Polarization observables for two-pion production off the nucleon

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We develop polarization observables for the processes $\gamma N \to \pi\pi N$ and $\pi N \to \pi\pi N$, using both a helicity and hybrid helicity-transversity basis. Such observables are crucial if processes that produce final states consisting of a spin-1/2 baryon and two pseudoscalar mesons are to be fully exploited for baryon spectroscopy. We derive relationships among the observables, as well as inequalities that they must satisfy. We also discuss the observables that must be measured in "complete" experiments and briefly examine the prospects for measurement of some of these observables in the near future.

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I. INTRODUCTION AND MOTIVATION

Polarization asymmetries are an essential ingredient in the interpretation of various meson production reactions in terms of the various resonances that contribute to the processes as real or virtual intermediate states. For instance, much of the information that we have on the light baryon resonances has been garnered from pion-nucleon and kaon-nucleon scattering experiments. In addition, photoproduction experiments have augmented the database of measurements that provide information on these resonances. The differential and total cross sections for these reactions, together with various polarization observables, are used to extract the amplitudes for the process, and these are then interpreted as arising from a number of resonant and nonresonant contributions [1,2].

For processes in which the final state consists of a nucleon (or a spin-1/2 baryon, in general) and a pseudoscalar meson, the polarization observables, their relationship to helicity or transversity amplitudes, and the measurements needed to extract each observable are all well documented [1–6]. For processes in which the final state contains two pseudoscalar mesons (along with a nucleon), the state of development is much less complete. For the most part, the final state with two pseudoscalar mesons and a nucleon (mainly $N\pi\pi$) has been treated as arising from either of the quasi-two-body states $\Delta\pi$ or $N\rho$, followed by the decay of the Δ or the ρ [4,7]. The $N\rho$ channel in particular, or more generally, the NV channel, where V is a vector meson, has received some attention in recent years [8]. A number of authors have formulated treatments based on more general quasi-two-body approaches [9].

This approach has been reasonably successful in the past, as the available data came from high-energy experiments. With today's facilities running at all energies from threshold up to relatively high energies, a more complete description of polarization observables for the three-body final state such as we have been describing is warranted. Indeed, such a description is essential to fully exploit the high-precision data that will be forthcoming. It must be stressed that experiments with more than a single pseudoscalar in the final state have been touted as our best hope for finding the so-called missing resonances [10]. It is therefore timely and crucial that the polarization observables for such processes be elucidated in a more general framework, one that goes beyond the quasi-two-body assumption.

The importance of polarization observables cannot be overstated. In the case of photoproduction of a single pseudoscalar meson, four complex amplitudes of some sort—helicity, transversity, or Lorentz covariant, for example—are required to describe the process. Because one phase will always remain ambiguous, this means that seven "numbers" are required at each kinematic point. The differential cross section provides information only on the sum of the absolute squares of these amplitudes. Polarization observables allow extraction of more information, including phases. For production of two pseudoscalar mesons, the same holds true. The process is described in terms of a number of amplitudes, and the differential cross section, in the form of mass distributions, angular distributions, or even fivefold differential distributions, provides information only on the sum of the absolute squares of these amplitudes. This is woefully inadequate for arriving at an unambiguous description of the process. As with the processes in which a single pseudoscalar meson is produced, polarization information is crucial.

The rest of this article is organized as follows. For definiteness, we refer to the final state that we treat as $N\pi\pi$, but the formalism we present is valid for any final state that consists of a spin-1/2 baryon and two pseudoscalar mesons. In addition, we also discuss final states with a single pseudoscalar meson in the final state, for the sake of comparison and completeness. In the next section we discuss the kinematics for the two- and three-body final states that we consider. In Sec. III we introduce the formalism and notation by discussing the processes $\pi N \to \pi N$ and $\pi N \to \pi\pi N$. In Sec. IV, we turn our attention to the processes $\gamma N \to \pi N$ and $\gamma N \to \pi\pi N$. In Sec. V, we discuss the prospects for measurements of some of these observables at present facilities, especially the Thomas Jefferson National Accelerator Facility, as well as our conclusions.

II. KINEMATICS AND CROSS SECTIONS

A. Introduction

For all of the processes we discuss, we work in the center-of-mass (c.m.) frame and define the momentum of the beam particle (pion or photon) to be

$$k = (\omega, 0, 0, K) \equiv (\omega, \mathbf{K}), \tag{1}$$

with the momentum of the target nucleon being

$$p_1 = (\sqrt{s} - \omega, 0, 0, -K). \tag{2}$$

For the recoil nucleon, we choose

$$p_2 = (E, -Q\sin\theta, 0, -Q\cos\theta) \equiv (E, -\mathbf{Q}). \tag{3}$$

The momentum of the single pion (for $N\pi$ final states) or the pair of pions (for $N\pi\pi$ final states) is chosen to be

$$q = (\sqrt{s} - E, \mathbf{Q}). \tag{4}$$

For the $N\pi\pi$ final state, the pion momenta are denoted q_1 and q_2, with $q = q_1 + q_2$. The x, y, and z axes are then defined as

$$\hat{z} = \hat{\mathbf{K}}, \quad \hat{y} = \frac{\mathbf{K}\times\mathbf{Q}}{|\mathbf{K}\times\mathbf{Q}|}, \quad \hat{x} = \hat{y}\times\hat{z}. \tag{5}$$

It is also useful to define a set of axes in which the z' axis is parallel to the momentum of the recoil nucleon. In this system, the y axis coincides with the y axis of the collision plane. In terms of momenta, the axes for this system are

$$\hat{z}' = \hat{\mathbf{P}}, \quad \hat{y}' = \frac{\mathbf{P}\times\mathbf{K}}{|\mathbf{P}\times\mathbf{K}|}, \quad \hat{x}' = \hat{y}'\times\hat{z}'. \tag{6}$$

B. $\pi N \to \pi N$

In the c.m. frame, we choose the momenta of the initial pion and nucleon as in Eqs. (1) and (2), whereas those for the final pion and nucleon are, respectively,

$$\begin{aligned} q &= (\omega, Q\sin\theta, 0, Q\cos\theta), \\ p_2 &= (\sqrt{s} - \omega, -Q\sin\theta, 0, -Q\cos\theta), \quad Q = K, \end{aligned} \tag{7}$$

and $\sqrt{s}$ is the total center-of-mass energy. The Mandelstam variables of interest are

$$\begin{aligned} s &= (k + p_1)^2 = (q + p_2)^2, \\ t &= (q - k)^2 = (p_1 - p_2)^2 \\ &= 2m^2 - 2(\sqrt{s} - \omega)^2 + 2K^2\cos\theta \\ &= -2K^2(1 - \cos\theta), \end{aligned} \tag{8}$$

where m is the nucleon mass. The energies and momenta are

$$\begin{aligned} \omega &= \frac{s + \mu^2 - m^2}{2\sqrt{s}}, \\ Q &= K = \lambda^{1/2}(s, m^2, \mu^2)/(2\sqrt{s}), \end{aligned} \tag{9}$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$ is Källen's function and μ is the pion mass.

The cross section for the process is

$$\begin{aligned} d\sigma = {} & \frac{\overline{|\mathcal{M}_{fi}|^2}}{4\sqrt{(p_1.k)^2 - m^2\mu^2}}(2\pi)^4\delta^4(p_1 + k - p_2 - q) \\ & \times \frac{d^3p_2}{(2\pi)^3 2E}\frac{d^3q}{(2\pi)^3 2\omega}. \end{aligned} \tag{10}$$

After integration, this yields

$$d\sigma = \frac{\overline{|\mathcal{M}_{fi}|^2}d\Omega_2}{16(2\pi)^2 s}.$$

C. $\gamma N \to \pi N$

In the c.m. frame, for real photons, the beam and target momenta are again as in Eqs. (1) and (2), but now $K = \omega$. For the pion and nucleon in the final state, the momenta are

$$\begin{aligned} q &= (\omega', Q\sin\theta, 0, Q\cos\theta), \\ p_2 &= (\sqrt{s} - \omega', -Q\sin\theta, 0, -Q\cos\theta). \end{aligned} \tag{11}$$

The Mandelstam variable s has the same definition as before, but t now takes the form

$$\begin{aligned} t &= (q - k)^2 = (p_1 - p_2)^2 \\ &= 2m^2 - 2(\sqrt{s} - \omega)(\sqrt{s} - \omega') + 2QK\cos\theta, \\ K &= \omega = \frac{s - m^2}{2\sqrt{s}}, \\ \omega' &= \frac{s + \mu^2 - m^2}{2\sqrt{s}}, \quad Q = \lambda^{1/2}(s, m^2, \mu^2)/(2\sqrt{s}). \end{aligned} \tag{12}$$

After phase-space integrations, the cross section for this process is

$$d\sigma = \frac{Q\,\overline{|\mathcal{M}_{fi}|^2}d\Omega_2}{16(2\pi)^2 sK}.$$

D. $\pi N \to \pi\pi N$

For this process, k, p_1, and p_2 are defined as in the $\pi N \to \pi N$ process. q_1 and q_2 are the momenta of the two final state pions, and momentum conservation gives

$$p_1 + k = p_2 + q_1 + q_2. \tag{13}$$

The momentum of the recoiling nucleon is taken to be

$$p_2 = \left(\frac{s - s_{\pi\pi} + m^2}{2\sqrt{s}}, -Q\sin\theta, 0, -Q\cos\theta\right), \tag{14}$$

where

$$s_{\pi\pi} = (q_1 + q_2)^2 \tag{15}$$

and

$$Q = \frac{\lambda^{1/2}(s, s_{\pi\pi}, m^2)}{2\sqrt{s}}. \tag{16}$$

Here, we are using the recoiling nucleon, or, more precisely, the recoiling pair of pions, to define the collision plane. The momentum of one of the pions may be written

$$\vec{q}_1 = Q_1(\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1), \tag{17}$$

where Q_1, θ_1, and ϕ_1 can be written in terms of $s, s_{\pi\pi}, \theta$, and the angles describing the motion of the pair of pions in their c.m. frame. The complicated expressions that result are not not reproduced here, particularly as they are not crucial for the discussion that follows.

We define the Mandelstam variables s and t as

$$s = (k + p_1)^2, \qquad t = (p_1 - p_2)^2. \tag{18}$$

In addition, we may define a number of other Mandelstam variables as

$$\begin{aligned} s_{N\pi_1} &= (p_2 + q_1)^2, & s_{N\pi_2} &= (p_2 + q_2)^2, \\ t_1 &= (k - q_1)^2, & t_2 &= (k - q_2)^2. \end{aligned} \tag{19}$$

Note that $s_{\pi\pi}$, $s_{N\pi_1}$, and $s_{N\pi_2}$ are not all independent, as they satisfy

$$s_{\pi\pi} + s_{N\pi_1} + s_{N\pi_2} = s + m^2 + 2\mu^2. \tag{20}$$

The differential cross section for this process is described in terms of five kinematic variables. These may be, for instance, two Lorentz invariants and three angles. One obvious choice for one of the invariants is s. The choice of the other four quantities can be fairly arbitrary and will depend on what information is being presented. One choice is the scattering angle of the nucleon, θ, or equivalently, t. For the other three variables, we can choose, for example, $s_{\pi\pi}$ and $d\Omega^\star_{\pi\pi} \equiv d\theta^\star_{\pi\pi} d\phi^\star_{\pi\pi}$, consistent with the way we define the momenta. Another equally valid choice would be $s_{N\pi_1}$ and $d\Omega^\star_{N\pi_1}$, where the solid angle is defined in the rest frame of the nucleon-pion pair. In this work, we choose the kinematic variables $s, t, s_{\pi\pi}$, and $d\Omega^\star_{\pi\pi}$.

The differential cross section is

$$\begin{aligned} d\sigma = &\frac{\overline{|\mathcal{M}|^2}}{4\sqrt{(k\cdot p_1)^2 - m^2\mu^2}} \\ &\times (2\pi)^4 \delta^4(p_1 + k - p_2 - q_1 - q_2) \\ &\times \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3q_1}{(2\pi)^3 2\omega_1} \frac{d^3q_2}{(2\pi)^3 2\omega_2}, \end{aligned} \tag{21}$$

where $\mathcal{M}$ is the amplitude for the transition, E_2 is the energy of the recoiling nucleon, and ω_i is the energy of the ith pion. Carrying out the integrations using standard techniques yields

$$\begin{aligned} \frac{d^5\sigma}{ds_{\pi\pi} d\Omega^\star_{\pi\pi} d\cos\theta} = &\frac{1}{4} \frac{1}{128(2\pi)^4 s^{3/2} K s_{\pi\pi}} \\ &\times \overline{|\mathcal{M}|^2} \lambda^{\frac{1}{2}}(s_{\pi\pi}, \mu^2, \mu^2) \lambda^{\frac{1}{2}}(s, s_{\pi\pi}, m^2), \end{aligned} \tag{22}$$

where $4\mu^2 \leqslant s_{\pi\pi} \leqslant (s - m)^2$ and $K = \lambda^{1/2}(s, m^2, \mu^2)/(2\sqrt{s})$.

E. $\gamma N \to \pi\pi N$

The kinematic treatment of this process is very much the same as for the process $\pi N \to \pi\pi N$. The main difference arises in the fact that, for the initial photon, $k^2 = 0$. We therefore do not discuss the kinematics of this process any further at this point, except to write down the form for the differential cross section. This is

$$\begin{aligned} \frac{d^5\sigma}{ds_{\pi\pi} d\Omega^\star_{\pi\pi} d\cos\theta} = &\frac{1}{4} \frac{1}{128(2\pi)^4 (s - m^2) s s_{\pi\pi}} \\ &\times \overline{|\mathcal{M}|^2} \lambda^{\frac{1}{2}}(s_{\pi\pi}, \mu^2, \mu^2) \lambda^{\frac{1}{2}}(s, s_{\pi\pi}, m^2). \end{aligned} \tag{23}$$

III. OBSERVABLES IN $\pi N \to \pi N$ AND $\pi N \to \pi\pi N$

For the processes $\pi N \to \pi N$ and $\pi N \to \pi\pi N$, the matrix element $\mathcal{M}$ can be written

$$i\mathcal{M} = \chi^\dagger(\mathcal{A} + \vec{\sigma}\cdot\vec{\mathcal{B}})\phi, \tag{24}$$

where χ and ϕ are the Pauli spinors representing the final and initial nucleons, respectively. Here, the quantity $\mathcal{A} + \vec{\sigma}\cdot\vec{\mathcal{B}}$ is the most general 2×2 matrix that can be constructed, and $\mathcal{A}$ and $\vec{\mathcal{B}}$ are quantities that will contain all of the details of the "model" used to describe the particular reaction being considered. At this point, their exact form is of no consequence. For both processes, we choose the nucleon momenta as defined in Eqs. (2) and (7).

These two processes may be described in either the helicity or transversity basis. In the helicity basis, the axis of quantization of the spin of each nucleon is its direction of motion. For the initial nucleon, the helicity spinors are

$$\phi^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \phi^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{25}$$

whereas for the final nucleon, they are

$$\chi^+ = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \qquad \chi^- = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}. \tag{26}$$

In the transversity basis, the axis of quantization of the spin of each nucleon is the y axis, which is as previously defined. For either initial or final nucleon, the transversity spinors are

$$\phi_T^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \qquad \phi_T^- = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \tag{27}$$

where the plus and minus refer to the spin projection relative to the y axis.

