Pressure inequalities for nuclear and neutron matter

Dean Lee

Department of Physics, North Carolina State University, Raleigh, North Carolina 27695, USA

(Received 9 August 2004; published 18 April 2005)

We prove several inequalities using lowest-order effective field theory for nucleons which give an upper bound on the pressure of asymmetric nuclear matter and neutron matter. We prove two types of inequalities, one based on convexity and another derived from shifting an auxiliary field.

I. INTRODUCTION

In the effective field theory description of low-energy nuclear matter, nucleons are treated as point particles rather than composite objects. While much of the work in the community has focused on few-body systems, there has also been recent interest in lattice simulations of bulk nuclear matter using effective field theory [1–6]. In parallel with this computational effort, effective field theory was also recently used to prove inequalities for the correlation function of two-nucleon operators in low-energy symmetric nuclear matter [7]. It was shown that the $S = 1$, $I = 0$ channel must have the lowest energy and longest correlation length in the two-nucleon sector. These results were shown to be valid at nonzero density and temperature and could be checked in effective field theory lattice simulations. The proof relied on positivity of the Euclidean functional integral measure and is similar in spirit to several quantum chromodynamics (QCD) inequalities proved using quark-gluon degrees of freedom [8–16].

In this work we prove several new inequalities using effective field theory which give an upper bound on the pressure of asymmetric nuclear matter and neutron matter. We prove two types of inequalities, one based on convexity and one derived from shifting an auxiliary Hubbard-Stratonovich field. We consider two general types of systems, one with two fermion species and an SU(2) symmetry and another with four fermion species and an SU(2) × SU(2) symmetry. The results we prove are quite general. In addition to nuclear and neutron matter, our inequalities apply to systems of cold, dilute gases of fermionic atoms [17–21] which can be described by the same lowest-order effective field theory.

II. LOWER BOUND

Before deriving pressure upper bounds, we first state a general lower bound for the pressure. The result is simple and perhaps obvious, but the derivation is useful to help set our notation. Consider any system in thermodynamic equilibrium that is invariant under a symmetry group S. Let μ be a symmetric chemical potential which preserves the group S. Let μ_3 be an asymmetric chemical potential which breaks S and flips sign $\mu_3 \rightarrow -\mu_3$ under some element of S. This means that the pressure P is an even function of μ_3.

Our condition of thermodynamic equilibrium requires that the system is stable and not further separating into regions with

more widely different values of μ_3. This implies the convexity condition,

$$\frac{\partial^2 P(\mu, \mu_3)}{\partial \mu_3^2} \geqslant 0. \tag{1}$$

Combining this with the fact that P is even in μ_3, we derive the lower bound

$$P(\mu, \mu_3) \geqslant P(\mu, 0). \tag{2}$$

This lower bound holds for all the systems we consider here.

III. TWO FERMION STATES—SU(2)

We consider an effective theory with two species of interacting fermion fields and an SU(2) symmetry. Let n be a doublet of fermion fields which we can regard as neutron spin states,

$$n = \begin{bmatrix} \uparrow \\ \downarrow \end{bmatrix}. \tag{3}$$

We can write the lowest-order Lagrange density in Euclidean space in two equivalent forms,

$$\mathcal{L}_E = -\bar{n} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu - \mu_3 \sigma_3 \right) \right] n - \frac{1}{2} C \bar{n} n \bar{n} n, \tag{4}$$

and

$$\mathcal{L}_E = -\bar{n} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3 \sigma_3 \right) \right] n \\ - \frac{1}{2} C' \bar{n} \vec{\sigma} n \cdot \bar{n} \vec{\sigma} n, \tag{5}$$

where

$$C' = -\tfrac{1}{3} C. \tag{6}$$

We use $\vec{\sigma}$ to represent Pauli matrices acting in spin space. μ is the symmetric chemical potential while μ_3 is the asymmetric chemical potential. We assume the interaction is attractive so that
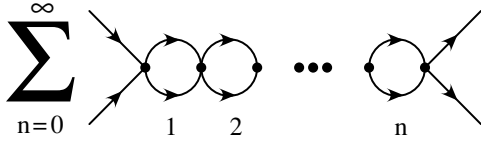
$$C < 0, C' > 0. \tag{7}$$

FIG. 1. Sum over two-particle bubble diagrams.

