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Number of spin I states of identical particles

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In this article we study the enumeration of number (denoted as D_I) of spin I states for fermions in a single-j shell and bosons with spin l. We show that D_I can be enumerated by the reduction from SU$(n+1)$ to SO(3). New regularities of D_I are discerned. As an example of our new algorithm, we obtained analytical expressions of D_I for four particles.

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The enumeration of number of spin I states (denoted as D_I) for fermions in a single-j shell or bosons with spin l (we use a convention that j is a half integer and l is an integer) is a very common practice in nuclear structure theory. One usually obtains this number by subtracting the combinatorial number of angular momentum projection $M = I + 1$ from that with $M = I$ [1]. More specifically, D_I equals the combinatorial number of $M = I$ subtracted by that of $M = I + 1$, where $M = m_1 + m_2 + \cdots + m_n$, with the requirement that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$ for bosons and $m_1 > m_2 > \cdots > m_n$ for fermions, where n is the number of particles. (This procedure is called process A in this article.) The combinatorial numbers of different M's look irregular, and such an enumeration would be prohibitively tedious when j and l are very large. The number of states of a few nucleons in a single-j shell is usually tabulated in textbooks, for sake of convenience.

Another well-known solution was given by Racah [2] in terms of the seniority scheme, where one has to introduce (usually by computer choice) additional quantum numbers. More than one decade ago, a third route was studied by Katriel *et al*. [3] and Sunko *et al*. [4], who constructed generating functions of the number of states for fermions in a single-j shell or bosons with spin l.

There were two efforts in constructing analytical formulas of D_I. In Ref. [5], D_0 for $n = 4$ was obtained analytically. In Ref. [6], D_I was constructed empirically for $n = 3$ and 4, and some D_I's for $n = 5$. It is therefore desirable to obtain a deeper insight into this difficult problem.

Equivalent to process A, we propose here another procedure, called process B and explained as follows. Let $\mathcal{P}(n, I_0)$ be the number of partitions of $I_0 = i_1 + i_2 + \cdots i_n$, with $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant 2j + 1 - n$ for fermions or $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant 2l$ for bosons. Here $I_{\max} = nj - [n(n-1)]/2$ for fermions in a single-j shell, and $I_{\max} = nl$ for bosons with spin l. One defines $\mathcal{P}(n, 0) = D_{I=I_{\max}} = 1$ for $I_0 = 0$. Then one has $D_{I=I_{\max}-I_0} = \mathcal{P}(n, I_0) - \mathcal{P}(n, I_0 - 1)$.

Now we look at D_I for $\bar{n}$ "bosons" of spin $L = n/2$, with $\bar{n} = 2l$ for bosons or $\bar{n} = 2j + 1 - n$ for fermions. $I_{\max}$ of these $\bar{n}$ "bosons" with spin L equals that of n bosons with spin l or that of n fermions in a single-j shell. Furthermore, $\mathcal{P}(n, I_0)$ of $I_0 = i_1 + i_2 + \cdots i_{\bar{n}}$ with the requirement $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{\bar{n}} \leqslant 2L = n$, always equals that of $I_0 = i_1 + i_2 + \cdots i_n$ with the requirement that $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant 2j + 1 - n$ for n fermions or $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant 2l$ for n bosons. This result can be explained from the fact as follows. The $\mathcal{P}(n, I_0)$ of $\bar{n}$ "bosons" with spin L corresponds to Young diagrams up to n rows, and $2l$ columns for bosons or $2j + 1 - n$ columns for fermions. The conjugates of these Young diagrams are those up to $2l$ rows for bosons or $2j + 1 - n$ rows for fermions, and up to n columns, which correspond to partitions in process B for n fermions in a single-j shell or bosons with spin l. Therefore, process B for $\bar{n}$ bosons with spin $L = n/2$ provides us with an alternative to construct D_I for n bosons with spin l or n fermions in a single-j shell.

This alternative (process B for $\bar{n}$ bosons with spin L) suggests the following identity. If $l = (2j + 1 - n)/2$ (n is even), i.e., $I_{\max}$ of bosons equals that of fermions, then D_I for bosons equals that of fermions. This identity can be easily confirmed. It means that one can obtain D_I of n fermions in a single-j shell by using that of n bosons with spin $l = (2j + 1 - n)/2$, or vice versa.

Process B for $\bar{n}$ bosons with spin $L = n/2$ is also useful in constructing formulas of D_I. One can see this point from the fact that process B involves SU$(n + 1)$ symmetry, which is *independent* of j and l, while in process A different j shell for fermions and spin l for bosons involve different symmetries [SU$(2j + 1)$ and SU$(2l + 1)$].

Below we exemplify our idea by $n = 4$. The relevant symmetry for process B of $\bar{n}$ bosons with spin L is SU(5) (i.e., $L = n/2 = 2$, d bosons). $\bar{n}$ equals $2l$ and $2j - 3$, for four bosons and four fermions, respectively.