For either of the processes being discussed,

$$\mathcal{A} + \vec{\sigma}\cdot\vec{\mathcal{B}} = \begin{pmatrix} \mathcal{A} + \mathcal{B}_3 & \mathcal{B}_- \\ \mathcal{B}_+ & \mathcal{A} - \mathcal{B}_3 \end{pmatrix} \equiv \begin{pmatrix} A_+ & B_- \\ B_+ & A_- \end{pmatrix}, \tag{28}$$

where we have defined

$$A_\pm \equiv \mathcal{A} \pm \mathcal{B}_3, \qquad B_\pm \equiv \mathcal{B}_1 \pm i\mathcal{B}_2, \tag{29}$$

and the $\mathcal{B}_i$ are the Cartesian components of $\vec{\mathcal{B}}$.

In terms of these, the four possible helicity amplitudes, $i\mathcal{M}_{\lambda_N\lambda'_N}$, are

$$\begin{aligned}
i\mathcal{M}_{++} &= A_- \cos\frac{\theta}{2} - B_- \sin\frac{\theta}{2} \equiv \mathcal{M}_1,\\
i\mathcal{M}_{+-} &= A_- \sin\frac{\theta}{2} + B_- \cos\frac{\theta}{2} \equiv \mathcal{M}_2,\\
i\mathcal{M}_{-+} &= -A_+ \sin\frac{\theta}{2} + B_+ \cos\frac{\theta}{2} \equiv \mathcal{M}_3,\\
i\mathcal{M}_{--} &= A_+ \cos\frac{\theta}{2} + B_+ \sin\frac{\theta}{2} \equiv \mathcal{M}_4.
\end{aligned} \tag{30}$$

In these equations, λ_N is the helicity of the target nucleon, whereas $\lambda_{N'}$ is that of the recoil nucleon. Note that in the form written above, A_- and B_- occur in one block of helicity amplitudes, whereas A_+ and B_+ occur in another block, with no "mixing" between the blocks. This makes inverting the equations very easy, yielding

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\left[(\mathcal{M}_1+\mathcal{M}_4)\cos\frac{\theta}{2} + (\mathcal{M}_2-\mathcal{M}_3)\sin\frac{\theta}{2}\right],\\
\mathcal{B}_1 &= \frac{1}{2}\left[(\mathcal{M}_2+\mathcal{M}_3)\cos\frac{\theta}{2} + (\mathcal{M}_4-\mathcal{M}_1)\sin\frac{\theta}{2}\right],\\
\mathcal{B}_2 &= \frac{i}{2}\left[(\mathcal{M}_2-\mathcal{M}_3)\cos\frac{\theta}{2} - (\mathcal{M}_1+\mathcal{M}_4)\sin\frac{\theta}{2}\right],\\
\mathcal{B}_3 &= \frac{1}{2}\left[(\mathcal{M}_4-\mathcal{M}_1)\cos\frac{\theta}{2} - (\mathcal{M}_2+\mathcal{M}_3)\sin\frac{\theta}{2}\right].
\end{aligned} \tag{31}$$

At this point, we have not yet used the parity properties of the helicity amplitudes. This is discussed later.

Two sets of transversity amplitudes may be defined. The first set are obtained by direct application of the transversity spinors defined above. We define these to be $ib_{\tau_N\tau'_N}$, where $\tau_i = \pm$ is the projection of the spin of the state i along the y axis (with the 1/2 suppressed), and these take the form

$$\begin{aligned}
ib_{++} &= \mathcal{A} + \mathcal{B}_2 \equiv b_1,\\
ib_{+-} &= -(\mathcal{B}_3 + i\mathcal{B}_1) \equiv b_2,\\
ib_{-+} &= -(\mathcal{B}_3 - i\mathcal{B}_1) \equiv b_3,\\
ib_{--} &= \mathcal{A} - \mathcal{B}_2 \equiv b_4.
\end{aligned} \tag{32}$$

The block structure is again apparent, and inverting these gives

$$\mathcal{A} = \frac{1}{2}(b_1+b_4), \qquad \mathcal{B}_1 = \frac{i}{2}(b_2-b_3),$$
$$\mathcal{B}_2 = \frac{1}{2}(b_1-b_4), \qquad \mathcal{B}_3 = -\frac{1}{2}(b_2+b_3).$$

For observables defined in terms of these transversity amplitudes, the x', y', and z' axes that define the Cartesian components of polarization observables coincide with the axes that define the initial state. This is because the transversity spinors contain no explicit information about the angles defining the motion of the recoil nucleon.

We can write the reaction rate I as

$$\rho_f I = I_0\big[1 + \vec{\Lambda}_i\cdot\vec{P} + \vec{\sigma}\cdot\vec{P}' + \Lambda_i^{\alpha}\sigma^{\beta'}\mathcal{O}_{\alpha\beta'}\big], \tag{33}$$

where $\vec{P}$ represents the polarization asymmetry that arises if the target nucleon is polarized, $\rho_f = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{P}')$ is the density matrix of the recoiling nucleon, and $\mathcal{O}_{\alpha\beta'}$ is the observable if the Cartesian α component of the target polarization is known and the β' component of the recoil polarization is measured. The primes indicate that the recoil observables, defined in terms of helicity amplitudes, are measured with respect to the set of axes x', y', and z', previously defined. If the observables are defined in terms of transversity amplitudes, the x', y', and z' axes are the same as the x, y, and z axes. $\vec{\Lambda}_i$ is the polarization of the initial nucleon.

The 16 polarization observables that are possible are shown in Table I. These 16 quantities are not all independent, as a number of relationships can be obtained among them. We first list six relationships that arise from considering the absolute magnitudes of the transversity amplitudes. These are

$$\begin{aligned}
&(P_{x'} \pm \mathcal{O}_{yx'})^2 + (P_{z'} \pm \mathcal{O}_{yz'})^2 = (1 \pm P_y)^2 - (P_{y'} \pm \mathcal{O}_{yy'})^2,\\
&(P_x \pm \mathcal{O}_{xy'})^2 + (P_z \pm \mathcal{O}_{zy'})^2 = (1 \pm P_{y'})^2 - (P_y \pm \mathcal{O}_{yy'})^2,\\
&(\mathcal{O}_{xx'} \pm \mathcal{O}_{zz'})^2 + (\mathcal{O}_{xz'} \mp \mathcal{O}_{zx'})^2 = (1 \pm \mathcal{O}_{yy'})^2 - (P_y \pm P_{y'})^2.
\end{aligned} \tag{34}$$

These six identities may be used to place limits on the absolute magnitudes of some of the observables. Because the left-hand sides of all six of these equations are positive definite, we obtain the inequalities

$$\begin{aligned}
|1 \pm P_y| &\geqslant |P_{y'} \pm \mathcal{O}_{yy'}|,\\
|1 \pm P_{y'}| &\geqslant |P_y \pm \mathcal{O}_{yy'}|,\\
|1 \pm \mathcal{O}_{yy'}| &\geqslant |P_y \pm P_{y'}|.
\end{aligned}$$

These are similar to the inequalities usually reported in the literature for pion photoproduction, for instance. In fact, simple rearrangement of the equations above allow a larger set of inequalities to be written. These are

$$\begin{aligned}
|1 \pm P_y| &\geqslant \{|P_{y'} \pm \mathcal{O}_{yy'}|, |P_{x'} \pm \mathcal{O}_{yx'}|, |P_{z'} \pm \mathcal{O}_{yz'}|\},\\
|1 \pm P_{y'}| &\geqslant \{|P_y \pm \mathcal{O}_{yy'}|, |P_x \pm \mathcal{O}_{xy'}|, |P_z \pm \mathcal{O}_{zy'}|\},\\
|1 \pm \mathcal{O}_{yy'}| &\geqslant \{|P_y \pm P_{y'}|, |\mathcal{O}_{xx'} \pm \mathcal{O}_{zz'}|, |\mathcal{O}_{xz'} \mp \mathcal{O}_{zx'}|\}.
\end{aligned}$$

Further manipulation of these inequalities leads to

$$\begin{aligned}
1 + P_y^2 &\geqslant \left\{P_{y'}^2 + \mathcal{O}_{yy'}^2, P_{x'}^2 + \mathcal{O}_{yx'}^2, P_{z'}^2 + \mathcal{O}_{yz'}^2\right\},\\
1 + P_{y'}^2 &\geqslant \left\{P_y^2 + \mathcal{O}_{yy'}^2, P_x^2 + \mathcal{O}_{xy'}^2, P_z^2 + \mathcal{O}_{zy'}^2\right\},\\
1 + \mathcal{O}_{yy'}^2 &\geqslant \left\{P_y^2 + P_{y'}^2, \mathcal{O}_{xx'}^2 + \mathcal{O}_{zz'}^2, \mathcal{O}_{xz'}^2 + \mathcal{O}_{zx'}^2\right\}.
\end{aligned}$$

Of the 16 observables, ten are therefore independent. We can further reduce the number of independent observables by using the relationships that exist among the phases of the transversity amplitudes. Because there will be one overall phase that is unmeasurable, only three of the relative phases are independent. This means that three relative phases can be eliminated, providing three more relationships among the observables, leaving seven independent observables. The three identities obtained this way can be displayed in a number of different ways, depending on, for instance, which phases

TABLE I. Polarization observables in single- and double-pion production using a pion beam, expressed in terms of helicity and transversity amplitudes. Variables labeled with a T require a polarized target with recoil polarization unmeasured, whereas those labeled with an R require an unpolarized target, but the recoil polarization is measured. Those denoted TR require polarized targets, with recoil polarization measured. The measurements required are shown by the pair $\{t, r\}$, which denote the component of the target (t) or recoil (r) polarization that must known or measured. For the target polarization, the x, y, and z axes are as defined in the text. The x', y', and z' axes are also defined in the text, as is the notation for the transversity amplitudes.

Observable	Helicity form	Transversity form	Expt. required	Type
I_0	$\|\mathcal{M}_1\|^2+\|\mathcal{M}_2\|^2+\|\mathcal{M}_3\|^2+\|\mathcal{M}_4\|^2$	$\|b_1\|^2+\|b_2\|^2+\|b_3\|^2+\|b_4\|^2$	$\{-;-\}$	
I_0P_x	$2\Re(\mathcal{M}_1\mathcal{M}_3^*+\mathcal{M}_2\mathcal{M}_4^*)$	$-2\Im(b_1b_3^*+b_2b_4^*)$	$\{x;-\}$	T
I_0P_y	$-2\Im(\mathcal{M}_1\mathcal{M}_3^*+\mathcal{M}_2\mathcal{M}_4^*)$	$\|b_1\|^2+\|b_2\|^2-\|b_3\|^2-\|b_4\|^2$	$\{y;-\}$	
I_0P_z	$-\|\mathcal{M}_1\|^2-\|\mathcal{M}_2\|^2+\|\mathcal{M}_3\|^2+\|\mathcal{M}_4\|^2$	$-2\Re(b_1b_3^*+b_2b_4^*)$	$\{z;-\}$	
$I_0P_{x'}$	$-2\Re(\mathcal{M}_1\mathcal{M}_2^*+\mathcal{M}_3\mathcal{M}_4^*)$	$2\Im(b_1b_2^*+b_3b_4^*)$	$\{-;x'\}$	R
$I_0P_{y'}$	$2\Im(\mathcal{M}_1\mathcal{M}_2^*+\mathcal{M}_3\mathcal{M}_4^*)$	$\|b_1\|^2-\|b_2\|^2+\|b_3\|^2-\|b_4\|^2$	$\{-;y'\}$	
$I_0P_{z'}$	$\|\mathcal{M}_1\|^2-\|\mathcal{M}_2\|^2+\|\mathcal{M}_3\|^2-\|\mathcal{M}_4\|^2$	$-2\Re(b_1b_2^*+b_3b_4^*)$	$\{-;z'\}$	
$I_0\mathcal{O}_{xx'}$	$-2\Re(\mathcal{M}_2\mathcal{M}_3^*+\mathcal{M}_1\mathcal{M}_4^*)$	$2\Re(-b_2b_3^*+b_1b_4^*)$	$\{x;x'\}$	TR
$I_0\mathcal{O}_{xy'}$	$2\Im(-\mathcal{M}_2\mathcal{M}_3^*+\mathcal{M}_1\mathcal{M}_4^*)$	$-2\Im(b_1b_3^*-b_2b_4^*)$	$\{x;y'\}$	
$I_0\mathcal{O}_{xz'}$	$2\Re(\mathcal{M}_1\mathcal{M}_3^*-\mathcal{M}_2\mathcal{M}_4^*)$	$2\Im(b_2b_3^*+b_1b_4^*)$	$\{x;z'\}$	
$I_0\mathcal{O}_{yx'}$	$2\Im(\mathcal{M}_2\mathcal{M}_3^*+\mathcal{M}_1\mathcal{M}_4^*)$	$2\Im(b_1b_2^*-b_3b_4^*)$	$\{y;x'\}$	
$I_0\mathcal{O}_{yy'}$	$2\Re(-\mathcal{M}_2\mathcal{M}_3^*+\mathcal{M}_1\mathcal{M}_4^*)$	$\|b_1\|^2-\|b_2\|^2-\|b_3\|^2+\|b_4\|^2$	$\{y;y'\}$	
$I_0\mathcal{O}_{yz'}$	$-2\Im(\mathcal{M}_1\mathcal{M}_3^*-\mathcal{M}_2\mathcal{M}_4^*)$	$2\Re(-b_1b_2^*+b_3b_4^*)$	$\{y;z'\}$	
$I_0\mathcal{O}_{zx'}$	$2\Re(\mathcal{M}_1\mathcal{M}_2^*-\mathcal{M}_3\mathcal{M}_4^*)$	$2\Im(b_2b_3^*-b_1b_4^*)$	$\{z;x'\}$	
$I_0\mathcal{O}_{zy'}$	$-2\Im(\mathcal{M}_1\mathcal{M}_2^*-\mathcal{M}_3\mathcal{M}_4^*)$	$2\Re(-b_1b_3^*+b_2b_4^*)$	$\{z;y'\}$	
$I_0\mathcal{O}_{zz'}$	$-\|\mathcal{M}_1\|^2+\|\mathcal{M}_2\|^2+\|\mathcal{M}_3\|^2-\|\mathcal{M}_4\|^2$	$2\Re(b_2b_3^*+b_1b_4^*)$	$\{z;z'\}$	

(or phase differences) are chosen to be the independent ones. Writing $b_i=\rho_i e^{i\phi_i}$ and defining all phase differences relative to ϕ_4, we obtain

$$
\begin{aligned}
-\frac{P_{x'}+\mathcal{O}_{yx'}}{P_{z'}+\mathcal{O}_{yz'}}
&=\frac{(\mathcal{O}_{xz'}-\mathcal{O}_{zx'})(\mathcal{O}_{zy'}-P_z)-(\mathcal{O}_{xx'}+\mathcal{O}_{zz'})(\mathcal{O}_{xy'}-P_z)}{(\mathcal{O}_{xx'}+\mathcal{O}_{zz'})(\mathcal{O}_{zy'}-P_z)+(\mathcal{O}_{xz'}-\mathcal{O}_{zx'})(\mathcal{O}_{xy'}-P_x)},\\
\frac{P_x+\mathcal{O}_{xy'}}{P_z+\mathcal{O}_{zz'}}
&=\frac{(\mathcal{O}_{xz'}-\mathcal{O}_{zx'})(\mathcal{O}_{yz'}-P_{z'})-(\mathcal{O}_{xx'}+\mathcal{O}_{zz'})(P_{x'}-\mathcal{O}_{yx'})}{(\mathcal{O}_{xx'}+\mathcal{O}_{zz'})(\mathcal{O}_{yz'}-P_{z'})+(\mathcal{O}_{xz'}-\mathcal{O}_{zx'})(P_{x'}-\mathcal{O}_{yx'})},\\
\frac{\mathcal{O}_{xz'}+\mathcal{O}_{zx'}}{\mathcal{O}_{zz'}-\mathcal{O}_{xx'}}
&=\frac{(\mathcal{O}_{xy'}-P_x)(\mathcal{O}_{yz'}-P_{z'})-(P_{x'}-\mathcal{O}_{yx'})(\mathcal{O}_{zy'}-P_z)}{(\mathcal{O}_{zy'}-P_z)(\mathcal{O}_{yz'}-P_{z'})+(\mathcal{O}_{xy'}-P_x)(P_{x'}-\mathcal{O}_{yx'})}.
\end{aligned}
\tag{35}
$$

We emphasize here that we have considered only the relationships among the observables. The number 7 does not necessarily represent a "minimal set" that must be measured for the so-called "complete" experiment. We postpone such a discussion until later in this section.