A. Two-body operator coefficients

We can calculate C using a lattice regulator for various lattice spacings, which denote as a_{lattice}. For simplicity we take the temporal lattice spacing to be zero. We must sum all two-particle scattering bubble diagrams, as shown in Fig. 1, and locate the pole in the scattering amplitude. We then use Lüscher's formula for energy levels in a finite periodic box [5,22,23] and tune the coefficients to give the physically measured scattering lengths. From Lüscher's formula there should be a pole in the two-particle scattering amplitude with energy

$$E_{\text{pole}} = \frac{4\pi a_{\text{scatt}}}{m_N L^3} + \cdots, \tag{8}$$

where a_{scatt} is the scattering length. We can write the sum over bubble diagrams as a geometric series. In order to produce a pole at this energy we must have

$$\frac{1}{m_N C} = \frac{1}{4\pi a_{\text{scatt}}} - \lim_{L \to \infty} \frac{1}{a_{\text{lattice}} L^3}$$
$$\times \sum_{\vec{k} \neq 0} \frac{1}{6 - 2\cos\frac{2\pi k_1}{L} - 2\cos\frac{2\pi k_2}{L} - 2\cos\frac{2\pi k_3}{L}}, \tag{9}$$

where a_{lattice} is the lattice spacing, and the sum is over integer values k_1, k_2, k_3 from 0 to $L-1$. Solving for C gives

$$C \simeq \frac{1}{m_N \left(\frac{1}{4\pi a_{\text{scatt}}} - \frac{0.253}{a_{\text{lattice}}} \right)}. \tag{10}$$

For any chosen temperature and neutron density there is a corresponding maximum value for the lattice spacing, a_{lattice}. The requirements are that the kinetic energy for the highest momentum mode must exceed the temperature, and the lattice spacing must be less than the interparticle spacing. We therefore have

$$a_{\text{lattice}}^{-1} \gg \left(a_{\text{lattice}}^{-1} \right)_{\text{min}} = \max[\pi^{-1}\sqrt{2m_N T}, \rho^{1/3}]. \tag{11}$$

This sets an upper bound for the absolute value for the scale-dependent coupling C,

$$|C| \ll |C|_{\text{max}} \equiv \frac{1}{m_N \left| \frac{1}{4\pi a_{\text{scatt}}} - 0.253 \left(a_{\text{lattice}}^{-1} \right)_{\text{min}} \right|}. \tag{12}$$

This result will be useful for the shifted-field inequalities derived later.

B. Convexity inequality

The grand canonical partition function is given by

$$Z_G(\mu, \mu_3) = \int Dn \, D\bar{n} \exp(-S_E)$$
$$= \int Dn \, D\bar{n} \exp\left(\int d^4x \, \mathcal{L}_E \right), \tag{13}$$

where we use the expression (4) for $\mathcal{L}_E$,

$$\mathcal{L}_E = -\bar{n} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu - \mu_3\sigma_3 \right) \right] n - \frac{1}{2}C\bar{n}n\bar{n}n. \tag{14}$$

Using a Hubbard-Stratonovich transformation [24,25], we can rewrite Z_G as

$$Z_G \propto \int Dn \, D\bar{n} \, Df \exp\left(\int d^4x \, \mathcal{L}_E^f \right), \tag{15}$$

where

$$\mathcal{L}_E^f = -\bar{n} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu - \mu_3\sigma_3 \right) \right] n$$
$$+ Cf\bar{n}n + \frac{1}{2}Cf^2. \tag{16}$$

