Generalized Grodzins relation

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A relation connecting the spin dependence of the energies $E(I+2)$ of the members of the rotational ground-state band with the $E2$ reduced transition probabilities $B(E2;(I+2)_1 \rightarrow I_1)$ is derived based on the Bohr Hamiltonian. This relation generalizes the Grodzins relation from the 2_1^+ state to all members of the ground-state band.

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I. INTRODUCTION

Phenomenological models play an important role in the analysis of the experimental data. They are especially useful if it becomes possible to derive on their basis relations between observables that do not include free parameters. A well-known example of such a relation is the Grodzins relation between the excitation energy of the first 2^+ state and the $B(E2;2_1^+ \rightarrow 0_1^+)$ [1,2]. This relation shows that the γ-ray $E2$ transition probabilities from the first 2^+ states of even-even nuclei to the ground states are approximately inversely proportional to $E(2_1^+)$. The aim of the present paper is to generalize the Grodzins relation from the 2_1^+ state to all members of the ground-state band for nuclei which are well deformed or at least deformed and have a quasirotational ground band. This will be done using a sum rule approach. It is well known that such nuclei are well described by the Bohr Hamiltonian [3]. Thus we will base the following discussion of the sum rule on the Bohr Hamiltonian:

$$\hat{H} = \hat{T} + V(\alpha), \tag{1}$$

where

$$\hat{T} = -\frac{\hbar^2}{2B}\sum_{\mu}(-1)^{\mu}\frac{\partial^2}{\partial\alpha_{2\mu}\partial\alpha_{2-\mu}}, \tag{2}$$

B is a mass coefficient, V is a potential energy depending on $\alpha_{2\mu}$ only, and $\alpha_{2\mu}$ is the collective quadrupole variable which is proportional to the quadrupole moment operator $Q_{2\mu}$,

$$Q_{2\mu} = \frac{3}{4\pi}ZeR_0^2\alpha_{2\mu} \equiv q\alpha_{2\mu}. \tag{3}$$

Using Eqs. (1)–(3) we obtain by straightforward calculations the following basic commutation relation:

$$(-1)^{\mu}[[H, Q_{2\mu}], Q_{2-\mu}] = -\frac{\hbar^2 q^2}{B}. \tag{4}$$

II. SUM RULES

Relation (4) allows us to derive sum rules. We introduce a basis of the eigenstates of the Hamiltonian (1) $|InM\rangle$, where I is the angular momentum, M is its projection on the z axis of the laboratory frame, and where n is a set of all other quantum numbers which are necessary to characterize the eigenstates. Then we obtain a sum rule by taking an average of the relation (4) and by using a full set of the intermediate states:

$$\sum_{I'n'}[E(In) - E(I'n')]\frac{1}{(2I+1)}|\langle In\|Q_2\|I'n'\rangle|^2 = -\frac{5\hbar^2 q^2}{2B}. \tag{5}$$

Here, $E(In)$ is the eigenvalue of the Hamiltonian (1): $\hat{H}|InM\rangle = E(In)|InM\rangle$. We analyze the consequences of the relation (5) by taking the average with a state with quantum numbers In belonging to the ground state band $|I_{gr}\rangle$. Since we are going to consider only near-deformed or well-deformed nuclei, we restrict the summation in Eq. (5) by the states of the ground, β, and γ bands. Then we obtain

$$\sum_{I'_{gr}}[E(I_{gr}) - E(I'_{gr})]\frac{1}{(2I+1)}|\langle I_{gr}\|Q_2\|I'_{gr}\rangle|^2 + \sum_{I'_{\beta}}[E(I_{gr}) - E(I'_{\beta})]\frac{1}{(2I+1)}|\langle I_{gr}\|Q_2\|I'_{\beta}\rangle|^2 + \sum_{I'_{\gamma}}[E(I_{gr}) - E(I'_{\gamma})]\frac{1}{(2I+1)}|\langle I_{gr}\|Q_2\|I'_{\gamma}\rangle|^2 = -\frac{5\hbar^2 q^2}{2B}. \tag{6}$$