It is interesting to note that we can obtain a different set of relationships among the observables by consideration of the helicity amplitudes instead of the transversity ones. Proceeding in this way, the relationships obtained are

$$
\begin{aligned}
&(P_x\pm\mathcal{O}_{xz'})^2+(P_y\pm\mathcal{O}_{yz'})^2=(1\pm P_{z'})^2-(P_z\pm\mathcal{O}_{zz'})^2,\\
&(\mathcal{O}_{xx'}\pm\mathcal{O}_{yy'})^2+(\mathcal{O}_{xy'}\mp\mathcal{O}_{yx'})^2=(1\pm\mathcal{O}_{zz'})^2-(P_z\pm P_{z'})^2,\\
&(P_{x'}\pm\mathcal{O}_{zx'})^2+(P_{y'}\pm\mathcal{O}_{zy'})^2=(1\pm P_z)^2-(P_{z'}\pm\mathcal{O}_{zz'})^2.
\end{aligned}
\tag{36}
$$

The corresponding inequalities obtained from these are

$$
\begin{aligned}
|1\pm P_{z'}|&\geqslant\{|P_z\pm\mathcal{O}_{zz'}|,|P_x\pm\mathcal{O}_{xz'}|,|P_y\pm\mathcal{O}_{yz'}|\},\\
|1\pm\mathcal{O}_{zz'}|&\geqslant\{|P_z\pm P_{z'}|,|\mathcal{O}_{xx'}\pm\mathcal{O}_{yy'}|,|\mathcal{O}_{xy'}\mp\mathcal{O}_{yx'}|\},\\
|1\pm P_z|&\geqslant\{|P_{z'}\pm\mathcal{O}_{zz'}|,|P_{x'}\pm\mathcal{O}_{zx'}|,|P_{y'}\pm\mathcal{O}_{zy'}|\}
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
1+P_{z'}^2&\geqslant\left\{P_z^2+\mathcal{O}_{zz'}^2,\,P_x^2+\mathcal{O}_{xz'}^2,\,P_y^2+\mathcal{O}_{yz'}^2\right\},\\
1+\mathcal{O}_{zz'}^2&\geqslant\left\{P_z^2+P_{z'}^2,\,\mathcal{O}_{xx'}^2+\mathcal{O}_{yy'}^2,\,\mathcal{O}_{xy'}^2+\mathcal{O}_{yx'}^2\right\},\\
1+P_z^2&\geqslant\left\{P_{z'}^2+\mathcal{O}_{zz'}^2,\,P_{x'}^2+\mathcal{O}_{zx'}^2,\,P_{y'}^2+\mathcal{O}_{zy'}^2\right\}.
\end{aligned}
\tag{38}
$$

In a similar manner, a set of relationships may be obtained by considering the phases of the helicity amplitudes.

A. Required experimental measurements in $\pi N \to \pi\pi N$

Information on baryon spectroscopy is obtained from processes such as $\pi N \to \pi N$ by extracting the helicity or transversity (or partial wave) amplitudes for the process. These amplitudes are then interpreted in terms of baryon resonances. There is therefore a great deal of interest in knowing how many measurements must be made at each kinematic point to extract the amplitudes. For this discussion, we focus on the process $\pi N \to \pi\pi N$, because such discussions have already been documented for $\pi N \to \pi N$.

If we write $b_i = \rho_i e^{i\phi_i}$, then the quantities I_0, P_y, $P_{y'}$, and $\mathcal{O}_{yy'}$ must be measured at each kinematic point to provide the information needed to extract the ρ_i unambiguously. In the bilinear combinations of transversity amplitudes, there are six phase differences that occur, but only three of these are independent. Any three can be chosen, so we discuss $\phi_{12} \equiv \phi_1 - \phi_2$, $\phi_{34} \equiv \phi_3 - \phi_4$, and $\phi_{23} \equiv \phi_2 - \phi_3$.

To access ϕ_{12}, two of the four quantities $P_{x'}$, $P_{z'}$, $\mathcal{O}_{yx'}$, and $\mathcal{O}_{yz'}$ should be measured at each kinematic point. The pair of measurements $P_{x'}$ and $\mathcal{O}_{yx'}$, or $P_{z'}$ and $\mathcal{O}_{yz'}$, would provide "cleaner" solutions for ϕ_{12}. Note that these measurement would also provide ϕ_{34}, and both of these phase differences would be subject to the well-known "quadrant ambiguities" [6].

This leaves one more phase difference to be determined. If we choose this to be ϕ_{23}, then measurement of one of $\mathcal{O}_{xx'}$, $\mathcal{O}_{zx'}$, $\mathcal{O}_{xz'}$, or $\mathcal{O}_{zz'}$ will allow its extraction. To do this, however, the other phase that occurs in these observables, ϕ_{14}, will have to be written in terms of the two phases already extracted and ϕ_{23} as

$$\begin{aligned}\phi_{14} &= \phi_1 - \phi_4 = \phi_1 - \phi_2 + \phi_2 - \phi_3 + \phi_3 - \phi_4\\ &= \phi_{12} + \phi_{23} + \phi_{34}.\end{aligned} \tag{39}$$

Then the only unknown in the measured quantity would be ϕ_{23}.

A similar analysis can be made in terms of the helicity amplitudes. In this case, I_0, P_z, $P_{z'}$, and $\mathcal{O}_{zz'}$ must be measured at each kinematic point to determine the magnitudes of the helicity amplitudes. Two measurements from among $P_{x'}$, $P_{y'}$, $\mathcal{O}_{zx'}$, and $\mathcal{O}_{zy'}$ will provide enough information to determine two of the relative phases, for instance, and one measurement from among P_x, P_y, $\mathcal{O}_{xz'}$, and $\mathcal{O}_{yz'}$ will provide enough information to determine the last phase needed.

The bottom line is that to extract reliable information on baryon properties, the helicity or transversity amplitudes must be extracted with some degree of certainty, and this can be done only if at least seven judiciously chosen measurements are performed at each kinematic point. This also means that both single and double polarization measurements will be essential. This is similar to the conclusion of Ref. [6] in their analysis of pion photoproduction and is independent of whether the observables are described in terms of helicity, transversity, or other amplitudes.

B. Parity conservation

The properties of the helicity and transversity amplitudes for a process $a + b \to c + d$ are well known. For $\pi N \to \pi N$, the relationships among the helicity amplitudes are written [11]

$$\mathcal{M}_{-\lambda_N -\lambda'_N}(\theta) = (-1)^{\lambda_N - \lambda'_N} \mathcal{M}_{\lambda_N \lambda'_N}(\theta), \tag{40}$$

where θ is as defined in Eq. (7). The corresponding relationships for transversity amplitudes are [12]

$$b_{\tau_N \tau'_N}(\theta) = (-1)^{\tau_N - \tau'_N} b_{\tau_N \tau'_N}(\theta). \tag{41}$$

Parity conservation therefore means that some of the transversity amplitudes vanish exactly.

In general, a minimum of three angles are needed to describe the scattering amplitude for a process $a + b \to c + d + e$. For the specific case of $\pi N \to \pi\pi N$, we choose these angles to be as defined in Eqs. (14) and (17). The relationships that arise among the helicity amplitudes may then be written

$$\mathcal{M}_{-\lambda_N -\lambda'_N}(\theta, \theta_1, \phi_1) = (-1)^{\lambda_N - \lambda'_N} \mathcal{M}_{\lambda_N \lambda'_N}(\theta, \theta_1, 2\pi - \phi_1). \tag{42}$$

These relations cannot be used to decrease the number of independent helicity amplitudes, but they can be used to determine which of the observables are even or odd under the transformation $\phi_1 \leftrightarrow 2\pi - \phi_1$.

C. Construction of transition amplitudes

1. $\pi N \to \pi N$

For this process, $\mathcal{A}$ must be a scalar and $\vec{\mathcal{B}}$ an axial vector. With the kinematics for this process as previously defined, we can write $\mathcal{A}$ and $\vec{\mathcal{B}}$ as

$$\begin{aligned}\mathcal{A} &= \alpha,\\ \vec{\mathcal{B}} &= \beta \frac{\hat{k} \times \hat{q}}{|\hat{k} \times \hat{q}|},\end{aligned} \tag{43}$$

where α and β are scalar quantities that contain all of the details of whatever model is constructed to describe the process. These can be compared to the form usually written for this process [1], namely

$$i\mathcal{M} = \chi^\dagger (f + g\vec{\sigma} \cdot \hat{n})\phi, \tag{44}$$

where $\hat{n} = \hat{k} \times \hat{q}/|\hat{k} \times \hat{q}|$. This means that we can identify $\alpha = f$ and $\beta = g$. With these kinematics, $\mathcal{B}_1 = \mathcal{B}_3 = 0$, leading to

$$\mathcal{M}_1 = \mathcal{M}_4, \qquad \mathcal{M}_2 = -\mathcal{M}_3 \tag{45}$$

in the helicity basis or

$$b_2 = b_3 = 0 \tag{46}$$

in the transversity basis. Two of the transversity amplitudes (the "transversity-flip" amplitudes) vanish identically (as expected), meaning that this process is exactly "transversity conserving." The relationships among the helicity amplitudes expected from considerations of parity symmetry are therefore obtained. Many of the polarization observables now become redundant or vanish identically

$$\begin{aligned}P_x &= P_z = P_{x'} = P_{z'} = \mathcal{O}_{xy'} = \mathcal{O}_{yx'} = \mathcal{O}_{yz'} = \mathcal{O}_{zy'} = 0,\\ I_0 &= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 = |b_1|^2 + |b_2|^2 = \mathcal{O}_{yy'},\\ P_y &= 2\Im(\mathcal{M}_1\mathcal{M}_2^*) = |b_1|^2 - |b_2|^2 = P_{y'},\\ \mathcal{O}_{xx'} &= -|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 = 2\Re(b_1 b_4^*) = \mathcal{O}_{zz'},\\ \mathcal{O}_{xz'} &= -2\Re(\mathcal{M}_1\mathcal{M}_2^*) = 2\Im(b_1 b_4^*) = -\mathcal{O}_{zx'},\end{aligned} \tag{47}$$

and at this point there are four observables left. From consideration of the transversity amplitudes, it is "obvious" why $\mathcal{O}_{yy'}$ and I_0 are equal. The relationships among observables reduces to a single relationship in this case, namely

$$P_y^2 + \mathcal{O}_{xx'}^2 + \mathcal{O}_{xz'}^2 = 1. \quad (48)$$

This last equation means that only three of the remaining four observables are independent.

We note that the convention has been to choose both sets of axes for this process to be the same. This introduces explicit factors of $\cos\theta$ and $\sin\theta$ into the observables. If we choose unrotated primed axes, the relationships among the observables we have defined and those found in the literature are

$$R = \mathcal{O}_{xx'}\cos\theta - \mathcal{O}_{xz'}\sin\theta,$$
$$A = \mathcal{O}_{xx'}\sin\theta + \mathcal{O}_{xz'}\cos\theta.$$

In terms of A and R, the identity that must be satisfied is

$$P_y^2 + R^2 + A^2 = 1. \quad (49)$$

2. $\pi N \to \pi\pi N$

For this process, there are three independent three-momenta, which we can choose to be $\vec{k}$, $\vec{p}_2$, and $\vec{q}_1$, where $\vec{q}_1$ is the momentum of one of the final pions. These have all been defined previously. In this case, $\mathcal{A}$ must be a pseudoscalar quantity, whereas $\mathcal{B}$ must be a vector. The only possibilities are

$$\begin{aligned} \mathcal{A} &= \alpha\hat{k}\cdot\hat{p}_2\times\hat{q}_1, \\ \vec{\mathcal{B}} &= \beta_1\hat{k} + \beta_2\hat{p}_2 + \beta_3\hat{q}_1, \end{aligned} \quad (50)$$

where α and the β_i depend only on scalar products of the vectors in the problem. For this case, $\vec{\mathcal{B}}$ has x, y, and z components and all of the four helicity amplitudes are independent. Furthermore, none of the polarization observables vanish exactly, and none are redundant. However, using the properties of the helicity amplitudes, we can predict that the observables P_x, P_z, $P_{x'}$, $P_{z'}$, $\mathcal{O}_{xy'}$, $\mathcal{O}_{yx'}$, $\mathcal{O}_{yz'}$, and $\mathcal{O}_{zy'}$ are all odd under the transformation $\phi_1 \leftrightarrow 2\pi - \phi_1$. The other eight observables are all even under this transformation.

IV. OBSERVABLES IN $\pi N \to \pi N$ AND $\pi N \to \pi\pi N$

We can treat these two processes in a framework similar to that used for $\pi N \to \pi N$ and $\pi N \to \pi\pi N$ by writing

$$i\mathcal{M} = \chi^\dagger(\mathcal{A}_j + \sigma_i\mathcal{B}_{ij})\phi\varepsilon_j, \quad (51)$$

where $\vec{\varepsilon}$ is the polarization vector of the initial photon, $\mathcal{A}_j$ are the components of a vector ($\gamma N \to \pi\pi N$) or an axial vector ($\gamma N \to \pi N$), and $\mathcal{B}_{ij}$ is a tensor ($\gamma N \to \pi N$) or pseudotensor ($\gamma N \to \pi\pi N$). The amplitude takes the form

$$i\mathcal{M} = \chi^\dagger\begin{pmatrix} A_{+j} & B_{-j} \\ B_{+j} & A_{-j}\end{pmatrix}\phi\varepsilon_j, \quad (52)$$

where

$$A_{\pm j} \equiv \mathcal{A}_j \pm \mathcal{B}_{3j}, \qquad B_{\pm j} \equiv \mathcal{B}_{1j} \pm i\mathcal{B}_{2j}, \quad (53)$$

in analogy with our treatment of $\pi N \to \pi N$ and $\pi N \to \pi\pi N$.