Let us define $\mathbf{M}$ as the matrix for the part of $\mathcal{L}_E^f$ bilinear in the neutron field,

$$\mathbf{M} = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu - \mu_3\sigma_3 \right) \right] + Cf. \tag{17}$$

We observe that $\mathbf{M}$ has the block diagonal form,

$$\mathbf{M} = \begin{bmatrix} M(\mu + \mu_3) & 0 \\ 0 & M(\mu - \mu_3) \end{bmatrix}, \tag{18}$$

where

$$M(\mu) = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu \right) \right] + Cf. \tag{19}$$

Since M is real valued, $\det M$ must also be real.

Integrating over the fermion fields gives us

$$Z_G(\mu, \mu_3) \propto \int Dn \, D\bar{n} \, Df \exp\left(\int d^4x \, \mathcal{L}_E^f \right)$$
$$= \int D\Theta \det \mathbf{M}$$
$$= \int D\Theta \det M(\mu + \mu_3) \det M(\mu - \mu_3), \tag{20}$$

where $D\Theta$ is the positive measure

$$D\Theta = Df \exp\left(\frac{1}{2}C \int d^4x \, f^2 \right). \tag{21}$$

Using the Cauchy-Schwarz inequality we find

$$\left| \int D\Theta \det M(\mu + \mu_3) \det M(\mu - \mu_3) \right|$$

$$\leqslant \int D\Theta \left| \det M(\mu + \mu_3) \det M(\mu - \mu_3) \right|$$

$$\leqslant \sqrt{\int D\Theta \, [\det M(\mu + \mu_3)]^2} \sqrt{\int D\Theta \, [\det M(\mu - \mu_3)]^2}. \tag{22}$$

We can now compare the asymmetric partition function to the symmetric partition function at chemical potentials $\mu + \mu_3$ and $\mu - \mu_3$,

$$Z_G(\mu, \mu_3) \leqslant \sqrt{Z_G(\mu + \mu_3, 0)} \sqrt{Z_G(\mu - \mu_3, 0)}. \tag{23}$$

We now use the thermodynamic relation,

$$\ln Z_G = \frac{PV}{k_B T}, \tag{24}$$

where P is the pressure, V is the volume, and T is the temperature. We find the upper bound

$$P(\mu, \mu_3) \leqslant \tfrac{1}{2}[P(\mu + \mu_3, 0) + P(\mu - \mu_3, 0)]. \tag{25}$$

C. Shifted-field inequality

We start again with the grand canonical partition function

$$Z_G(\mu, \mu_3) = \int Dn \, D\bar{n} \, \exp(-S_E)$$
$$= \int Dn \, D\bar{n} \, \exp\left(\int d^4x \, \mathcal{L}_E\right). \tag{26}$$

This time we use the other expression (5) for $\mathcal{L}_E$,

$$\mathcal{L}_E = -\bar{n}\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3\sigma_3\right)\right]n$$
$$- \frac{1}{2}C'\bar{n}\vec{\sigma}n \cdot \bar{n}\vec{\sigma}n. \tag{27}$$

We can rewrite the grand canonical partition function using three Hubbard-Stratonovich fields,

$$Z_G \propto \int Dn \, D\bar{n} \, D\vec{\phi} \, \exp\left(\int d^4x \, \mathcal{L}_E^{\vec{\phi}}\right), \tag{28}$$

where

$$\mathcal{L}_E^{\vec{\phi}} = -\bar{n}\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3\sigma_3\right)\right]n$$
$$+ iC'\vec{\phi} \cdot \bar{n}\vec{\sigma}n - \frac{1}{2}C'\vec{\phi} \cdot \vec{\phi}. \tag{29}$$

Let $\mathbf{M}_0$ be the neutron matrix without the $\mu_3\sigma_3$ term,

$$\mathbf{M}_0 = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu\right)\right] + iC'\vec{\phi} \cdot \vec{\sigma}. \tag{30}$$