The last two equations are the main sum rule result. The next point is to evaluate these equations approximately. To understand the relative importance of the different terms in the left-hand side of Eq. (6) let us consider this relation in the limit of the strongly deformed nucleus whose rotational energies are strictly proportional to $I(I+1)$ and for which the β and γ vibrations are harmonic. In this case

$$E(I_{gr}) = \frac{\hbar^2}{2\Im}I(I+1), \tag{7}$$

$$E(I_{\beta}) = \hbar\sqrt{\frac{C_{\beta}}{B}} + \frac{\hbar^2}{2\Im}I(I+1), \tag{8}$$

$$E(I_\gamma) = \hbar\sqrt{\frac{C_\gamma}{B}} + \frac{\hbar^2}{2\Im}[I(I+1)-4], \tag{9}$$

where $\Im$ is the moment of inertia and the wave functions of the states of the ground, β, and γ bands are

$$\Psi_{gr}(IM) = \mathcal{N}_{gr} D^I_{M0}(\Omega) \exp\left(-\frac{1}{2}\frac{(\beta-\beta_0)^2}{\hbar/\sqrt{BC_\beta}} - \frac{1}{2}\frac{\gamma^2}{\hbar/\sqrt{BC_\gamma}}\right), \tag{10}$$

$$\Psi_\beta(IM) = \mathcal{N}_\beta D^I_{M0}(\Omega)(\beta-\beta_0) \times \exp\left(-\frac{1}{2}\frac{(\beta-\beta_0)^2}{\hbar/\sqrt{BC_\beta}} - \frac{1}{2}\frac{\gamma^2}{\hbar/\sqrt{BC_\gamma}}\right), \tag{11}$$

$$\Psi_\gamma(IM) = \mathcal{N}_\gamma \left[D^I_{M2}(\Omega) + D^I_{M-2}(\Omega)\right]\gamma \times \exp\left(-\frac{1}{2}\frac{(\beta-\beta_0)^2}{\hbar/\sqrt{BC_\beta}} - \frac{1}{2}\frac{\gamma^2}{\hbar/\sqrt{BC_\gamma}}\right), \tag{12}$$

where $\mathcal{N}_{gr}, \mathcal{N}_\beta$, and $\mathcal{N}_\gamma$ are the normalization coefficients, $D^I_{MK}(\Omega)$ is a Wigner function, C_β (C_γ) is a stiffness coefficient of the β (γ) oscillations, and β_0 is an equilibrium deformation. In the harmonic approximation the quadrupole moment operator takes the form

$$Q_{2\mu} = q\left(D^2_{\mu 0}\beta_0 + D^2_{\mu 0}(\beta-\beta_0) + \gamma\frac{1}{\sqrt{2}}\left(D^2_{\mu 2} + D^2_{\mu -2}\right)\right). \tag{13}$$

Using Eqs. (10)–(13) after straightforward calculations which can be done analytically, we obtain

$$\langle I_{gr}\|Q_2\|I'_{gr}\rangle^2 = q^2\beta_0^2(2I+1)\left(C^{I'0}_{I020}\right)^2, \tag{14}$$

$$\langle I_{gr}\|Q_2\|I'_\beta\rangle^2 = q^2\frac{\hbar}{2\sqrt{BC_\beta}}(2I+1)\left(C^{I'0}_{I020}\right)^2, \tag{15}$$

$$\langle I_{gr}\|Q_2\|I'_\gamma\rangle^2 = q^2\frac{\hbar}{\sqrt{BC_\gamma}}(2I+1)\left(C^{I'2}_{I022}\right)^2. \tag{16}$$

Above $C^{I'K}_{I02K}$ is the Clebsch-Gordan coefficient. The relations (14)–(16) express the Alaga rules for the $E2$ transitions within the ground-state rotational band or between the states belonging to different rotational bands. Using Eqs. (10)–(13) we can derive also the following expressions:

$$\frac{1}{(2I+1)}\sum_{I'_\beta}|\langle I_{gr}\|Q_2\|I'_\beta\rangle|^2 = q^2\langle I_{gr}\|(\beta-\beta_0)^2\|I_{gr}\rangle = q^2\frac{\hbar}{2\sqrt{BC_\beta}} \tag{17}$$

and

$$\frac{1}{(2I+1)}\sum_{I'_\gamma}|\langle I_{gr}\|Q_2\|I'_\gamma\rangle|^2 = q^2\langle I_{gr}\|\gamma^2\|I_{gr}\rangle = q^2\frac{\hbar}{\sqrt{BC_\gamma}} \tag{18}$$

from which it is seen that the factor $\hbar/2\sqrt{BC_\beta}$ describes the mean square fluctuations of β around the equilibrium value β_0, and the factor $\hbar/\sqrt{BC_\gamma}$ describes the mean square fluctuations of γ around $\gamma = 0$.