Defining the helicity amplitudes

$$\begin{aligned} i\mathcal{M}_{++}^\lambda &\equiv \mathcal{M}_1^\lambda, \\ i\mathcal{M}_{+-}^\lambda &\equiv \mathcal{M}_2^\lambda, \\ i\mathcal{M}_{-+}^\lambda &\equiv \mathcal{M}_3^\lambda, \\ i\mathcal{M}_{--}^\lambda &\equiv \mathcal{M}_4^\lambda, \end{aligned} \quad (54)$$

where λ is the helicity of the photon and the transversity amplitudes

$$\begin{aligned} ib_{++}^\lambda &\equiv b_1^\lambda, \\ ib_{+-}^\lambda &\equiv b_2^\lambda, \\ ib_{-+}^\lambda &\equiv b_3^\lambda, \\ ib_{--}^\lambda &\equiv b_4^\lambda, \end{aligned} \quad (55)$$

we find

$$\begin{aligned} \mathcal{A}_j\varepsilon_j(\lambda) &= \frac{1}{2}\left[\left(\mathcal{M}_1^\lambda + \mathcal{M}_4^\lambda\right)\cos\frac{\theta}{2} + \left(\mathcal{M}_2^\lambda - \mathcal{M}_3^\lambda\right)\sin\frac{\theta}{2}\right] \\ &= \frac{1}{2}\left(b_1^\lambda + b_4^\lambda\right), \\ \mathcal{B}_{1j}\varepsilon_j(\lambda) &= \frac{1}{2}\left[\left(\mathcal{M}_2^\lambda + \mathcal{M}_3^\lambda\right)\cos\frac{\theta}{2} + \left(\mathcal{M}_4^\lambda - \mathcal{M}_1^\lambda\right)\sin\frac{\theta}{2}\right] \\ &= \frac{i}{2}\left(b_2^\lambda - b_3^\lambda\right), \\ \mathcal{B}_{2j}\varepsilon_j(\lambda) &= \frac{i}{2}\left[\left(\mathcal{M}_2^\lambda - \mathcal{M}_3^\lambda\right)\cos\frac{\theta}{2} - \left(\mathcal{M}_1^\lambda + \mathcal{M}_4^\lambda\right)\sin\frac{\theta}{2}\right] \\ &= \frac{1}{2}\left(b_1^\lambda - b_4^\lambda\right), \\ \mathcal{B}_{3j}\varepsilon_j(\lambda) &= \frac{1}{2}\left[\left(\mathcal{M}_4^\lambda - \mathcal{M}_1^\lambda\right)\cos\frac{\theta}{2} - \left(\mathcal{M}_2^\lambda + \mathcal{M}_3^\lambda\right)\sin\frac{\theta}{2}\right] \\ &= \frac{1}{2}\left(b_2^\lambda + b_3^\lambda\right). \end{aligned} \quad (56)$$

Note that the amplitudes b_i^λ are not strictly transversity amplitudes, as the photon spin is still quantized along its direction of motion. Quantizing along the transverse direction requires construction of the combinations $\mathcal{D}_i^\pm = b_i^+ \pm b_i^-$.

The transversity spinors of Eq. (27) can be written as linear superpositions of the helicity spinors of Eqs. (25) and (26). The expressions are

$$\begin{aligned} \phi_T^+ &= \frac{1}{\sqrt{2}}(\phi^+ - i\phi^-) = \frac{1}{\sqrt{2}}e^{i\theta/2}(\chi^+ - i\chi^-), \\ \phi_T^- &= \frac{1}{\sqrt{2}}(\phi^+ + i\phi^-) = \frac{1}{\sqrt{2}}e^{-i\theta/2}(\chi^+ + i\chi^-). \end{aligned} \quad (57)$$

This allows yet another set of amplitudes, $i\mathcal{T}_{\tau_N\tau_N'}^{\lambda_\gamma}$, to be defined in terms of the helicity amplitudes. These are

$$
\begin{aligned}
i\mathcal{T}_{++}^{\pm} &\equiv \mathcal{T}_1^{\pm} = \frac{1}{2}e^{-i\theta/2}\left[\mathcal{M}_1^{\pm} + \mathcal{M}_4^{\pm} + i\left(\mathcal{M}_2^{\pm} - \mathcal{M}_3^{\pm}\right)\right],\\
i\mathcal{T}_{+-}^{\pm} &\equiv \mathcal{T}_2^{\pm} = \frac{1}{2}e^{i\theta/2}\left[\mathcal{M}_1^{\pm} - \mathcal{M}_4^{\pm} - i\left(\mathcal{M}_2^{\pm} + \mathcal{M}_3^{\pm}\right)\right],\\
i\mathcal{T}_{-+}^{\pm} &\equiv \mathcal{T}_3^{\pm} = \frac{1}{2}e^{-i\theta/2}\left[\mathcal{M}_1^{\pm} - \mathcal{M}_4^{\pm} + i\left(\mathcal{M}_2^{\pm} + \mathcal{M}_3^{\pm}\right)\right],\\
i\mathcal{T}_{--}^{\pm} &\equiv \mathcal{T}_4^{\pm} = \frac{1}{2}e^{i\theta/2}\left[\mathcal{M}_1^{\pm} + \mathcal{M}_4^{\pm} - i\left(\mathcal{M}_2^{\pm} - \mathcal{M}_3^{\pm}\right)\right].
\end{aligned} \tag{58}
$$

Full transversity amplitudes can be constructed from these as $\mathcal{D}_i^{\tau_\gamma} = \mathcal{T}_i^+ \pm \mathcal{T}_i^-$. For $\gamma N \to \pi N$, the resulting amplitudes are similar to those found in the literature, but the phases $e^{\pm i\theta/2}$ are absent from the published forms (see Ref. [3], p. 270).

We can write the reaction rate I as

$$
\begin{aligned}
\rho_f I = I_0\Big\{&\left(1 + \vec{\Lambda}_i\cdot\vec{P} + \vec{\sigma}\cdot\vec{P}' + \Lambda_i^\alpha\sigma^{\beta'}\mathcal{O}_{\alpha\beta'}\right)\\
&+ \delta_\odot\left(I^\odot + \vec{\Lambda}_i\cdot\vec{P}^\odot + \vec{\sigma}\cdot\vec{P}^{\odot\prime} + \Lambda_i^\alpha\sigma^{\beta'}\mathcal{O}_{\alpha\beta'}^\odot\right)\\
&+ \delta_\ell\Big[\sin 2\beta\left(I^s + \vec{\Lambda}_i\cdot\vec{P}^s + \vec{\sigma}\cdot\vec{P}^{s\prime} + \Lambda_i^\alpha\sigma^{\beta'}\mathcal{O}_{\alpha\beta'}^s\right)\\
&+ \cos 2\beta\left(I^c + \vec{\Lambda}_i\cdot\vec{P}^c + \vec{\sigma}\cdot\vec{P}^{c\prime} + \Lambda_i^\alpha\sigma^{\beta'}\mathcal{O}_{\alpha\beta'}^c\right)\Big]\Big\},
\end{aligned}
$$

where $\vec{P}$ represents the polarization asymmetry that arises if the target nucleon is polarized, $\rho_f = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{P}')$ is the density matrix of the recoiling nucleon, and $\mathcal{O}_{\alpha\beta'}$ is the observable if both the target and recoil polarization are measured. The primes indicate that the recoil observables are measured with respect to a set of axes x', y', and z', in which z' is along the direction of motion of the recoiling nucleon and $y' = y$. $\delta_\odot$ is the degree of circular polarization in the photon beam, whereas δ_ℓ is the degree of linear polarization, with the direction of polarization being at an angle β to the x axis.

The polarization observables that arise for these two processes are given in Tables II–V. Note that there are 64 polarization observables in general.

As was the case with the pion-induced reactions, there are a number of relationships among these 64 polarization observables. In fact, there are 28 relations that arise from consideration of the absolute magnitudes of the helicity or transversity amplitudes and another 21 that arise from consideration of their phases, leaving 15 independent quantities. We list here the relations that arise from considerations of the absolute magnitudes of the amplitudes b_i:

$$
\begin{aligned}
&\left[P_{x'} + \xi\mathcal{O}_{yx'} + \zeta\left(P_{x'}^\odot + \xi\mathcal{O}_{yx'}^\odot\right)\right]^2 + \left[P_{z'} + \xi\mathcal{O}_{yz'} + \zeta\left(P_{z'}^\odot + \xi\mathcal{O}_{yz'}^\odot\right)\right]^2\\
&\quad = \left[1 + \xi P_y + \zeta\left(I^\odot + \xi P_y^\odot\right)\right]^2 - \left[P_{y'} + \xi\mathcal{O}_{yy'} + \zeta\left(P_{y'}^\odot + \xi\mathcal{O}_{yy'}^\odot\right)\right]^2,\\
&\left[P_x + \xi\mathcal{O}_{xy'} + \zeta\left(P_x^\odot + \xi\mathcal{O}_{xy'}^\odot\right)\right]^2 + \left[P_z + \xi\mathcal{O}_{zy'} + \zeta\left(P_z^\odot + \xi\mathcal{O}_{zy'}^\odot\right)\right]^2\\
&\quad = \left[1 + \xi P_{y'} + \zeta\left(I^\odot + \xi P_{y'}^\odot\right)\right]^2 - \left[P_y + \xi\mathcal{O}_{yy'} + \zeta\left(P_y^\odot + \xi\mathcal{O}_{yy'}^\odot\right)\right]^2,\\
&\left[\mathcal{O}_{xz'} - \xi\mathcal{O}_{zx'} + \zeta\left(\mathcal{O}_{xz'}^\odot - \xi\mathcal{O}_{zx'}^\odot\right)\right]^2 + \left[\mathcal{O}_{xx'} + \xi\mathcal{O}_{zz'} - \zeta\left(\mathcal{O}_{xx'}^\odot + \xi\mathcal{O}_{zz'}^\odot\right)\right]^2\\
&\quad = \left[1 + \xi\mathcal{O}_{yy'} - \zeta\left(I^\odot + \xi\mathcal{O}_{yy'}^\odot\right)\right]^2 - \left[P_y + \xi P_{y'} - \zeta\left(P_y^\odot + \xi P_{y'}^\odot\right)\right]^2,\\
&\left[I^s + \xi\mathcal{O}_{yy'}^s + \zeta\left(P_y^s + \xi P_{y'}^s\right)\right]^2 + \left[I^c + \xi\mathcal{O}_{yy'}^c + \zeta\left(P_y^c + \xi P_{y'}^c\right)\right]^2\\
&\quad = \left[1 + \xi\mathcal{O}_{yy'} + \zeta(P_y + \xi P_{y'})\right]^2 - \left[I^\odot + \xi\mathcal{O}_{yy'}^\odot + \zeta\left(P_y^\odot + \xi P_{y'}^\odot\right)\right]^2,\\
&\left[P_{x'}^s + \xi\mathcal{O}_{yx'}^s + \zeta\left(P_{z'}^c + \xi\mathcal{O}_{yz'}^c\right)\right]^2 + \left[P_{z'}^s + \xi\mathcal{O}_{yz'}^s - \zeta\left(P_{x'}^c + \xi\mathcal{O}_{yx'}^c\right)\right]^2\\
&\quad = \left[1 + \xi P_y + \zeta\left(P_{y'}^\odot + \xi\mathcal{O}_{yy'}^\odot\right)\right]^2 - \left[I^\odot + \xi P_y^\odot + \zeta(P_{y'} + \xi\mathcal{O}_{yy'})\right]^2,\\
&\left[P_x^s + \xi\mathcal{O}_{xy'}^s + \zeta\left(P_z^c + \xi\mathcal{O}_{zy'}^c\right)\right]^2 + \left[P_z^s + \xi\mathcal{O}_{zy'}^s - \zeta\left(P_x^c + \xi\mathcal{O}_{xy'}^c\right)\right]^2\\
&\quad = \left[1 + \xi P_{y'} - \zeta\left(P_y^\odot + \xi\mathcal{O}_{yy'}^\odot\right)\right]^2 - \left[I^\odot + \xi P_{y'}^\odot - \zeta(P_y + \xi\mathcal{O}_{yy'})\right]^2,\\
&\left[\mathcal{O}_{xx'}^s + \xi\mathcal{O}_{zz'}^s + \zeta\left(\mathcal{O}_{xz'}^c - \xi\mathcal{O}_{zx'}^c\right)\right]^2 + \left[\mathcal{O}_{xz'}^s - \xi\mathcal{O}_{zx'}^s - \zeta\left(\mathcal{O}_{xx'}^c + \xi\mathcal{O}_{zz'}^c\right)\right]^2\\
&\quad = \left[1 + \xi\mathcal{O}_{yy'} + \zeta\left(P_{y'}^\odot + \xi P_y^\odot\right)\right]^2 - \left[I^\odot + \xi\mathcal{O}_{yy'}^\odot + \zeta(P_{y'} + \xi P_y)\right]^2.
\end{aligned} \tag{59}
$$

In each of these equations, ξ and ζ can independently take either of the values ± 1, meaning that the seven equations shown above actually represent 28 identities. We can also obtain another 21 identities from considering the phases of the transversity amplitudes, but we do not display these here. We also point out that the equations above were obtained from considering the transversity amplitudes. Had we considered the helicity amplitudes instead, we would obtain a different set of 28 identities among the observables from the magnitudes of the amplitudes and another 21 identities from their phases. In either case, we are left with 15 independent observables.

As was done in the case of $\pi N \to \pi\pi N$, we can use the identities above to write a number of inequalities that the polarization observables for $\gamma N \to \pi\pi N$ must satisfy. These inequalities are

TABLE II. Polarization observables of single- and double-pion photoproduction in terms of the helicity and transversity amplitudes. These observables arise with an unpolarized photon beam. Variables labeled with a T require a polarized target with recoil polarization unmeasured, whereas those labeled with an R require an unpolarized target, but the recoil polarization is measured. Those denoted TR require polarized targets, with recoil polarization measured. The measurements required are shown by the set $\{b, t, r\}$, which denote the component of the beam (b), target (t) or recoil (r) polarization that must known or measured. For the target polarization, the x, y, and z axes are as defined in the text. The x', y', and z' axes are also defined in the text, as is the notation for the transversity amplitudes.