Our first result is that D_I of four bosons with spin l always equals that of four fermions in a single j shell when $l = (2j - 3)/2$. Our second result is that we can derive D_I of four bosons with spin l by this new method. Here one needs D_I of d bosons with $\bar{n} = 2l$. This problem was studied in the interacting boson model, suggested by Arima and Iachello [7] in the 1970's. Below we revisit

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the enumeration of D_I for d bosons with particle number $\bar{n} = 2l$.

Let us follow the notation of Ref. [7] and define $\bar{n} = 2l = 2\nu + v = 2\nu + 3n_\delta + \lambda$. D_I of $\bar{n}$ d bosons is enumerated via the procedure as follows: (1) v takes the value $2l, 2l-2, 2l-4, \ldots, 0$, which corresponds to $\nu = 0, 1, 2, \ldots, n/2 = l$, respectively. (2) For each value of v, n_Δ takes value from 0 to $[\frac{v}{3}]$. (3) For each set of v and n_Δ, λ is determined by $v - 3n_\Delta$. (4) For each λ obtained in step (3), the allowed spin is given by $\lambda, \lambda+1, \lambda+2, \ldots, 2\lambda-3, 2\lambda-2, 2\lambda$. Note that there is no state with $2\lambda - 1$. One easily sees that there is no $I = 1$ states for d bosons, because $\lambda = 1$ presents $I = 2$ state ($2\lambda - 1$ is missing).

In order to obtain D_I, it is necessary to know the number of λ appearing in the above process for each I. Let us call this number f_λ and define $\bar{n} = 2l = 6k + \kappa$, $\kappa = 0, 2, 4$, and $k \geqslant 1$. Below we exemplify how we obtain f_λ by the case of $\kappa = 0$. We have the following hierarchy:

λ	f_λ	v
0	$k+1$	$0, 6, 12, \ldots, 6k$
1	k	$4, 10, 16, \ldots, 6k-2$
2	k	$2, 8, 14, \ldots, 6k-4$
3	k	$6, 12, 18, \ldots, 6k$
4	k	$4, 10, 16, \ldots, 6k-2$
5	$k-1$	$8, 14, 20, \ldots, 6k-4$
6	k	$6, 12, 18, \ldots, 6k$
7	$k-1$	$10, 16, 22, \ldots, 6k-2$
8	$k-1$	$8, 14, 20, \ldots, 6k-4$
9	$k-1$	$12, 18, 24, \ldots, 6k$
10	$k-1$	$10, 16, 22, \ldots, 6k-2$
11	$k-2$	$14, 20, 26, \ldots, 6k-4$
12	$k-1$	$12, 18, 24, \ldots, 6k$
13	$k-2$	$16, 22, 28, \ldots, 6k-2$
14	$k-2$	$14, 20, 26, \ldots, 6k-4$
15	$k-2$	$18, 24, 30, \ldots, 6k$
16	$k-2$	$16, 22, 28, \ldots, 6k-2$
17	$k-3$	$20, 26, 32, \ldots, 6k-4$
$\vdots$	$\vdots$	$\vdots$.

From this tabulation we have $f_\lambda = k + \delta_{m0} - \delta_{m5} - [\frac{\lambda}{6}]$, where m is equal to λ mod 6 when $\kappa = 0$, and [ ] means to take the largest integer not exceeding the value inside.

For the sake of simplicity we define $I = 2I_0$ for even values of I and $I = 2I_0 + 3$ for odd values of I. For $I_0 \leqslant l$,

$$D_{I=2I_0} = \sum_{\lambda=I_0}^{2I_0} f_\lambda. \tag{1}$$

For $\kappa = 0$ and $I_0 \leqslant l$ ($I = 2I_0 \leqslant 2l$),

$$D_{I=2I_0} = (I_0+1)k - (9K^2 - K + 3K\mathcal{K} + (2\mathcal{K}-5)\theta(2\mathcal{K}-5)) + \delta_{\mathcal{K}0}, \tag{2}$$

where $K = [\frac{I_0}{6}]$, $\mathcal{K} = (I_0 \bmod 6)$. $\theta(x) = 1$ if $x > 0$ and zero otherwise. One can repeat the same procedure for $\kappa = 2$ and 4. We list these results as below.