We note that

$$\sigma_2 \mathbf{M}_0 \sigma_2 = \mathbf{M}_0^*, \tag{31}$$

where $\mathbf{M}_0^*$ is the complex conjugate of $\mathbf{M}_0$. This means that $\mathbf{M}_0$ is either singular, in which case $\det \mathbf{M}_0 = 0$, or has the same eigenvalues as $\mathbf{M}_0^*$. In all cases $\det \mathbf{M}_0$ is real. Furthermore the fact that σ_2 is antisymmetric means that the real eigenvalues of $\mathbf{M}_0$ are doubly degenerate, and so $\det \mathbf{M}_0 \geqslant 0$ [26].

We now concentrate on the part of $\mathcal{L}_E^{\vec{\phi}}$ that contains μ_3 and ϕ_3,

$$-\tfrac{1}{2}C'\phi_3^2 + iC'\phi_3\bar{n}\sigma_3 n + \mu_3\bar{n}\sigma_3 n. \tag{32}$$

We can rewrite this as

$$-\frac{1}{2}C'\phi_3'^2 - i\mu_3\phi_3' + iC'\phi_3'\bar{n}\sigma_3 n + \frac{1}{2}\frac{\mu_3^2}{C'} \tag{33}$$

where

$$\phi_3' = \phi_3 - i\frac{\mu_3}{C'}. \tag{34}$$

The original contour of integration for ϕ_3' is off the real axis, but we can deform the contour onto the real axis. For notational convenience we now drop the prime on ϕ_3' and have

$$\mathcal{L}_E^{\vec{\phi}} = -\bar{n}\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu\right)\right]n$$
$$+ iC'\vec{\phi} \cdot \bar{n}\vec{\sigma}n - \frac{1}{2}C'\vec{\phi} \cdot \vec{\phi} - i\mu_3\phi_3 + \frac{1}{2}\frac{\mu_3^2}{C'}. \tag{35}$$

The neutron matrix is now $\mathbf{M}_0$, which we have shown has a non-negative determinant. The complex phase is contained entirely in the local expression $-i\mu_3\phi_3$.

We now have

$$Z_G \propto \int D\Theta \, \exp\left\{\int d^4x \left[-i\mu_3\phi_3 + \frac{1}{2}\frac{\mu_3^2}{C'}\right]\right\}$$
$$= \exp\left(\frac{V\mu_3^2}{2C'k_B T}\right) \int D\Theta \, \exp\left(-i\mu_3 \int d^4x \, \phi_3\right), \tag{36}$$

where $D\Theta$ is the normalized positive measure

$$D\Theta = \frac{D\vec{\phi} \, \det \mathbf{M}_0 \exp\left(-\int d^4x \, \mathcal{V}(\vec{\phi})\right)}{\int D\vec{\phi} \, \det \mathbf{M}_0 \exp\left(-\int d^4x \, \mathcal{V}(\vec{\phi})\right)} \tag{37}$$

with

$$-\mathcal{V}(\vec{\phi}) = -\tfrac{1}{2}C'\vec{\phi} \cdot \vec{\phi}. \tag{38}$$

Using Eq. (24) we find

$$P(\mu, \mu_3) - P(\mu, 0)$$
$$= \frac{k_B T}{V} \ln\left[\exp\left(\frac{V\mu_3^2}{2C'k_B T}\right) \int D\Theta \, \exp\left(-i\mu_3 \int d^4x \, \phi_3\right)\right]$$
$$= \frac{\mu_3^2}{2C'} + \frac{k_B T}{V} \ln\left[\int D\Theta \, \exp\left(-i\mu_3 \int d^4x \, \phi_3\right)\right]. \tag{39}$$

So we conclude that

$$P(\mu, \mu_3) \leqslant P(\mu, 0) + \frac{\mu_3^2}{2C'}. \tag{40}$$

This upper bound is unusual in that it relates physical observables independent of the cutoff scale to the scale-dependent coupling C'. By taking the lattice spacing as large as possible, we have

$$C' = \tfrac{1}{3}|C|_{max}, \tag{41}$$

where $|C|_{\max}$ was defined in Eq. (12), and therefore

$$P(\mu, \mu_3) \leqslant P(\mu, 0) + \frac{3\mu_3^2}{2 |C|_{\max}}. \tag{42}$$

As a rough estimate of the quantities involved, we note that for $\rho \sim 0.1\rho_N$ and $T < 10$ MeV, $|C|_{\max}$ is about 3 fm^2 .