Substituting Eqs. (7)–(16) into Eq. (6) we obtain

$$\sum_{I'_{gr}}\frac{\hbar^2}{2\Im}[I(I+1)-I'(I'+1)]\beta_0^2\left(C^{I'0}_{I020}\right)^2 + \sum_{I'_\beta}\left(-\hbar\sqrt{\frac{C_\beta}{B}} + \frac{\hbar^2}{2\Im}[I(I+1)-I'(I'+1)]\right) \times \frac{\hbar}{2\sqrt{BC_\beta}}\left(C^{I'0}_{I020}\right)^2 + \sum_{I'_\gamma}\left(-\hbar\sqrt{\frac{C_\gamma}{B}} + \frac{\hbar^2}{2\Im}[I(I+1) - I'(I'+1)+4]\right)\frac{\hbar}{\sqrt{BC_\gamma}}\left(C^{I'2}_{I022}\right)^2 = -\frac{5\hbar^2}{2B}. \tag{19}$$

All sums in Eq. (19) can be calculated exactly after substitution of algebraic expressions for the Clebsch-Gordan coefficients. The final result is

$$\Im = 3B\beta_0^2\left(1 + \frac{\hbar}{2\beta_0^2\sqrt{BC_\beta}} + \frac{\hbar}{3\beta_0^2\sqrt{BC_\gamma}}\right), \tag{20}$$

where

$$\frac{\hbar}{2\sqrt{BC_\beta}} = \langle gr|(\beta-\beta_0)^2|gr\rangle \tag{21}$$

and

$$\frac{\hbar}{\sqrt{BC_\gamma}} = \langle gr|\gamma^2|gr\rangle. \tag{22}$$

Thus, the second and third terms in Eq. (20) represent the ratios of the amplitudes of the β and γ fluctuations near the equilibrium values to β_0:

$$\Im = 3B\beta_0^2\left(1 + \frac{\langle gr|(\beta-\beta_0)^2|gr\rangle}{\beta_0^2} + \frac{\langle gr|\gamma^2|gr\rangle}{3\beta_0^2}\right). \tag{23}$$

In deformed nuclei these ratios are small and we obtain by neglecting their contributions the following relation:

$$\Im = 3B\beta_0^2, \tag{24}$$

which is the expression for the moment of inertia in the Bohr-Mottelson model. The last result (24) can be obtained directly from Eq. (19) if we neglect in the second and third terms in the left-hand side of Eq. (19) a rotational contribution to the energy differences which is small in comparison with a vibrational one.

We will now consider a situation when the Alaga rules are not satisfied perfectly, however, nuclei under consideration are deformed. In this case we can neglect as above the rotational contributions into the energy differences in the second and third terms in the left-hand side of Eq. (6). It is necessary to mention that the neglected terms are angular momentum dependent and increase as angular momentum increases. At the same time these terms have both positive and negative signs. For this reason, it is difficult to estimate their effect on the validity of the approximation which neglect the rotational contribution into the energy differences considered above.

If the rotational contributions into the γ-transition energies between β, γ bands and the ground band is relatively small, we neglect them and obtain

$$\sum_{I'_{gr}}[E(I_{gr}) - E(I'_{gr})]\frac{1}{(2I+1)}|\langle I_{gr}\|Q_2\|I'_{gr}\rangle|^2 - \hbar\sqrt{\frac{C_\beta}{B}}\frac{1}{(2I+1)}\sum_{I'_\beta}|\langle I_{gr}\|Q_2\|I'_\beta\rangle|^2 - \hbar\sqrt{\frac{C_\gamma}{B}}\frac{1}{(2I+1)} \times \sum_{I'_\gamma}|\langle I_{gr}\|Q_2\|I'_\gamma\rangle|^2 = -\frac{5\hbar^2 q^2}{2B}. \quad (25)$$

Substituting Eqs. (17) and (18) into Eq. (25) and taking into account that in Eq. (25) $I'_{gr} = I_{gr} \pm 2$, we obtain

$$[E(I+2) - E(I)]\frac{1}{(2I+1)}|\langle I\|Q_2\|I+2\rangle|^2 - [E(I) - E(I-2)]\frac{1}{(2I+1)}|\langle I\|Q_2\|I-2\rangle|^2 = \frac{\hbar^2 q^2}{B}. \quad (26)$$

Since only states of the ground band are presented in Eq. (26), the notation *gr* is omitted there.

Using a definition of the reduced transition probabilities $B(E2)$ we can rewrite Eq. (26) as

$$[E(I+2) - E(I)][2(I+2)+1]B(E2; I+2 \to I) - [E(I) - E(I-2)](2I+1)B(E2; I \to I-2) = \frac{\hbar^2 q^2}{B}(2I+1). \quad (27)$$

III. GENERALIZED GRODZINS RELATION

The relation (27) is a finite differences equation whose solution is the searched-for generalization of the Grodzins relation:

$$[E(I+2) - E(I)]B(E2; I+2 \to I) = \frac{\hbar^2 q^2}{2B}\frac{(I+2)(I+1)}{(2I+5)}. \quad (28)$$

Using the expression for the square of the Clebsch-Gordan coefficient $(C^{I0}_{I+2020})^2$ we can present Eq. (28) as

$$[E(I+2) - E(I)]B(E2; I+2 \to I) = \frac{\hbar^2 q^2}{3B}(2I+3)\left(C^{I0}_{I+2020}\right)^2. \quad (29)$$

These Eqs. (28) and (29) are the generalization of the Grodzins relation from the 2^+_1 state to all members of the ground state band. This generalization of the Grodzins relation is the main result of this paper.