Observable	Helicity form	Transversity form	Expt.	Type
I_0	$\lvert\mathcal{M}_1^-\rvert^2 + \lvert\mathcal{M}_1^+\rvert^2 + \lvert\mathcal{M}_2^-\rvert^2 + \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^-\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 + \lvert\mathcal{M}_4^-\rvert^2 + \lvert\mathcal{M}_4^+\rvert^2$	$\lvert b_1^-\rvert^2 + \lvert b_1^+\rvert^2 + \lvert b_2^-\rvert^2 + \lvert b_2^+\rvert^2 + \lvert b_3^-\rvert^2 + \lvert b_3^+\rvert^2 + \lvert b_4^-\rvert^2 + \lvert b_4^+\rvert^2$	$\{-;-;-\}$	
$I_0 P_x$	$2\Re(\mathcal{M}_1^-\mathcal{M}_3^{-*} + \mathcal{M}_1^+\mathcal{M}_3^{+*} + \mathcal{M}_2^-\mathcal{M}_4^{-*} + \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^-b_3^{-*} + b_1^+b_3^{+*} + b_2^-b_4^{-*} + b_2^+b_4^{+*})$	$\{-;x;-\}$	T
$I_0 P_y$	$-2\Im(\mathcal{M}_1^-\mathcal{M}_3^{-*} + \mathcal{M}_1^+\mathcal{M}_3^{+*} + \mathcal{M}_2^-\mathcal{M}_4^{-*} + \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$\lvert b_1^-\rvert^2 + \lvert b_1^+\rvert^2 + \lvert b_2^-\rvert^2 + \lvert b_2^+\rvert^2 - \lvert b_3^-\rvert^2 - \lvert b_3^+\rvert^2 - \lvert b_4^-\rvert^2 - \lvert b_4^+\rvert^2$	$\{-;y;-\}$	
$I_0 P_z$	$-\lvert\mathcal{M}_1^-\rvert^2 - \lvert\mathcal{M}_1^+\rvert^2 - \lvert\mathcal{M}_2^-\rvert^2 - \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^-\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 + \lvert\mathcal{M}_4^-\rvert^2 + \lvert\mathcal{M}_4^+\rvert^2$	$-2\Re(b_1^-b_3^{-*} + b_1^+b_3^{+*} + b_2^-b_4^{-*} + b_2^+b_4^{+*})$	$\{-;z;-\}$	
$I_0 P_{x'}$	$-2\Re(\mathcal{M}_1^-\mathcal{M}_2^{-*} + \mathcal{M}_1^+\mathcal{M}_2^{+*} + \mathcal{M}_3^-\mathcal{M}_4^{-*} + \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$2\Im(b_1^-b_2^{-*} + b_1^+b_2^{+*} + b_3^-b_4^{-*} + b_3^+b_4^{+*})$	$\{-;-;x'\}$	R
$I_0 P_{y'}$	$2\Im(\mathcal{M}_1^-\mathcal{M}_2^{-*} + \mathcal{M}_1^+\mathcal{M}_2^{+*} + \mathcal{M}_3^-\mathcal{M}_4^{-*} + \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$\lvert b_1^-\rvert^2 + \lvert b_1^+\rvert^2 - \lvert b_2^-\rvert^2 - \lvert b_2^+\rvert^2 + \lvert b_3^-\rvert^2 + \lvert b_3^+\rvert^2 - \lvert b_4^-\rvert^2 - \lvert b_4^+\rvert^2$	$\{-;-;y'\}$	
$I_0 P_{z'}$	$\lvert\mathcal{M}_1^-\rvert^2 + \lvert\mathcal{M}_1^+\rvert^2 - \lvert\mathcal{M}_2^-\rvert^2 - \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^-\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 - \lvert\mathcal{M}_4^-\rvert^2 - \lvert\mathcal{M}_4^+\rvert^2$	$-2\Re(b_1^-b_2^{-*} + b_1^+b_2^{+*} + b_3^-b_4^{-*} + b_3^+b_4^{+*})$	$\{-;-;z'\}$	
$I_0\mathcal{O}_{xx'}$	$-2\Re(\mathcal{M}_2^-\mathcal{M}_3^{-*} + \mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^-\mathcal{M}_4^{-*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$2\Re(-b_2^-b_3^{-*} - b_2^+b_3^{+*} + b_1^-b_4^{-*} + b_1^+b_4^{+*})$	$\{-;x;x'\}$	TR
$I_0\mathcal{O}_{xy'}$	$2\Im(-\mathcal{M}_2^-\mathcal{M}_3^{-*} - \mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^-\mathcal{M}_4^{-*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^-b_3^{-*} + b_1^+b_3^{+*} - b_2^-b_4^{-*} - b_2^+b_4^{+*})$	$\{-;x;y'\}$	
$I_0\mathcal{O}_{xz'}$	$2\Re(\mathcal{M}_1^-\mathcal{M}_3^{-*} + \mathcal{M}_1^+\mathcal{M}_3^{+*} - \mathcal{M}_2^-\mathcal{M}_4^{-*} - \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$2\Im(b_2^-b_3^{-*} + b_2^+b_3^{+*} + b_1^-b_4^{-*} + b_1^+b_4^{+*})$	$\{-;x;z'\}$	
$I_0\mathcal{O}_{yx'}$	$2\Im(\mathcal{M}_2^-\mathcal{M}_3^{-*} + \mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^-\mathcal{M}_4^{-*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$2\Im(b_1^-b_2^{-*} + b_1^+b_2^{+*} - b_3^-b_4^{-*} - b_3^+b_4^{+*})$	$\{-;y;x'\}$	
$I_0\mathcal{O}_{yy'}$	$2\Re(-\mathcal{M}_2^-\mathcal{M}_3^{-*} - \mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^-\mathcal{M}_4^{-*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$\lvert b_1^-\rvert^2 + \lvert b_1^+\rvert^2 - \lvert b_2^-\rvert^2 - \lvert b_2^+\rvert^2 - \lvert b_3^-\rvert^2 - \lvert b_3^+\rvert^2 + \lvert b_4^-\rvert^2 + \lvert b_4^+\rvert^2$	$\{-;y;y'\}$	
$I_0\mathcal{O}_{yz'}$	$-2\Im(\mathcal{M}_1^-\mathcal{M}_3^{-*} + \mathcal{M}_1^+\mathcal{M}_3^{+*} - \mathcal{M}_2^-\mathcal{M}_4^{-*} - \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$2\Re(-b_1^-b_2^{-*} - b_1^+b_2^{+*} + b_3^-b_4^{-*} + b_3^+b_4^{+*})$	$\{-;y;z'\}$	
$I_0\mathcal{O}_{zx'}$	$2\Re(\mathcal{M}_1^-\mathcal{M}_2^{-*} + \mathcal{M}_1^+\mathcal{M}_2^{+*} - \mathcal{M}_3^-\mathcal{M}_4^{-*} - \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$2\Im(b_2^-b_3^{-*} + b_2^+b_3^{+*} - b_1^-b_4^{-*} - b_1^+b_4^{+*})$	$\{-;z;x'\}$	
$I_0\mathcal{O}_{zy'}$	$-2\Im(\mathcal{M}_1^-\mathcal{M}_2^{-*} + \mathcal{M}_1^+\mathcal{M}_2^{+*} - \mathcal{M}_3^-\mathcal{M}_4^{-*} - \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$2\Re(-b_1^-b_3^{-*} - b_1^+b_3^{+*} + b_2^-b_4^{-*} + b_2^+b_4^{+*})$	$\{-;z;y'\}$	
$I_0\mathcal{O}_{zz'}$	$-\lvert\mathcal{M}_1^-\rvert^2 - \lvert\mathcal{M}_1^+\rvert^2 + \lvert\mathcal{M}_2^-\rvert^2 + \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^-\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 - \lvert\mathcal{M}_4^-\rvert^2 - \lvert\mathcal{M}_4^+\rvert^2$	$2\Re(b_2^-b_3^{-*} + b_2^+b_3^{+*} + b_1^-b_4^{-*} + b_1^+b_4^{+*})$	$\{-;z;z'\}$	

$$
\begin{aligned}
&\left|1 + \xi P_y + \zeta\left(I^{\odot} + \xi P_y^{\odot}\right)\right| \geqslant \left\{\left|P_{y'} + \xi \mathcal{O}_{yy'} + \zeta\left(P_{y'}^{\odot} + \xi \mathcal{O}_{yy'}^{\odot}\right)\right|,\right.\\
&\left.\left|P_{x'} + \xi \mathcal{O}_{yx'} + \zeta\left(P_{x'}^{\odot} + \xi \mathcal{O}_{yx'}^{\odot}\right)\right|, \quad \left|P_{z'} + \xi \mathcal{O}_{yz'} + \zeta\left(P_{z'}^{\odot} + \xi \mathcal{O}_{yz'}^{\odot}\right)\right|\right\},\\
&\left|1 + \xi P_{y'} + \zeta\left(I^{\odot} + \xi P_{y'}^{\odot}\right)\right| \geqslant \left\{\left|P_{y} + \xi \mathcal{O}_{yy'} + \zeta\left(P_{y}^{\odot} + \xi \mathcal{O}_{yy'}^{\odot}\right)\right|,\right.\\
&\left.\left|P_{x} + \xi \mathcal{O}_{xy'} + \zeta\left(P_{x}^{\odot} + \xi \mathcal{O}_{xy'}^{\odot}\right)\right|, \quad \left|P_{z} + \xi \mathcal{O}_{zy'} + \zeta\left(P_{z}^{\odot} + \xi \mathcal{O}_{zy'}^{\odot}\right)\right|\right\},\\
&\left|1 + \xi \mathcal{O}_{yy'} - \zeta\left(I^{\odot} + \xi \mathcal{O}_{yy'}^{\odot}\right)\right| \geqslant \left\{\left|P_{y} + \xi P_{y'} - \zeta\left(P_{y}^{\odot} + \xi P_{y'}^{\odot}\right)\right|,\right.\\
&\left.\left|\mathcal{O}_{xz'} - \xi \mathcal{O}_{zx'} + \zeta\left(\mathcal{O}_{xz'}^{\odot} - \xi \mathcal{O}_{zx'}^{\odot}\right)\right|, \quad \left|\mathcal{O}_{xx'} + \xi \mathcal{O}_{zz'} - \zeta\left(\mathcal{O}_{xx'}^{\odot} + \xi \mathcal{O}_{zz'}^{\odot}\right)\right|\right\},\\
&\left|1 + \xi \mathcal{O}_{yy'} + \zeta(P_y + \xi P_{y'})\right| \geqslant \left\{\left|I^{\odot} + \xi \mathcal{O}_{yy'}^{\odot} + \zeta\left(P_{y}^{\odot} + \xi P_{y'}^{\odot}\right)\right|,\right.\\
&\left.\left|I^{s} + \xi \mathcal{O}_{yy'}^{s} + \zeta\left(P_{y}^{s} + \xi P_{y'}^{s}\right)\right|, \quad \left|I^{c} + \xi \mathcal{O}_{yy'}^{c} + \zeta\left(P_{y}^{c} + \xi P_{y'}^{c}\right)\right|\right\},\\
&\left|1 + \xi P_y + \zeta\left(P_{y'}^{\odot} + \xi \mathcal{O}_{yy'}^{\odot}\right)\right| \geqslant \left\{\left|I^{\odot} + \xi P_{y}^{\odot} + \zeta(P_{y'} + \xi \mathcal{O}_{yy'})\right|,\right.\\
&\left.\left|P_{x'}^{s} + \xi \mathcal{O}_{yx'}^{s} + \zeta\left(P_{z'}^{c} + \xi \mathcal{O}_{yz'}^{c}\right)\right|, \quad \left|P_{z'}^{s} + \xi \mathcal{O}_{yz'}^{s} - \zeta\left(P_{x'}^{c} + \xi \mathcal{O}_{yx'}^{c}\right)\right|\right\}\\
&\left|1 + \xi P_{y'} - \zeta\left(P_{y}^{\odot} + \xi \mathcal{O}_{yy'}^{\odot}\right)\right| \geqslant \left\{\left|I^{\odot} + \xi P_{y'}^{\odot} - \zeta(P_{y} + \xi \mathcal{O}_{yy'})\right|,\right.\\
&\left.\left|P_{x}^{s} + \xi \mathcal{O}_{xy'}^{s} + \zeta\left(P_{z}^{c} + \xi \mathcal{O}_{zy'}^{c}\right)\right|, \quad \left|P_{z}^{s} + \xi \mathcal{O}_{zy'}^{s} - \zeta\left(P_{x}^{c} + \xi \mathcal{O}_{xy'}^{c}\right)\right|\right\},\\
&\left|1 + \xi \mathcal{O}_{yy'} + \zeta\left(P_{y'}^{\odot} + \xi P_{y}^{\odot}\right)\right| \geqslant \left\{\left|I^{\odot} + \xi \mathcal{O}_{yy'}^{\odot} + \zeta(P_{y'} + \xi P_{y})\right|,\right.\\
&\left.\left|\mathcal{O}_{xx'}^{s} + \xi \mathcal{O}_{zz'}^{s} + \zeta\left(\mathcal{O}_{xz'}^{c} - \xi \mathcal{O}_{zx'}^{c}\right)\right|, \quad \left|\mathcal{O}_{xz'}^{s} - \xi \mathcal{O}_{zx'}^{s} - \zeta\left(\mathcal{O}_{xx'}^{c} + \xi \mathcal{O}_{zz'}^{c}\right)\right|\right\}.
\end{aligned}
\tag{60}
$$

TABLE III. Polarization observables of single- and double-pion photoproduction in terms of the helicity and transversity amplitudes. These observables arise with circularly polarized photons. The notation is as in Table II. $B_\odot$ indicates that a circularly polarized photon beam is needed for these measurements.