As C' decreases the upper bound in Eq. (40) increases. But at the same time the tightness of the bound becomes poorer as complex phase oscillations due to the term

$$\exp \left[\int d^4x \left(-\frac{1}{2}C'\phi_3^2 - i\mu_3\phi_3 \right) \right] \tag{43}$$

become more significant. The average phase for our functional integral is given by

$$\begin{aligned}
\langle \text{phase} \rangle &= \int D\Theta \exp \left(-i\mu_3 \int d^4x \ \phi_3 \right) \\
&= \exp \left[\frac{V}{k_B T} \left(P(\mu, \mu_3) - P(\mu, 0) - \frac{\mu_3^2}{2C'} \right) \right]. \tag{44}
\end{aligned}$$

Given an estimate of the pressure difference, this relation can be used to predict the feasibility of a numerical simulation using this representation of the functional integral. In cases where the phase problem is not too severe we can use hybrid Monte Carlo to generate Hubbard-Stratonovich field configurations according to the relative probability weight $\det \mathbf{M}_0$. The phase of the configuration can then be included as an observable using the local expression $-i\mu_3\phi_3$. This local expression for the phase could increase algorithmic speed by several orders of magnitude. The only known way to compute the phase of matrix determinants is LU decomposition, an algorithm which writes a matrix as a product of lower and upper triangular matrices. The number of operations for LU decomposition scales as N^3, where N is the dimension of the matrix. For an L^4 lattice the scaling is thus L^{12}.

IV. FOUR FERMION STATES—SU(2) × SU(2)

We now consider an effective theory with four species of interacting fermions and an SU(2) × SU(2) symmetry. Let N be a quartet of fermion states, which we can regard as nucleon fields,

$$N = \begin{bmatrix} p \\ n \end{bmatrix} \otimes \begin{bmatrix} \uparrow \\ \downarrow \end{bmatrix}. \tag{45}$$

We use $p(n)$ to represent protons(neutrons) and $\uparrow(\downarrow)$ to represent up(down) spins. We use $\vec{\tau}$ to represent Pauli matrices acting in isospin space and $\vec{\sigma}$ to represent Pauli matrices acting in spin space. We assume exact isospin and spin symmetry in the absence of symmetry-breaking chemical potentials, and so the symmetry group is SU(2)$_I$ × SU(2)$_S$.

In the nonrelativistic limit and below the threshold for pion production, we can write the lowest-order terms in the effective Lagrangian in two equivalent ways,

$$\begin{aligned}
\mathcal{L}_E = &-\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^0 - \mu \right) \right] N - \frac{1}{2}C_S(\bar{N}N)^2 \\
&- \frac{1}{2}C_T \bar{N}\vec{\sigma}N \cdot \bar{N}\vec{\sigma}N - \frac{1}{3!}C_3(\bar{N}N)^3 - \frac{1}{4!}C_4(\bar{N}N)^4,
\end{aligned} \tag{46}$$

or

$$\begin{aligned}
\mathcal{L}_E = &-\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0'} - \mu \right) \right] N - \frac{1}{2}C_S'(\bar{N}N)^2 \\
&- \frac{1}{2}C_U' \bar{N}\vec{\tau}N \cdot \bar{N}\vec{\tau}N - \frac{1}{3!}C_3(\bar{N}N)^3 - \frac{1}{4!}C_4(\bar{N}N)^4.
\end{aligned} \tag{47}$$

We will introduce symmetry breaking chemical potentials later. We have included both three-body and four-body forces. The SU(4)-symmetric three-nucleon force is needed for consistent renormalization and has been shown to be the dominant three-body force contribution [27–29].

