable $\overline{Q}(t)$, $\tau_{\rm slow}\sim\sqrt{M/|C|}$. In this case one has to consider, in principle, nonlinear dynamics of $\overline{Q}$, arising either by the coordinate-dependent nuclear mass parameter M or by the higher-order corrections [in $(\overline{Q}-Q_0)\sim\dot{Q}_0\tau_{\rm slow}$] to the response force (28).

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APPENDIX: SPLITTING OF THE RETARDED FORCE

In this Appendix, we demonstrate how the retarded force in the equation of motion for the slow variable

$$M\ddot{\overline{Q}}=-C(\overline{Q}-Q_0)-\int_0^t\overline{\gamma}(t-t')\dot{\overline{Q}}(t')dt' \quad \text{(A1)}$$

can be split into the conservative and friction forces,

$$\int_0^t\overline{\gamma}(t-t')\dot{\overline{Q}}(t')dt'=\mathcal{C}(t)[\overline{Q}(t)-Q_0]+\mathcal{G}(t)\dot{\overline{Q}}(t). \quad \text{(A2)}$$

First, we find the analytical solution to the equation of motion with the help of the Laplace transformation,

$$\overline{Q}(t)=Q_0+A(t)Q_0+B(t)\dot{Q}_0, \quad \text{(A3)}$$

where

$$A(t)=\int_0^{+\infty}\frac{s}{Ms^2+\hat{\gamma}(s)s+C}e^{st}ds, \quad \text{(A4)}$$

$$B(t)=\int_0^{+\infty}\frac{1}{Ms^2+\hat{\gamma}(s)s+C}e^{st}ds, \quad \text{(A5)}$$

and

$$\hat{\gamma}(s)=\int_0^{+\infty}\overline{\gamma}(t)e^{-st}dt. \quad \text{(A6)}$$

Then, we construct a second-order differential equation, which has the same solution (A3) as the integro-differential equation (A1). Let us write it in the form

$$M\ddot{\overline{Q}}=-C(\overline{Q}-Q_0)-\mathcal{C}(t)(\overline{Q}-Q_0)-\mathcal{G}(t)\dot{\overline{Q}}, \quad \text{(A7)}$$

with some unknown functions $\mathcal{C}(t)$ and $\mathcal{G}(t)$. Since Eq. (A7) has two linearly independent solutions $A(t)Q_0$ and $B(t)\dot{Q}_0$, one obtains

$$M\ddot{A}=-CA-\mathcal{C}A-\mathcal{G}\dot{A},$$
$$M\ddot{B}=-CB-\mathcal{C}B-\mathcal{G}\dot{B},$$

which defines $\mathcal{C}$ and $\mathcal{G}$. Solving this system of equations, we get

$$\mathcal{C}(t) = M\frac{\ddot{A}(t)\dot{B}(t) - \dot{A}(t)\ddot{B}(t)}{\dot{A}(t)B(t) - \dot{B}(t)A(t)} - C, \tag{A8}$$

and

$$\mathcal{G}(t) = M\frac{\ddot{B}(t)A(t) - \dot{A}(t)B(t)}{\dot{A}(t)B(t) - \dot{B}(t)A(t)}. \tag{A9}$$

Formal comparison of Eqs. (A1) and (A7), having the same solution (A3), leads to an expression (A2) for the retarded force $\int_0^t \overline{\gamma}(t-t')\dot{\overline{Q}}(t')dt'$.

One can check the result (A2) in the two limiting cases: for the Markovian dynamics, $\overline{\gamma}(t-t') = 2\gamma_0\delta(t-t')$ [when the retarded force becomes an ordinary friction force $\gamma_0 \cdot \dot{\overline{Q}}(t)$], and in the opposite limit of the constant memory kernel, $\overline{\gamma}(t-t') = C_0$ [when the retarded force is given by the pure conservative force $C_0 \cdot [\overline{Q}(t) - Q_0]$. In the first case, we obtain from Eqs. (A8) and (A9),

$$\mathcal{C}(t) = C_0, \quad \mathcal{G}(t) = 0, \tag{A10}$$

while in the second case,

$$\mathcal{C}(t) = 0, \quad \mathcal{G}(t) = \gamma_0, \tag{A11}$$

as it should be.

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