Observable	Helicity form	Transversity form	Expt.	Type
$I_0 I^\odot$	$-\lvert\mathcal{M}_1^-\rvert^2+\lvert\mathcal{M}_1^+\rvert^2-\lvert\mathcal{M}_2^-\rvert^2+\lvert\mathcal{M}_2^+\rvert^2$ $-\lvert\mathcal{M}_3^-\rvert^2+\lvert\mathcal{M}_3^+\rvert^2-\lvert\mathcal{M}_4^-\rvert^2+\lvert\mathcal{M}_4^+\rvert^2$	$-\lvert b_1^-\rvert^2+\lvert b_1^+\rvert^2-\lvert b_2^-\rvert^2+\lvert b_2^+\rvert^2$ $-\lvert b_3^-\rvert^2+\lvert b_3^+\rvert^2-\lvert b_4^-\rvert^2+\lvert b_4^+\rvert^2$	$\{c;-;-\}$	$B_\odot$
$I_0 P_x^\odot$	$2\Re(-\mathcal{M}_1^-\mathcal{M}_3^{-*}+\mathcal{M}_1^+\mathcal{M}_3^{+*}-\mathcal{M}_2^-\mathcal{M}_4^{-*}+\mathcal{M}_2^+\mathcal{M}_4^{+*})$	$2\Im(b_1^-b_3^{-*}-b_1^+b_3^{+*}+b_2^-b_4^{-*}-b_2^+b_4^{+*})$	$\{c;x;-\}$	$B_\odot$T
$I_0 P_y^\odot$	$2\Im(\mathcal{M}_1^-\mathcal{M}_3^{-*}-\mathcal{M}_1^+\mathcal{M}_3^{+*}+\mathcal{M}_2^-\mathcal{M}_4^{-*}-\mathcal{M}_2^+\mathcal{M}_4^{+*})$	$-\lvert b_1^-\rvert^2+\lvert b_1^+\rvert^2-\lvert b_2^-\rvert^2+\lvert b_2^+\rvert^2$ $+\lvert b_3^-\rvert^2-\lvert b_3^+\rvert^2+\lvert b_4^-\rvert^2-\lvert b_4^+\rvert^2$	$\{c;y;-\}$	
$I_0 P_z^\odot$	$\lvert\mathcal{M}_1^-\rvert^2-\lvert\mathcal{M}_1^+\rvert^2+\lvert\mathcal{M}_2^-\rvert^2-\lvert\mathcal{M}_2^+\rvert^2$ $-\lvert\mathcal{M}_3^-\rvert^2+\lvert\mathcal{M}_3^+\rvert^2-\lvert\mathcal{M}_4^-\rvert^2+\lvert\mathcal{M}_4^+\rvert^2$	$2\Re(b_1^-b_3^{-*}-b_1^+b_3^{+*}+b_2^-b_4^{-*}-b_2^+b_4^{+*})$	$\{c;z;-\}$	
$I_0 P_{x'}^\odot$	$2\Re(\mathcal{M}_1^-\mathcal{M}_2^{-*}-\mathcal{M}_1^+\mathcal{M}_2^{+*}+\mathcal{M}_3^-\mathcal{M}_4^{-*}-\mathcal{M}_3^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^-b_2^{-*}-b_1^+b_2^{+*}+b_3^-b_4^{-*}-b_3^+b_4^{+*})$	$\{c;-;x'\}$	$B_\odot$R
$I_0 P_{y'}^\odot$	$-2\Im(\mathcal{M}_1^-\mathcal{M}_2^{-*}-\mathcal{M}_1^+\mathcal{M}_2^{+*}+\mathcal{M}_3^-\mathcal{M}_4^{-*}-\mathcal{M}_3^+\mathcal{M}_4^{+*})$	$-\lvert b_1^-\rvert^2+\lvert b_1^+\rvert^2+\lvert b_2^-\rvert^2-\lvert b_2^+\rvert^2$ $-\lvert b_3^-\rvert^2+\lvert b_3^+\rvert^2+\lvert b_4^-\rvert^2-\lvert b_4^+\rvert^2$	$\{c;-;y'\}$	
$I_0 P_{z'}^\odot$	$-\lvert\mathcal{M}_1^-\rvert^2+\lvert\mathcal{M}_1^+\rvert^2+\lvert\mathcal{M}_2^-\rvert^2-\lvert\mathcal{M}_2^+\rvert^2$ $-\lvert\mathcal{M}_3^-\rvert^2+\lvert\mathcal{M}_3^+\rvert^2+\lvert\mathcal{M}_4^-\rvert^2-\lvert\mathcal{M}_4^+\rvert^2$	$2\Re(b_1^-b_2^{-*}-b_1^+b_2^{+*}+b_3^-b_4^{-*}-b_3^+b_4^{+*})$	$\{c;-;z'\}$	
$I_0\mathcal{O}_{xx'}^\odot$	$2\Re(\mathcal{M}_2^-\mathcal{M}_3^{-*}-\mathcal{M}_2^+\mathcal{M}_3^{+*}+\mathcal{M}_1^-\mathcal{M}_4^{-*}-\mathcal{M}_1^+\mathcal{M}_4^{+*})$	$2\Re(b_2^-b_3^{-*}-b_2^+b_3^{+*}-b_1^-b_4^{-*}+b_1^+b_4^{+*})$	$\{c;x;x'\}$	$B_\odot$TR
$I_0\mathcal{O}_{xy'}^\odot$	$2\Im(\mathcal{M}_2^-\mathcal{M}_3^{-*}-\mathcal{M}_2^+\mathcal{M}_3^{+*}-\mathcal{M}_1^-\mathcal{M}_4^{-*}+\mathcal{M}_1^+\mathcal{M}_4^{+*})$	$2\Im(b_1^-b_3^{-*}-b_1^+b_3^{+*}-b_2^-b_4^{-*}+b_2^+b_4^{+*})$	$\{c;x;y'\}$	
$I_0\mathcal{O}_{xz'}^\odot$	$2\Re(-\mathcal{M}_1^-\mathcal{M}_3^{-*}+\mathcal{M}_1^+\mathcal{M}_3^{+*}+\mathcal{M}_2^-\mathcal{M}_4^{-*}-\mathcal{M}_2^+\mathcal{M}_4^{+*})$	$-2\Im(b_2^-b_3^{-*}-b_2^+b_3^{+*}+b_1^-b_4^{-*}-b_1^+b_4^{+*})$	$\{c;x;z'\}$	
$I_0\mathcal{O}_{yx'}^\odot$	$-2\Im(\mathcal{M}_2^-\mathcal{M}_3^{-*}-\mathcal{M}_2^+\mathcal{M}_3^{+*}+\mathcal{M}_1^-\mathcal{M}_4^{-*}-\mathcal{M}_1^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^-b_2^{-*}-b_1^+b_2^{+*}-b_3^-b_4^{-*}+b_3^+b_4^{+*})$	$\{c;y;x'\}$	
$I_0\mathcal{O}_{yy'}^\odot$	$2\Re(\mathcal{M}_2^-\mathcal{M}_3^{-*}-\mathcal{M}_2^+\mathcal{M}_3^{+*}-\mathcal{M}_1^-\mathcal{M}_4^{-*}+\mathcal{M}_1^+\mathcal{M}_4^{+*})$	$-\lvert b_1^-\rvert^2+\lvert b_1^+\rvert^2+\lvert b_2^-\rvert^2-\lvert b_2^+\rvert^2$ $+\lvert b_3^-\rvert^2-\lvert b_3^+\rvert^2-\lvert b_4^-\rvert^2+\lvert b_4^+\rvert^2$	$\{c;y;y'\}$	
$I_0\mathcal{O}_{yz'}^\odot$	$2\Im(\mathcal{M}_1^-\mathcal{M}_3^{-*}-\mathcal{M}_1^+\mathcal{M}_3^{+*}-\mathcal{M}_2^-\mathcal{M}_4^{-*}+\mathcal{M}_2^+\mathcal{M}_4^{+*})$	$2\Re(b_1^-b_2^{-*}-b_1^+b_2^{+*}-b_3^-b_4^{-*}+b_3^+b_4^{+*})$	$\{c;y;z'\}$	
$I_0\mathcal{O}_{zx'}^\odot$	$2\Re(-\mathcal{M}_1^-\mathcal{M}_2^{-*}+\mathcal{M}_1^+\mathcal{M}_2^{+*}+\mathcal{M}_3^-\mathcal{M}_4^{-*}-\mathcal{M}_3^+\mathcal{M}_4^{+*})$	$-2\Im(b_2^-b_3^{-*}-b_2^+b_3^{+*}-b_1^-b_4^{-*}+b_1^+b_4^{+*})$	$\{c;z;x'\}$	
$I_0\mathcal{O}_{zy'}^\odot$	$2\Im(\mathcal{M}_1^-\mathcal{M}_2^{-*}-\mathcal{M}_1^+\mathcal{M}_2^{+*}-\mathcal{M}_3^-\mathcal{M}_4^{-*}+\mathcal{M}_3^+\mathcal{M}_4^{+*})$	$2\Re(b_1^-b_3^{-*}-b_1^+b_3^{+*}-b_2^-b_4^{-*}+b_2^+b_4^{+*})$	$\{c;z;y'\}$	
$I_0\mathcal{O}_{zz'}^\odot$	$\lvert\mathcal{M}_1^-\rvert^2-\lvert\mathcal{M}_1^+\rvert^2-\lvert\mathcal{M}_2^-\rvert^2+\lvert\mathcal{M}_2^+\rvert^2$ $-\lvert\mathcal{M}_3^-\rvert^2+\lvert\mathcal{M}_3^+\rvert^2+\lvert\mathcal{M}_4^-\rvert^2-\lvert\mathcal{M}_4^+\rvert^2$	$2\Re(-b_2^-b_3^{-*}+b_2^+b_3^{+*}-b_1^-b_4^{-*}+b_1^+b_4^{+*})$	$\{c;z;z'\}$	

These inequalities can also be manipulated (as was done for $\pi N\to\pi\pi N$) to lead to

$$
\begin{aligned}
&1+P_y^2+(I^\odot)^2+\left(P_y^\odot\right)^2\geqslant\big\{P_{y'}^2+\mathcal{O}_{yy'}^2+\left(P_{y'}^\odot\right)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2,\\
&P_{x'}^2+\mathcal{O}_{yx'}^2+\left(P_{x'}^\odot\right)^2+\left(\mathcal{O}_{yx'}^\odot\right)^2,\quad P_{z'}^2+\mathcal{O}_{yz'}^2+\left(P_{z'}^\odot\right)^2+\left(\mathcal{O}_{yz'}^\odot\right)^2\big\},\\
&1+P_{y'}^2+(I^\odot)^2+\left(P_{y'}^\odot\right)^2\geqslant\big\{P_y^2+\mathcal{O}_{yy'}^2+\left(P_y^\odot\right)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2,\\
&P_x^2+\mathcal{O}_{xy'}^2+\left(P_x^\odot\right)^2+\left(\mathcal{O}_{xy'}^\odot\right)^2,\quad P_z^2+\mathcal{O}_{zy'}^2+\left(P_z^\odot\right)^2+\left(\mathcal{O}_{zy'}^\odot\right)^2\big\},\\
&1+\mathcal{O}_{yy'}^2+(I^\odot)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2\geqslant\big\{P_y^2+P_{y'}^2+\left(P_y^\odot\right)^2+\left(P_{y'}^\odot\right)^2,\\
&\mathcal{O}_{xz'}^2+\mathcal{O}_{zx'}^2+\left(\mathcal{O}_{xz'}^\odot\right)^2+\left(\mathcal{O}_{zx'}^\odot\right)^2,\quad \mathcal{O}_{xx'}^2+\mathcal{O}_{zz'}^2+\left(\mathcal{O}_{xx'}^\odot\right)^2+\left(\mathcal{O}_{zz'}^\odot\right)^2\big\},\\
&1+\mathcal{O}_{yy'}^2+P_y^2+P_{y'}^2\geqslant\big\{(I^\odot)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2+\left(P_y^\odot\right)^2+\left(P_{y'}^\odot\right)^2,\\
&(I^s)^2+\left(\mathcal{O}_{yy'}^s\right)^2+\left(P_y^s\right)^2+\left(P_{y'}^s\right)^2,\quad (I^c)^2+\left(\mathcal{O}_{yy'}^c\right)^2+\left(P_y^c\right)^2+\left(P_{y'}^c\right)^2\big\},\\
&1+P_y^2+\left(P_{y'}^\odot\right)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2\geqslant\big\{(I^\odot)^2+\left(P_y^\odot\right)^2+P_{y'}^2+\mathcal{O}_{yy'}^2,\\
&\left(P_{x'}^s\right)^2+\left(\mathcal{O}_{yx'}^s\right)^2+\left(P_{z'}^c\right)^2+\left(\mathcal{O}_{yz'}^c\right)^2,\quad \left(P_{z'}^s\right)^2+\left(\mathcal{O}_{yz'}^s\right)^2+\left(P_{x'}^c\right)^2+\left(\mathcal{O}_{yx'}^c\right)^2\big\}\\
&1+P_{y'}^2+\left(P_y^\odot\right)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2\geqslant\big\{(I^\odot)^2+\left(P_{y'}^\odot\right)^2+P_y^2+\mathcal{O}_{yy'}^2,\\
&\left(P_x^s\right)^2+\left(\mathcal{O}_{xy'}^s\right)^2+\left(P_z^c\right)^2+\left(\mathcal{O}_{zy'}^c\right)^2,\quad \left(P_z^s\right)^2+\left(\mathcal{O}_{zy'}^s\right)^2+\left(P_x^c\right)^2+\left(\mathcal{O}_{xy'}^c\right)^2\big\},\\
&1+\mathcal{O}_{yy'}^2+\left(P_{y'}^\odot\right)^2+\left(P_y^\odot\right)^2\geqslant\big\{(I^\odot)^2+\left(\mathcal{O}_{yy'}^\odot\right)^2+P_{y'}^2+P_y^2,\\
&\left(\mathcal{O}_{xx'}^s\right)^2+\left(\mathcal{O}_{zz'}^s\right)^2+\left(\mathcal{O}_{xz'}^c\right)^2+\left(\mathcal{O}_{zx'}^c\right)^2,\quad \left(\mathcal{O}_{xz'}^s\right)^2+\left(\mathcal{O}_{zx'}^s\right)^2+\left(\mathcal{O}_{xx'}^c\right)^2+\left(\mathcal{O}_{zz'}^c\right)^2\big\}.
\end{aligned}
\tag{61}
$$

TABLE IV. Polarization observables of single- and double-pion photoproduction in terms of the helicity and transversity amplitudes. These observables arise with linearly polarized photons, and are proportional to $\sin 2\beta$ in the cross section. The notation is as in Table II. B_ℓ indicates that a circularly polarized photon beam is needed for these measurements. $L(\theta_1, \theta_2)$ indicates that the measurements require photon beams that are linearly polarized at angles θ_1 and θ_2 to the scattering plane.

Obs.	Helicity form	Transversity form	Expt.	Type
$I_0 I^s$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_1^{-*} + \mathcal{M}_2^+\mathcal{M}_2^{-*} + \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$-2\Im(b_1^+b_1^{-*} + b_2^+b_2^{-*} + b_3^+b_3^{-*} + b_4^+b_4^{-*})$	$\{L(\pm\frac{\pi}{4}); -; -\}$	B_ℓ
$I_0 P_x^s$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_3^{-*} - \mathcal{M}_1^-\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{-*} - \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Re(-b_1^+b_3^{-*} + b_1^-b_3^{+*} - b_2^+b_4^{-*} + b_2^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); x; -\}$	B_ℓT
$I_0 P_y^s$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_3^{-*} + \mathcal{M}_1^-\mathcal{M}_3^{+*} - \mathcal{M}_2^+\mathcal{M}_4^{-*} + \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_1^{-*} + b_2^+b_2^{-*} - b_3^+b_3^{-*} - b_4^+b_4^{-*})$	$\{L(\pm\frac{\pi}{4}); y; -\}$	
$I_0 P_z^s$	$2\Im(\mathcal{M}_1^+\mathcal{M}_1^{-*} + \mathcal{M}_2^+\mathcal{M}_2^{-*} - \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Im(b_1^+b_3^{-*} - b_1^-b_3^{+*} + b_2^+b_4^{-*} - b_2^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); z; -\}$	
$I_0 P_{x'}^s$	$2\Im(\mathcal{M}_1^+\mathcal{M}_2^{-*} - \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} - \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Re(b_1^+b_2^{-*} - b_1^-b_2^{+*} + b_3^+b_4^{-*} - b_3^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); -; x'\}$	B_ℓR
$I_0 P_{y'}^s$	$2\Re(\mathcal{M}_1^+\mathcal{M}_2^{-*} - \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} - \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_1^{-*} - b_2^+b_2^{-*} + b_3^+b_3^{-*} - b_4^+b_4^{-*})$	$\{L(\pm\frac{\pi}{4}); -; y'\}$	
$I_0 P_{z'}^s$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_1^{-*} - \mathcal{M}_2^+\mathcal{M}_2^{-*} + \mathcal{M}_3^+\mathcal{M}_3^{-*} - \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Im(b_1^+b_2^{-*} - b_1^-b_2^{+*} + b_3^+b_4^{-*} - b_3^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); -; z'\}$	
$I_0 \mathcal{O}_{xx'}^s$	$2\Im(\mathcal{M}_2^+\mathcal{M}_3^{-*} - \mathcal{M}_2^-\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{-*} - \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Im(b_2^+b_3^{-*} - b_2^-b_3^{+*} - b_1^+b_4^{-*} + b_1^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); x; x'\}$	B_ℓTR
$I_0 \mathcal{O}_{xy'}^s$	$2\Re(-\mathcal{M}_2^+\mathcal{M}_3^{-*} + \mathcal{M}_2^-\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{-*} - \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Re(-b_1^+b_3^{-*} + b_1^-b_3^{+*} + b_2^+b_4^{-*} - b_2^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); x; y'\}$	
$I_0 \mathcal{O}_{xz'}^s$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_3^{-*} - \mathcal{M}_1^-\mathcal{M}_3^{+*} - \mathcal{M}_2^+\mathcal{M}_4^{-*} + \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Re(b_2^+b_3^{-*} - b_2^-b_3^{+*} + b_1^+b_4^{-*} - b_1^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); x; z'\}$	
$I_0 \mathcal{O}_{yx'}^s$	$2\Re(\mathcal{M}_2^+\mathcal{M}_3^{-*} - \mathcal{M}_2^-\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{-*} - \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Re(b_1^+b_2^{-*} - b_1^-b_2^{+*} - b_3^+b_4^{-*} + b_3^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); y; x'\}$	
$I_0 \mathcal{O}_{yy'}^s$	$2\Im(\mathcal{M}_2^+\mathcal{M}_3^{-*} - \mathcal{M}_2^-\mathcal{M}_3^{+*} - \mathcal{M}_1^+\mathcal{M}_4^{-*} + \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_1^{-*} - b_2^+b_2^{-*} - b_3^+b_3^{-*} + b_4^+b_4^{-*})$	$\{L(\pm\frac{\pi}{4}); y; y'\}$	
$I_0 \mathcal{O}_{yz'}^s$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_3^{-*} + \mathcal{M}_1^-\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{-*} - \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Im(b_1^+b_2^{-*} - b_1^-b_2^{+*} - b_3^+b_4^{-*} + b_3^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); y; z'\}$	
$I_0 \mathcal{O}_{zx'}^s$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_2^{-*} - \mathcal{M}_1^-\mathcal{M}_2^{+*} - \mathcal{M}_3^+\mathcal{M}_4^{-*} + \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Re(b_2^+b_3^{-*} - b_2^-b_3^{+*} - b_1^+b_4^{-*} + b_1^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); z; x'\}$	
$I_0 \mathcal{O}_{zy'}^s$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_2^{-*} + \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} - \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Im(b_1^+b_3^{-*} - b_1^-b_3^{+*} - b_2^+b_4^{-*} + b_2^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); z; y'\}$	
$I_0 \mathcal{O}_{zz'}^s$	$2\Im(\mathcal{M}_1^+\mathcal{M}_1^{-*} - \mathcal{M}_2^+\mathcal{M}_2^{-*} - \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$-2\Im(b_2^+b_3^{-*} - b_2^-b_3^{+*} + b_1^+b_4^{-*} - b_1^-b_4^{+*})$	$\{L(\pm\frac{\pi}{4}); z; z'\}$	

A. Required experimental measurements in $\gamma N \to \pi\pi N$

As in the case of $\pi N \to \pi\pi N$, we can examine which observables need to be measured to extract information on the helicity or transversity amplitudes. As there are eight such amplitudes, a minimum of eight measurements must be made at each kinematic point (recall that these observables depend on five kinematic variables) to obtain the absolute magnitudes of the helicity or transversity amplitudes. In terms of our choice of transversity basis, these measurements are the differential cross section, along with P_y, $P_{y'}$, $\mathcal{O}_{yy'}$, $I^{\odot}$, $P_y^{\odot}$, $P_{y'}^{\odot}$, and $\mathcal{O}_{yy'}^{\odot}$.