With four distinct fermion species there are two irreducible representations of SU(2)$_I$ × SU(2)$_S$ for two fermions in an s wave, a spin-singlet isospin-triplet ($S = 0$) or an isospin-singlet spin-triplet ($I = 0$). One can show that [7]

$$C_U' = -C_T, \quad C_S' = C_S - 2C_T. \tag{48}$$

In the case of nucleons, one finds that both of the s-wave channels are attractive, with the $I = 0$ channel being more strongly attractive,

$$\frac{1}{a_{\text{scatt}}^{I=0}} > \frac{1}{a_{\text{scatt}}^{S=0}}. \tag{49}$$

This implies that [7]

$$C_S < 3C_T, \quad C_T < 0, \tag{50}$$

$$C_S' < -C_U', \quad C_U' > 0. \tag{51}$$

For a more general system with four fermion states and an SU(2) × SU(2) symmetry, we can interchange the isospin and spin labels so that, without loss of generality,

$$\frac{1}{a_{\text{scatt}}^{I=0}} \geqslant \frac{1}{a_{\text{scatt}}^{S=0}}. \tag{52}$$

In the special case when the scattering lengths are equal, the symmetry group is the full Wigner SU(4) symmetry [30], and the isospin and spin labels can be interchanged.

A. Two-body operator coefficients

We determine the two-body operator coefficients in the same manner as before. The only difference is that there are now two s-wave channels. The coefficient C in Eq. (9) is replaced by $C^{S=0}$ and $C^{I=0}$, where

$$C^{S=0} = C_S' + C_U', \tag{53}$$

$$C^{I=0} = C_S' - 3C_U'. \tag{54}$$

We then find

$$C_S' \simeq \frac{3}{4m_N \left(\frac{1}{4\pi a_{\text{scatt}}^{S=0}} - \frac{0.253}{a_{\text{lattice}}} \right)} + \frac{1}{4m_N \left(\frac{1}{4\pi a_{\text{scatt}}^{I=0}} - \frac{0.253}{a_{\text{lattice}}} \right)}, \quad (55)$$

$$C_U' \simeq \frac{1}{4m_N \left(\frac{1}{4\pi a_{\text{scatt}}^{S=0}} - \frac{0.253}{a_{\text{lattice}}} \right)} - \frac{1}{4m_N \left(\frac{1}{4\pi a_{\text{scatt}}^{I=0}} - \frac{0.253}{a_{\text{lattice}}} \right)}. \quad (56)$$

For any chosen temperature and nucleon density there is again a corresponding maximum value for the lattice spacing,

$$a_{\text{lattice}}^{-1} \gg \left(a_{\text{lattice}}^{-1} \right)_{\text{min}} = \max[\pi^{-1}\sqrt{2m_N T}, \rho^{1/3}]. \quad (57)$$

This sets a maximum value for the absolute value of the coupling C_U',

$$|C_U'| \ll |C_U'|_{\text{max}} \equiv \frac{\left| \frac{1}{4\pi a_{\text{scatt}}^{I=0}} - \frac{1}{4\pi a_{\text{scatt}}^{S=0}} \right|}{4m_N \left| \left(\frac{1}{4\pi a_{\text{scatt}}^{S=0}} - 0.253 \left(a_{\text{lattice}}^{-1} \right)_{\text{min}} \right) \left(\frac{1}{4\pi a_{\text{scatt}}^{I=0}} - 0.253 \left(a_{\text{lattice}}^{-1} \right)_{\text{min}} \right) \right|}. \quad (58)$$

A similar bound for C_S' can be made but is not needed in our analysis.