One can directly derive this relation for the rigid rotor. In this case one finds for the transition energy the relation

$$E(I+2) - E(I) = \frac{\hbar^2}{\Im}(2I+3) = \frac{\hbar^2}{3B\beta_0^2}(2I+3), \quad (30)$$

and the $E2$ reduced transition probabilities $B[E2; (I+2)_1 \to I_1]$ are

$$B(E2; I+2 \to I) = q^2\beta_0^2\frac{3}{2}\frac{(I+2)(I+1)}{(2I+3)(2I+5)}. \quad (31)$$

By taking the product we indeed recover Eq. (28). Thus there is an easy derivation for the generalized Grodzins relation in the case of the axially symmetric rigid rotor. In many cases the energies and the $B(E2)$'s of the axially symmetric rigid rotor have been tested empirically. Thus we can claim that the generalization of the Grodzins relation is also empirically checked for the axially symmetric rigid rotor.

We note that there are many nuclei for which the rigid rotor formula does not describe the energies of the rotational band, however. In this case a number of generalizations of the $I(I+1)$ dependence have been suggested. We mention in particular the Bohr-Mottelson expansion in a power series in $I(I+1)$ [3], the variable moment of inertia model [4,5], the Lipas factor [6,7], the Ejiri expansion in powers of I [8], and the soft rotor [9]. Also the recently proposed confined Beta-soft rotor model [10] provides analytical expressions for the ground-band energies of deformed nuclei with $2.9 < R_{4/2} < 3.33$. These generalizations have been proven to be quite successful for the energies of the ground band of quasirotational nuclei. The generalized Grodzins formula gives to each of these models a predicted spin dependence of the $E2$ reduced transition probabilities $B[E2; (I+2)_1 \to I_1]$. These predictions have to be checked.

An alternative and parameter free form of this relation is found by putting in Eq. (28) $I = 0$. We obtain

$$E\left(2^+_1\right)B\left(E2; 2^+_1 \to 0^+_1\right) = \frac{\hbar^2 q^2}{5B}. \quad (32)$$

With this result we can rewrite Eq. (28) as

$$[E(I+2) - E(I)]B(E2; I+2 \to I)\frac{(2I+5)}{(I+1)(I+2)} = \frac{5}{2}E\left(2^+_1\right)B\left(E2; 2^+_1 \to 0^+_1\right). \quad (33)$$

This—parameter free—relation connects the spin dependence of the transition energies $[(E(I+2) - E(I)]$ of the members of the quasirotational ground-state band with the $E2$ reduced transition probabilities $B[E2; (I+2)_1 \to I_1]$. We note that Eq. (33) has the form of a plot. Namely, the rhs of this equation is a constant and is independent of the spin I. We want to discuss now the range of applicability of the generalized Grodzins relation, which we have derived from the Bohr Hamiltonian. There were two crucial assumptions: (1) the mass parameter B was taken as a constant and (2) it was assumed that the rotational energies are small compared to the vibrational energies of the β and γ band. This means $E(2^+_1) < 0.1E(2^+_2)$. These are strong assumptions which restrict the applicability of the relation. In particular, it does not work for the vibrator or the γ unstable Jean-Wilets rotor. But it presumably works for nuclei with an $R(4/2) = E(4)/E(2) \geqslant 2.9$. We note that the assumed constancy of the mass parameter B with spin is a strong assumption. One can consider the success of the $X(5)$ model of Iachello [11] and of the CBS rotor [10] as a test of this assumption for the strongly deformed nuclei.

IV. CONCLUSIONS

Summing up, we have suggested energy-weighted sum rules for the $B(E2)$ for nuclei with a quasirotational ground-state band. These truncated sum rules were derived based on the Bohr Hamiltonian. We also used the assumption that the rotational contribution to the γ-transition energies between the β, γ bands and the ground band is small compared to the vibrational contribution. These relations are parameter free and they allow us to calculate the spin dependence of the $B(E2)$'s of the ground band from the energies of the ground band. One can consider these relations as a generalization of the Grodzins relation to higher spins.

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