The eight phases of the transversity amplitude mean that there are seven independent phase differences that can be extracted, and seven measurements are needed for this. For instance, the relative phases (in what should be an obvious notation) $\phi_1^- - \phi_2^-$, $\phi_1^+ - \phi_2^+$, $\phi_3^- - \phi_4^-$, and $\phi_3^+ - \phi_4^+$ require measurement of any four of the eight observables $P_{x'}$, $P_{z'}$, $\mathcal{O}_{yx'}$, $\mathcal{O}_{yz'}$, $P_{x'}^{\odot}$, $P_{z'}^{\odot}$, $\mathcal{O}_{yx'}^{\odot}$, and $\mathcal{O}_{yz'}^{\odot}$. $\phi_1^- - \phi_3^-$, and $\phi_1^- + \phi_3^+$ may then be extracted from measurement of any two observables from among P_x, P_z, $\mathcal{O}_{xy'}$, $\mathcal{O}_{zy'}$, $P_x^{\odot}$, $P_z^{\odot}$, $\mathcal{O}_{xy'}^{\odot}$, and $\mathcal{O}_{zy'}^{\odot}$, along with use of the identities $\phi_2^\pm - \phi_4^\pm = (\phi_2^\pm - \phi_1^\pm) + (\phi_1^\pm - \phi_3^\pm) + (\phi_3^\pm - \phi_4^\pm)$. The remaining independent phase can then be extracted from one of the observables that arise from linearly polarized photons. A "complete" set of experiments will therefore require measurement of single, double, and triple polarization observables, in addition to the differential cross section.

B. Parity conservation

For the process $\gamma N \to \pi N$, parity conservation leads to the relationships

$$\mathcal{M}_{-\lambda_N -\lambda'_N}^{-\lambda_\gamma}(\theta) = (-1)^{\lambda_\gamma - \lambda_N + \lambda'_N} \mathcal{M}_{\lambda_N \lambda'_N}^{\lambda_\gamma}(\theta). \qquad (62)$$

The relationships that arise among the helicity amplitudes for $\gamma N \to \pi\pi N$ are

$$\mathcal{M}_{-\lambda_N -\lambda'_N}^{-\lambda_\gamma}(\theta, \theta_1, \phi_1) = (-1)^{\lambda_\gamma - \lambda_N + \lambda'_N} \mathcal{M}_{\lambda_N \lambda'_N}^{\lambda_\gamma}(\theta, \theta_1, 2\pi - \phi_1). \qquad (63)$$

As was the case with $\pi N \to \pi\pi N$, these relations cannot be used to decrease the number of independent helicity amplitudes, but they can be used to determine which observables are even or odd under the transformation $\phi_1 \leftrightarrow 2\pi - \phi_1$.

TABLE V. Polarization observables of single- and double-pion photoproduction in terms of the helicity and transversity amplitudes. These observables arise with linearly polarized photons, and are proportional to $\cos 2\beta$ in the cross section. The notation is as in Table II. B_ℓ indicates that a circularly polarized photon beam is needed for these measurements. $L(\theta_1, \theta_2)$ indicates that the measurements require photon beams that are linearly polarized at angles θ_1 and θ_2 to the scattering plane.

Obs.	Helicity form	Transversity form	Expt.	Type
$I_0 I^c$	$-2\Re(\mathcal{M}_1^+\mathcal{M}_1^{-*} + \mathcal{M}_2^+\mathcal{M}_2^{-*} + \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Re[-(b_1^+b_1^{-*}) - b_2^+b_2^{-*} - b_3^+b_3^{-*} - b_4^+b_4^{-*}]$	$\{L(\frac{\pi}{2},0);-;-\}$	B_ℓ
$I_0 P_x^c$	$-2\Re(\mathcal{M}_1^+\mathcal{M}_3^{-*} + \mathcal{M}_1^-\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{-*} + \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Im(b_1^+b_3^{-*} + b_1^-b_3^{+*} + b_2^+b_4^{-*} + b_2^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;-\}$	B_ℓT
$I_0 P_y^c$	$2\Im(\mathcal{M}_1^+\mathcal{M}_3^{-*} + \mathcal{M}_1^-\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{-*} + \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Re[-(b_1^+b_1^{-*}) - b_2^+b_2^{-*} + b_3^+b_3^{-*} + b_4^+b_4^{-*}]$	$\{L(\frac{\pi}{2},0);y;-\}$	
$I_0 P_z^c$	$2\Re(\mathcal{M}_1^+\mathcal{M}_1^{-*} + \mathcal{M}_2^+\mathcal{M}_2^{-*} - \mathcal{M}_3^+\mathcal{M}_3^{-*} - \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Re(b_1^+b_3^{-*} + b_1^-b_3^{+*} + b_2^+b_4^{-*} + b_2^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);z;-\}$	
$I_0 P_{x'}^c$	$2\Re(\mathcal{M}_1^+\mathcal{M}_2^{-*} + \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} + \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_2^{-*} + b_1^-b_2^{+*} + b_3^+b_4^{-*} + b_3^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);-;x'\}$	B_ℓR
$I_0 P_{y'}^c$	$-2\Im(\mathcal{M}_1^+\mathcal{M}_2^{-*} + \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} + \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Re[-(b_1^+b_1^{-*}) + b_2^+b_2^{-*} - b_3^+b_3^{-*} + b_4^+b_4^{-*}]$	$\{L(\frac{\pi}{2},0);-;y'\}$	
$I_0 P_{z'}^c$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_1^{-*} + \mathcal{M}_2^+\mathcal{M}_2^{-*} - \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Re(b_1^+b_2^{-*} + b_1^-b_2^{+*} + b_3^+b_4^{-*} + b_3^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);-;z'\}$	
$I_0\mathcal{O}_{xx'}^c$	$2\Re(\mathcal{M}_2^+\mathcal{M}_3^{-*} + \mathcal{M}_2^-\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{-*} + \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Re(b_2^+b_3^{-*} + b_2^-b_3^{+*} - b_1^+b_4^{-*} - b_1^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;x'\}$	B_ℓTR
$I_0\mathcal{O}_{xy'}^c$	$2\Im(\mathcal{M}_2^+\mathcal{M}_3^{-*} + \mathcal{M}_2^-\mathcal{M}_3^{+*} - \mathcal{M}_1^+\mathcal{M}_4^{-*} - \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Im(b_1^+b_3^{-*} + b_1^-b_3^{+*} - b_2^+b_4^{-*} - b_2^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;y'\}$	
$I_0\mathcal{O}_{xz'}^c$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_3^{-*} - \mathcal{M}_1^-\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{-*} + \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$-2\Im(b_2^+b_3^{-*} + b_2^-b_3^{+*} + b_1^+b_4^{-*} + b_1^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;z'\}$	
$I_0\mathcal{O}_{yx'}^c$	$-2\Im(\mathcal{M}_2^+\mathcal{M}_3^{-*} + \mathcal{M}_2^-\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{-*} + \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_2^{-*} + b_1^-b_2^{+*} - b_3^+b_4^{-*} - b_3^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;x'\}$	
$I_0\mathcal{O}_{yy'}^c$	$2\Re(\mathcal{M}_2^+\mathcal{M}_3^{-*} + \mathcal{M}_2^-\mathcal{M}_3^{+*} - \mathcal{M}_1^+\mathcal{M}_4^{-*} - \mathcal{M}_1^-\mathcal{M}_4^{+*})$	$2\Re[-(b_1^+b_1^{-*}) + b_2^+b_2^{-*} + b_3^+b_3^{-*} - b_4^+b_4^{-*}]$	$\{L(\frac{\pi}{2},0);x;y'\}$	
$I_0\mathcal{O}_{yz'}^c$	$2\Im(\mathcal{M}_1^+\mathcal{M}_3^{-*} + \mathcal{M}_1^-\mathcal{M}_3^{+*} - \mathcal{M}_2^+\mathcal{M}_4^{-*} - \mathcal{M}_2^-\mathcal{M}_4^{+*})$	$2\Re(b_1^+b_2^{-*} + b_1^-b_2^{+*} - b_3^+b_4^{-*} - b_3^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);x;z'\}$	
$I_0\mathcal{O}_{zx'}^c$	$2\Re(-\mathcal{M}_1^+\mathcal{M}_2^{-*} - \mathcal{M}_1^-\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{-*} + \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Im(-b_2^+b_3^{-*} - b_2^-b_3^{+*} + b_1^+b_4^{-*} + b_1^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);z;x'\}$	
$I_0\mathcal{O}_{zy'}^c$	$2\Im(\mathcal{M}_1^+\mathcal{M}_2^{-*} + \mathcal{M}_1^-\mathcal{M}_2^{+*} - \mathcal{M}_3^+\mathcal{M}_4^{-*} - \mathcal{M}_3^-\mathcal{M}_4^{+*})$	$2\Re(b_1^+b_3^{-*} + b_1^-b_3^{+*} - b_2^+b_4^{-*} - b_2^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);z;y'\}$	
$I_0\mathcal{O}_{zz'}^c$	$2\Re(\mathcal{M}_1^+\mathcal{M}_1^{-*} - \mathcal{M}_2^+\mathcal{M}_2^{-*} - \mathcal{M}_3^+\mathcal{M}_3^{-*} + \mathcal{M}_4^+\mathcal{M}_4^{-*})$	$2\Re(-b_2^+b_3^{-*} - b_2^-b_3^{+*} - b_1^+b_4^{-*} - b_1^-b_4^{+*})$	$\{L(\frac{\pi}{2},0);z;z'\}$	

C. Construction of transition amplitudes

1. $\gamma N \to \pi N$

In this case, there are two independent vectors $\vec{k}$, the momentum of the photon, and $\vec{q}$, the momentum of the pion. $\mathcal{A}$ must be an axial vector, whereas $\mathcal{B}_{ij}$ must be a tensor. For real photons, $\vec{\varepsilon}\cdot\vec{k} = 0$, so there can be no k_j terms in $\mathcal{B}_{ij}$. The forms that can be written are

$$\vec{\mathcal{A}} = \alpha\hat{k}\times\hat{q}, \qquad \mathcal{B}_{ij} = \beta_1\delta_{ij} + \beta_2\hat{k}_i\hat{q}_j + \beta_3\hat{q}_i\hat{q}_j. \tag{64}$$

Comparing this with the form written by Chew, Goldberger, Low, and Nambu [13]

$$i\mathcal{M} = \chi^\dagger(F_1\vec{\sigma}\cdot\vec{\varepsilon} + iF_2\vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\hat{k}\times\vec{\varepsilon} + F_3\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\vec{\varepsilon} + F_4\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\vec{\varepsilon})\phi \tag{65}$$

means that we can identify

$$\alpha = iF_2, \qquad \beta_1 = F_1 - \hat{k}\cdot\hat{q}F_2, \qquad \beta_2 = F_2 + F_3, \qquad \beta_3 = F_4. \tag{66}$$

From the explicit forms for $\hat{k}$, $\hat{q}$, and $\vec{\varepsilon}(\lambda)$, we can use Eq. (56) to obtain $\mathcal{M}_1^\mp = \mathcal{M}_4^\pm$, $\mathcal{M}_3^\mp = -\mathcal{M}_2^\pm$ (or, equivalently, $b_4^\mp = b_1\pm$, $b_3^\mp = -b_2^\pm$), leaving 4 independent helicity amplitudes, as expected. These helicity amplitudes are related to those of Storrow [3], for example, by $N = \mathcal{M}_2^+$, $S_1 = \mathcal{M}_4^+$, $S_2 = \mathcal{M}_1^+$, and $D = \mathcal{M}_3^+$. Of the 64 observables, 32 vanish identically. Furthermore, all 13 remaining triple-polarization observables are related to double- or single-polarization observables, or the differential cross section, and three of the remaining 15 double-polarization observables are related to single-polarization observables, leaving a total of 16 observables. The remaining observables are given in terms of the helicity and transversity amplitudes in Table VI.

The relationships among these observables, obtained from consideration of the transversity amplitudes, are

$$\begin{aligned}
\left(P_{x'}^{\odot} \mp P_{z'}^{s}\right)^2 + \left(P_{z'}^{\odot} \pm P_{x'}^{s}\right)^2 &= (1 \pm P_y)^2 - (P_{y'} \pm \mathcal{O}_{yy'})^2,\\
\left(P_{x}^{\odot} \pm P_{z}^{s}\right)^2 + \left(P_{z}^{\odot} \mp P_{x}^{s}\right)^2 &= (1 \pm P_{y'})^2 - (P_{y} \pm \mathcal{O}_{yy'})^2,\\
(\mathcal{O}_{xz'} \mp \mathcal{O}_{zx'})^2 + (\mathcal{O}_{xx'} \pm \mathcal{O}_{zz'})^2 &= (1 \pm \mathcal{O}_{yy'})^2 - (P_y \pm P_{y'})^2.
\end{aligned} \tag{67}$$

TABLE VI. Polarization observables of single-pion photoproduction expressed as bilinear forms of the helicity amplitudes.