B. Convexity inequality for μ_3^S

We first consider the case when an asymmetric chemical potential μ_3^S is coupled to the nucleon spins. The grand canonical partition function is given by

$$Z_G = \int DN D\bar{N} \exp\left(-S_E\right) = \int DN D\bar{N} \exp\left(\int d^4x \, \mathcal{L}_E\right), \quad (59)$$

where we take the form of $\mathcal{L}_E$ given in Eq. (47) with an asymmetric spin chemical potential,

$$\mathcal{L}_E = -\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3^S \sigma_3\right) \right] N$$
$$- \frac{1}{2}C_S'(\bar{N}N)^2 - \frac{1}{2}C_U' \bar{N}\vec{\tau}N \cdot \bar{N}\vec{\tau}N$$
$$- \frac{1}{3!}C_3(\bar{N}N)^3 - \frac{1}{4!}C_4(\bar{N}N)^4. \quad (60)$$

Using Hubbard-Stratonovich transformations we can rewrite Z_G as

$$Z_G \propto \int DN D\bar{N} Df D\vec{\phi} \exp\left(\int d^4x \, \mathcal{L}_E^{f,\vec{\phi}}\right), \quad (61)$$

where

$$\mathcal{L}_E^{f,\vec{\phi}} = -\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3^S \sigma_3\right) \right] N$$
$$+ f\bar{N}N + iC_U'\vec{\phi} \cdot \bar{N}\vec{\tau}N + g(f) - \frac{1}{2}C_U'\vec{\phi} \cdot \vec{\phi}. \quad (62)$$

In Ref. [31] it was shown that three-body and four-body forces can be introduced without spoiling positivity of the functional integral measure. The only requirements are that the three-body force is not too strong and the four-body force is not too repulsive. Estimates of the three- and four-body forces suggest that these conditions are satisfied. For our analysis here we assume that to be the case, and the function $g(f)$ is

a real-valued function which produces the two-, three-, and four-body force terms involving $\bar{N}N$.

The nucleon matrix $\mathbf{M}$ has the block diagonal structure

$$\mathbf{M} = \begin{bmatrix} M\left(\mu + \mu_3^S\right) & 0 \\ 0 & M\left(\mu - \mu_3^S\right) \end{bmatrix}, \quad (63)$$

where the upper block is for up spins and the lower block is for down spins. M is a matrix in isospin space,

$$M(\mu) = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu\right) \right] + f + iC_U'\vec{\phi} \cdot \vec{\tau}. \quad (64)$$

We note that

$$\tau_2 M \tau_2 = M^*, \quad (65)$$

and so $\det M \geqslant 0$.

Integrating over the fermion fields gives us

$$Z_G\left(\mu, \mu_3^S\right) \propto \int DN D\bar{N} Df D\vec{\phi} \exp\left(\int d^4x \, \mathcal{L}_E^{f,\vec{\phi}}\right)$$
$$= \int D\Theta \det \mathbf{M}$$
$$= \int D\Theta \det M\left(\mu + \mu_3^S\right) \det M\left(\mu - \mu_3^S\right), \quad (66)$$

where

$$D\Theta = Df D\vec{\phi} \exp\left(-\int d^4x \, \mathcal{V}(f, \vec{\phi})\right) \quad (67)$$

with

$$-\mathcal{V}(f, \vec{\phi}) = g(f) - \tfrac{1}{2}C_U'\vec{\phi} \cdot \vec{\phi}. \quad (68)$$

From the Cauchy-Schwarz inequality we get

$$Z_G(\mu, \mu_3) \leqslant \sqrt{Z_G\left(\mu + \mu_3^S, 0\right)}\sqrt{Z_G\left(\mu - \mu_3^S, 0\right)}. \quad (69)$$

We therefore find an upper bound for the pressure,

$$P\left(\mu, \mu_3^S\right) \leqslant \tfrac{1}{2}\left[P\left(\mu + \mu_3^S, 0\right) + P\left(\mu - \mu_3^S, 0\right)\right]. \quad (70)$$