For $\kappa = 2$ and $I_0 \leqslant l$,

$$D_{I=2I_0} = (I_0+1)k - (9K^2 - K + 3K\mathcal{K} + (2\mathcal{K}-5) \times \theta(2\mathcal{K}-5)) + \left[\frac{I_0+3}{6}\right] + \left[\frac{I_0+5}{6}\right] + \delta_{\mathcal{K}0} - \delta_{\mathcal{K}3}. \tag{3}$$

For $\kappa = 4$ and $I_0 \leqslant l$,

$$D_{I=2I_0} = (I_0+1)(k+1) - (9K^2 - K + 3K\mathcal{K} + (2\mathcal{K}-5) \times \theta(2\mathcal{K}-5)) - \left[\frac{I_0+3}{6}\right] - \left[\frac{I_0+4}{6}\right] + \delta_{\mathcal{K}4}. \tag{4}$$

For I is odd and $I \leqslant 2l$, we use a relation $D_{I=2I_0} - D_{I=2I_0+3} = [\frac{I_0}{2}] + 1$. This relation was obtained empirically in Ref. [6] and can be obtained mathematically by calculating

$$D_{I=2I_0+3} = \sum_{\lambda=I_0+3}^{2I_0+3} f_\lambda$$

and comparing with $D_{I=2I_0}$.

For the case with $I \geqslant 2l$, we define $I = I_{\max} - 2I_0$ for even I and $I = I_{\max} - 2I_0 - 3$ for odd I. $f_{\lambda=I_0} = [\frac{I_0}{6}] - \delta_{(I_0 \bmod 6),0}$. We obtain

$$\begin{aligned} D_{I_{\max}-2I_0} &= D_{I_{\max}-2I_0-3} \\ &= 3\left[\frac{I_0}{6}\right]\left(\left[\frac{I_0}{6}\right]+1\right) - \left[\frac{I_0}{6}\right] + \left(\left[\frac{I_0}{6}\right]+1\right) \\ &\quad \times ((I_0 \bmod 6) + 1) + \delta_{(I_0 \bmod 6),0} - 1. \end{aligned} \tag{5}$$

Thus we solve the problem of enumeration of D_I for four bosons with spin l or four fermions in a single-j shell by using the new enumeration procedure. One may obtain D_I of other n (n is even) cases by applying this method similarly, if the reduction rule of SU($n+1$) $\rightarrow$SO(3) is available.

A question arises when we apply this method to odd n cases, for which spin L of $\bar{n}$ bosons involved in process B is not an integer ($L = n/2$). These bosons are therefore not "realistic." For such cases I of n bosons with spin l cannot equal that of n fermions in a single-j shell. Namely, there is no similar correspondence of D_I between bosons and fermions when n is odd [8].However, D_I of $\bar{n}$ fictitious bosons with spin $n/2$ (n is odd) obtained by process A *equals* that of n bosons with spin l or that of n fermions in a single-j shell, where $\bar{n} = 2l$ (even value) and $2j + 1 - n$ (odd value) for bosons and fermions, respectively. In other words, D_I of $\bar{n}$ fictitious bosons with spin $n/2$ equals that of n bosons with spin l if $\bar{n} = 2l$ or that of n fermions in a single-j shell if $\bar{n} = 2j + 1 - n$, here n is odd. Further discussion is warranted on this problem.

To summarize, We have presented in this article an alternative to enumerate the number of spin I states, D_I, for n

fermions in a single-j shell or n bosons with spin l. We proved that D_I of n bosons with spin l equals that of n fermions in a single-j shell when $2l = 2j + 1 - n$, where n is even. We have also exemplified the usefulness of this new method in constructing analytical formulas of D_I by $n = 4$.

For odd n, the procedure of our new method involves half integer spin L for "bosons." Further consideration of this fictitious situation is necessary.

Finally, it might be helpful to compare the present formulas with earlier results of Refs. [5,6]. According to our discussion of the $n = 4$ case, D_I for $n = 4$ equals that of d bosons with $\bar{n} = 2l = 2j + 1 - n = 2j - 3$. $I_0 = K = \mathcal{K} = 0$ in Eqs. (2)–(4). In such cases these formulas can be readily reduced to

$$D_{I=0} = k + 1 = \left[\frac{\bar{n}}{6}\right] + 1 = \left[\frac{2j-3}{6}\right] + 1 = \left[\frac{2j+3}{6}\right].$$

This is identical to the expression in Ref. [5]. Equations (2)–(4) are also consistent with those in Ref. [6]. The enumeration of d boson states in Ref. [7] and process B suggested in this article provide a microscopic foundation of Table I in Ref. [6], i.e., the staggering of D_I for $n = 4$. For $I \geqslant 2l$, D_I in this article is identical to that of Ref. [6]; for $I \leqslant 2l$, D_I obtained here always equals that of Ref. [6], although they seem different.

Note added in proof: After this paper was accepted, Dr. Igal Talmi informed us of his mathematical proof of Eq. (2) in Ref. [6], where the case with $n = 3$ and $I \geqslant j$ was empirically obtained. The case with $n = 3$ and $I = j$ was also discussed in a recent preprint, nucl-th/0502062, by Drs. L. Zamick and A. Escuderos. The number of states with arbitrary spin I for $n = 3$ can be analytically obtained by studying fictitious bosons with spin 3/2 and applying the new recipe presented in this paper.

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