Observable		Usual name	Helicity form	Transversity form	Measurements
I_0	$(-\mathcal{O}^c_{yy'})$	I_0	$\lvert\mathcal{M}_1^+\rvert^2 + \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 + \lvert\mathcal{M}_4^+\rvert^2$	$\lvert b_1^+\rvert^2 + \lvert b_2^+\rvert^2 + \lvert b_3^+\rvert^2 + \lvert b_4^+\rvert^2$	$\{-;-;-\}, \{L(\frac{\pi}{2},0);x;y'\}$
P_y	$(-P^c_{y'})$	T	$-2\Im(\mathcal{M}_1^+\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$\lvert b_1^+\rvert^2 + \lvert b_2^+\rvert^2 - \lvert b_3^+\rvert^2 - \lvert b_4^+\rvert^2$	$\{-;y;-\}, \{L(\frac{\pi}{2},0);-;y'\}$
$P_{y'}$	$(-P^c_y)$	P	$2\Im(\mathcal{M}_1^+\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$\lvert b_1^+\rvert^2 - \lvert b_2^+\rvert^2 + \lvert b_3^+\rvert^2 - \lvert b_4^+\rvert^2$	$\{-;-;y'\}, \{L(\frac{\pi}{2},0);y;-\}$
$\mathcal{O}_{xx'}$	$(-\mathcal{O}^c_{zz'})$	T_x	$2\Re(-\mathcal{M}_2^+\mathcal{M}_3^{+*} - \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$2\Re(-b_2^+b_3^{+*} + b_1^+b_4^{+*})$	$\{-;x;x'\}, \{L(\frac{\pi}{2},0);z;z'\}$
$\mathcal{O}_{xz'}$	$(\mathcal{O}^c_{zx'})$	T_z	$2\Re(\mathcal{M}_1^+\mathcal{M}_3^{+*} - \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$-2\Im(-b_2^+b_3^{+*} - b_1^+b_4^{+*})$	$\{-;x;z'\}, \{L(\frac{\pi}{2},0);z;x'\}$
$\mathcal{O}_{yy'}$	$(-I^c)$	Σ	$2\Re(-\mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$\lvert b_1^+\rvert^2 - \lvert b_2^+\rvert^2 - \lvert b_3^+\rvert^2 + \lvert b_4^+\rvert^2$	$\{-;y;y'\}, \{L(\frac{\pi}{2},0);-;-\}$
$\mathcal{O}_{zx'}$	$(\mathcal{O}^c_{xz'})$	L_x	$2\Re(\mathcal{M}_1^+\mathcal{M}_2^{+*} - \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$-2\Im(-b_2^+b_3^{+*} + b_1^+b_4^{+*})$	$\{-;z;x'\}, \{L(\frac{\pi}{2},0);x;z'\}$
$\mathcal{O}_{zz'}$	$(-\mathcal{O}^c_{xx'})$	L_z	$-\lvert\mathcal{M}_1^+\rvert^2 + \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 - \lvert\mathcal{M}_4^+\rvert^2$	$2\Re(b_2^+b_3^{+*} + b_1^+b_4^{+*})$	$\{-;z;z'\}, \{L(\frac{\pi}{2},0);x;x'\}$
$P_x^\odot$	$(\mathcal{O}^s_{zy'})$	F	$2\Re(\mathcal{M}_1^+\mathcal{M}_3^{+*} + \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_3^{+*} + b_2^+b_4^{+*})$	$\{c;x;-\}, \{L(\pm\frac{\pi}{4});z;y'\}$
$P_z^\odot$	$(-\mathcal{O}^s_{xy'})$	E	$-\lvert\mathcal{M}_1^+\rvert^2 - \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 + \lvert\mathcal{M}_4^+\rvert^2$	$2\Re(-b_1^+b_3^{+*} - b_2^+b_4^{+*})$	$\{c;z;-\}, \{L(\pm\frac{\pi}{4});x;y'\}$
$P_{x'}^\odot$	$(-\mathcal{O}^s_{yz'})$	C_x	$-2\Re(\mathcal{M}_1^+\mathcal{M}_2^{+*} + \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$-2\Im(-b_1^+b_2^{+*} - b_3^+b_4^{+*})$	$\{c;-;x'\}, \{L(\pm\frac{\pi}{4});y;z'\}$
$P_{z'}^\odot$	$(\mathcal{O}^s_{yx'})$	C_z	$\lvert\mathcal{M}_1^+\rvert^2 - \lvert\mathcal{M}_2^+\rvert^2 + \lvert\mathcal{M}_3^+\rvert^2 - \lvert\mathcal{M}_4^+\rvert^2$	$2\Re(-b_1^+b_2^{+*} - b_3^+b_4^{+*})$	$\{c;-;z'\}, \{L(\pm\frac{\pi}{4});y;x'\}$
P_x^s	$(-\mathcal{O}^\odot_{zy'})$	H	$2\Im(\mathcal{M}_1^+\mathcal{M}_2^{+*} - \mathcal{M}_3^+\mathcal{M}_4^{+*})$	$2\Re(b_1^+b_3^{+*} - b_2^+b_4^{+*})$	$\{L(\pm\frac{\pi}{4});x;-\}, \{c;z;y'\}$
P_z^s	$(\mathcal{O}^\odot_{xy'})$	G	$2\Im(-\mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_3^{+*} - b_2^+b_4^{+*})$	$\{L(\pm\frac{\pi}{4});z;-\}, \{c;x;y'\}$
$P_{x'}^s$	$(\mathcal{O}^\odot_{yz'})$	O_x	$-2\Im(\mathcal{M}_1^+\mathcal{M}_3^{+*} - \mathcal{M}_2^+\mathcal{M}_4^{+*})$	$2\Re(-b_1^+b_2^{+*} + b_3^+b_4^{+*})$	$\{L(\pm\frac{\pi}{4});-;x'\}, \{c;y;z'\}$
$P_{z'}^s$	$(-\mathcal{O}^\odot_{yx'})$	O_z	$-2\Im(\mathcal{M}_2^+\mathcal{M}_3^{+*} + \mathcal{M}_1^+\mathcal{M}_4^{+*})$	$-2\Im(b_1^+b_2^{+*} - b_3^+b_4^{+*})$	$\{L(\pm\frac{\pi}{4});-;z'\}, \{c;y;x'\}$

These lead to the inequalities

$$\begin{aligned}
|1 \pm P_y| &\geqslant \left\{\left|P_{x'}^\odot \mp P_{z'}^s\right|,\quad \left|P_{z'}^\odot \pm P_{x'}^s\right|,\quad |P_{y'} \pm \mathcal{O}_{yy'}|\right\},\\
|1 \pm P_{y'}| &\geqslant \left\{\left|P_x^\odot \pm P_z^s\right|,\quad \left|P_z^\odot \mp P_x^s\right|,\quad |P_y \pm \mathcal{O}_{yy'}|\right\},\\
|1 \pm \mathcal{O}_{yy'}| &\geqslant \{|\mathcal{O}_{xz'} \mp \mathcal{O}_{zx'}|,\quad |\mathcal{O}_{xx'} \pm \mathcal{O}_{zz'}|,\quad |P_y \pm P_{y'}|\},
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
1 + P_y^2 &\geqslant \left\{\left(P_{x'}^\odot\right)^2 + \left(P_{z'}^s\right)^2,\quad \left(P_{z'}^\odot\right)^2 + \left(P_{x'}^s\right)^2,\quad P_{y'}^2 + \mathcal{O}_{yy'}^2\right\},\\
1 + P_{y'}^2 &\geqslant \left\{\left(P_x^\odot\right)^2 + \left(P_z^s\right)^2,\quad \left(P_z^\odot\right)^2 + \left(P_x^s\right)^2,\quad P_y^2 + \mathcal{O}_{yy'}^2\right\},\\
1 + \mathcal{O}_{yy'}^2 &\geqslant \left\{\mathcal{O}_{xz'}^2 + \mathcal{O}_{zx'}^2,\quad \mathcal{O}_{xx'}^2 + \mathcal{O}_{zz'}^2,\quad P_y^2 + P_{y'}^2\right\}.
\end{aligned} \tag{69}$$

2. $\gamma N \to \pi\pi N$

For this process, we have three independent vectors, $\hat{k}$, $\hat{p}_2$, and $\hat{q}_1$ with which to construct a vector for $\vec{\mathcal{A}}$ and a pseudotensor for $\mathcal{B}_{ij}$. However, using these leads to the difficulty that there are too many structures left in $\mathcal{B}_{ij}$. To avoid this problem, we define an axial vector $\hat{n}$ as $\hat{n} = \hat{k} \times \hat{p}_2/N$, with $N = |\hat{k} \times \hat{p}_2| = \sin\theta$. $\hat{n}$ defines the y axis, whereas the x axis is defined by $\hat{n} \times \hat{k} = (\hat{p}_2 - \hat{k}\hat{k}\cdot\hat{p}_2)/N$. We can now write

$$\begin{aligned}
\hat{q}_1 &= \hat{q}_1 \cdot \hat{k}\hat{k} + \hat{q}_1 \cdot \hat{n}\hat{n} + (\hat{q}_1 \cdot \hat{p}_2 - \hat{q}_1 \cdot \hat{k}\hat{k} \cdot \hat{p}_2)\\
&\quad \times (\hat{p}_2 - \hat{k}\hat{k}\cdot\hat{p}_2)/N^2
\end{aligned} \tag{70}$$

and use the axial vector $\hat{n}$ and the pseudoscalar $\mathcal{P} = \hat{q}_1 \cdot \hat{k} \times \hat{p}_2 = N\hat{q}_1 \cdot \hat{n}$ instead of $\hat{q}_1$ to build the structures that make up A^i and B^{ij}. $\hat{n}$ and $\mathcal{P}$ can appear only once in these structures because, by expanding the product of Levi-Civita tensors, it is easy to show that $\mathcal{P}^2$ is a scalar that depends only on quantities already defined

$$\begin{aligned}
\mathcal{P}^2 = 1 &- (\hat{k}\cdot\hat{p}_2)^2 - (\hat{k}\cdot\hat{q}_1)^2 - (\hat{p}_2\cdot\hat{q}_1)^2\\
&- 2\hat{k}\cdot\hat{p}_2\hat{k}\cdot\hat{q}_1\hat{p}_2\cdot\hat{q}_1,
\end{aligned} \tag{71}$$

whereas $\hat{n}^i\hat{n}^j$ can be expressed as

$$\hat{n}^i\hat{n}^j = \delta^{ij} - \frac{\hat{p}_2^i\hat{p}_2^j + \hat{k}^i\hat{k}^j + \hat{k}\cdot\hat{p}_2\left(\hat{k}^i\hat{p}_2^j + \hat{p}_2^i\hat{k}^j\right)}{N^2}. \tag{72}$$

The vectors that can make up A^i are k^i, q_1^i and $\mathcal{P}\hat{n}^i$. Because $\varepsilon \cdot \hat{k} = 0$, only two structures remain. Similarly B^{ij} can be expressed as a sum of $\hat{n}^i\hat{p}_2^j$, $\hat{n}^i\hat{k}^j$, $\hat{p}_2^i\hat{n}^j$, $\mathcal{P}\hat{p}_2^i\hat{p}_2^j$, $\mathcal{P}\hat{p}_2^i\hat{k}^j$, $\mathcal{P}\delta^{ij}$, $\epsilon^{ijk}\hat{k}^k$, $\epsilon^{ijk}\hat{p}_2^k$, and $\mathcal{P}\epsilon^{ijk}\hat{n}^k$.

Expressing ε^i as

$$\begin{aligned}
\varepsilon^i &= \varepsilon\cdot\hat{k}\hat{k}^i + \varepsilon\cdot\hat{n}\hat{n}^i + (\varepsilon\cdot\hat{p}_2 - \varepsilon\cdot\hat{k}\hat{k}\cdot\hat{p}_2)\\
&\quad \times\left(\hat{p}_2^i - \hat{k}^i\hat{k}\cdot\hat{p}_2\right)/N^2 && (73)\\
&= \varepsilon\cdot\hat{n}\hat{n}^i + \varepsilon\cdot\hat{p}_2\left(\hat{p}_2^i - \hat{k}^i\hat{k}\cdot\hat{p}_2\right)/N^2, && (74)
\end{aligned}$$

it is easy to show that the last three structures can be expressed as

$$\epsilon^{ijk}\hat{k}^k\varepsilon^i = \frac{\hat{n}^i\hat{p}_2^j - \hat{n}^i\hat{k}^j\hat{p}_2\cdot\hat{k} - \hat{p}_2^i\hat{n}^j}{N}\varepsilon^i \tag{75}$$

$$\epsilon^{ijk}\hat{p}_2^k\varepsilon^i = \frac{\hat{n}^i\hat{p}_2^j\hat{k}\cdot\hat{p}_2 - \hat{n}^i\hat{k}^j + \hat{p}_2^i\hat{n}^j\hat{k}\cdot\hat{p}_2}{N}\varepsilon^i \tag{76}$$

$$\mathcal{P}\epsilon^{ijk}\hat{n}^k\varepsilon^i = \mathcal{P}\frac{\hat{p}_2^i\hat{k}^j}{N}\varepsilon^i. \tag{77}$$

These three structures can therefore be omitted from the construction of the amplitude. Finally, we write

$$\begin{aligned}\vec{\mathcal{A}} &= \alpha_1\hat{q}_1 + \alpha_2\mathcal{P}\hat{n},\\ \mathcal{B}^{ij} &= \beta_1\hat{n}^i\hat{p}_2^j + \beta_2\hat{n}^i\hat{k}^j + \beta_3\hat{p}_2^i\hat{n}^j\\ &\quad + \beta_4\mathcal{P}\hat{p}_2^i\hat{p}_2^j + \beta_5\mathcal{P}\hat{p}_2^i\hat{k}^j + \beta_6\mathcal{P}\delta^{ij}. \qquad (78)\end{aligned}$$

As discussed previously, parity conservation can be used to tell which observables are even and which are odd under the transformation $\phi_1 \leftrightarrow 2\pi - \phi_1$. In the previous subsection, we listed the nonvanishing observables for $\gamma N \to N\pi$. The corresponding observables in $\gamma N \to N\pi\pi$ are all even under the transformation in ϕ_1. The variables that vanish in $\gamma N \to \pi N$ are nonzero for $\gamma N \to \pi\pi N$ but are odd under the ϕ_1 transformation.

V. CONCLUSION AND OUTLOOK

We have developed a set of polarization observables that are applicable to final states that contain two pseudoscalar mesons and a spin-1/2 baryon, such as $N\pi\pi$, and have examined the observables that arise using both photon and pion (or other pseudoscalar meson) beams. We have written these observables in terms of both helicity and transversity amplitudes, obtained relationships among them, and used these to list inequalities that these observables satisfy. We have also indicated the measurements that are needed for each observable. The framework that we have used is a very simple one: undoubtedly, the expressions for the observables and the relationships among them can be derived in a more elegant manner.

Although we have discussed helicity and one set of transversity amplitudes, there remains the possibility of defining yet another set of amplitudes and writing the observables in terms of these. In the c.m. frame, the momenta of the final particles satisfy $\vec{p}_2 + \vec{q}_1 + \vec{q}_2 = \vec{0}$, which means that they define a plane. The normal to this plane can be defined by $\vec{p}_2 \times \vec{q}_1$, and this offers another natural axis for quantization of the spin of the final nucleon. One possible advantage of using this axis is that it automatically incorporates information about the entire final state, not just the final nucleon. Whether this leads to any particular advantage, simplification, or insight into the observables, the relationships among them, or even in the amplitudes themselves awaits exploration.

As we stated at the start of this manuscript, polarization observables are crucial for extracting resonance information from scattering data. Differential cross sections, presented in whatever form will provide information only on the magnitudes of helicity or transversity amplitudes. Phase information is crucial, and this is available only from measurement of a number of different observables. This is well known for processes such as $\gamma N \to \pi N$. The same is true, or perhaps even more true, for processes such as $\gamma N \to \pi\pi N$. Models with quite different input can and will succeed in describing the total and differential cross section, but the polarization observables will serve to distinguish among such models.

A number of these observables can be measured in the near future at existing facilities, for a number of processes. Indeed, the photon polarization asymmetry $I^{\odot}$ has already been measured at Jefferson Laboratory [14] for $\gamma p \to p\pi^+\pi^-$, and the analysis is continuing at present. Clearly, this variable can be measured in other channels, and there are plans to do so for $\gamma p \to p\pi^0\pi^0$, at Bonn [15]. The existence of polarized targets means that P_x, P_y, and P_z are accessible, and coupling such targets with circularly polarized beams allows measurement of $P_x^{\odot}$, $P_y^{\odot}$, and $P_z^{\odot}$. The use of linearly polarized photons opens the door to measurements of $P_x^{s,c}$, $P_y^{s,c}$, $P_z^{s,c}$, and $I^{s,c}$. For processes with a hyperon in the final state, such as $\gamma N \to \pi K\Lambda$, the self-analyzing decay of the hyperon allows its polarization to be determined, in principle allowing many more observables to be measured, including a number of triple-polarization ones. For processes such as $\pi N \to \pi\pi N$, three of the observables are readily available with polarized targets. All others require the measurement of recoil polarization. Unfortunately, there are at present no existing hadronic beams facilities that will allow us to capitalize on these observables.

We have not attempted to explore the properties of the observables that we described herein, apart from a brief discussion of the oddness or evenness under the ϕ_1 transformation. In particular, we have said nothing on their values at special values of θ, for instance, such as $\theta = 0$ or π. This is left for a possible future manuscript. In the near future, we plan to explore a number of these observables in the framework of an existing model for the photoproduction of two pseudoscalar mesons off a nucleon target. In particular, the sensitivity of the observables to the details of the underlying dynamics, as well as the rich structure of these observables, are discussed.

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