C. Shifted-field inequality for μ_3^I

We now consider the case with an isospin chemical potential μ_3^I. We start with the Lagrange density in terms of the Hubbard-Stratonovich fields,

$$\mathcal{L}_E^{f,\vec{\phi}} = -\bar{N}\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu - \mu_3^I\tau_3\right)\right]N$$
$$+ f\bar{N}N + iC_U'\vec{\phi}\cdot\bar{N}\vec{\tau}N + g(f) - \frac{1}{2}C_U'\vec{\phi}\cdot\vec{\phi}. \tag{71}$$

Let $\mathbf{M}_0$ be the nucleon matrix without the $\mu_3^I\tau_3$ term,

$$\mathbf{M}_0 = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + \left(m_N^{0\prime} - \mu\right)\right] + f + iC_U'\vec{\phi}\cdot\vec{\tau}. \tag{72}$$

We note that

$$\tau_2\mathbf{M}_0\tau_2 = \mathbf{M}_0^*, \tag{73}$$

and so $\det\mathbf{M}_0 \geqslant 0$.

As we did for the two-fermion case, we now shift the ϕ_3 field and find the inequality

$$P\left(\mu,\mu_3^I\right) \leqslant P(\mu,0) + \frac{\left(\mu_3^I\right)^2}{2C_U'}. \tag{74}$$

If we take the lattice spacing as large as possible then

$$P\left(\mu,\mu_3^I\right) \leqslant P(\mu,0) + \frac{\left(\mu_3^I\right)^2}{2\left|C_U'\right|_{\max}}, \tag{75}$$

where $|C_U'|_{\max}$ was defined in Eq. (58). As a rough estimate of the quantities involved, we note that for $\rho \sim 0.1\rho_N$ and $T < 10$ MeV, $|C_U'|_{\max}$ is about 0.2 fm^2. In this case, however, the situation is complicated by nuclear saturation, and it is not clear that the pionless effective theory is applicable.

V. SUMMARY AND DISCUSSION

The main results we have shown are as follows. We first considered the two-fermion system with an attractive interaction and SU(2) symmetry. If μ is the symmetric chemical potential and μ_3 is the asymmetric chemical potential, we proved both the convexity inequality

$$P(\mu,0) \leqslant P(\mu,\mu_3) \leqslant \frac{1}{2}[P(\mu + \mu_3, 0) + P(\mu - \mu_3, 0)], \tag{76}$$

and the shift-field inequality

$$P(\mu,0) \leqslant P(\mu,\mu_3) \leqslant P(\mu,0) + \frac{3\mu_3^2}{2\left|C\right|_{\max}}. \tag{77}$$

We then analyzed the four-fermion system with an $SU(2)_I \times SU(2)_S$ symmetry. We considered the case when both s-wave channels are attractive and without loss of generality assumed the $I = 0$ channel to be more strongly

attractive. With μ as the symmetric chemical potential and μ_3^S as the asymmetric spin chemical potential we proved the convexity inequality

$$P(\mu,0) \leqslant P\left(\mu,\mu_3^S\right) \leqslant \frac{1}{2}\left[P\left(\mu + \mu_3^S, 0\right) + P\left(\mu - \mu_3^S, 0\right)\right]. \tag{78}$$

For nonzero asymmetric isospin chemical potential μ_3^I we proved the shifted-field inequality

$$P(\mu,0) \leqslant P\left(\mu,\mu_3^I\right) \leqslant P(\mu,0) + \frac{\left(\mu_3^I\right)^2}{2|C_U'|_{\max}}. \tag{79}$$

In the Wigner SU(4) symmetry limit, we note that the shift-field inequality (76) becomes meaningless since $|C_U'|_{\max} \to 0$. However in this limit we also have the convexity inequality for μ_3^I,

$$P(\mu,0) \leqslant P\left(\mu,\mu_3^I\right) \leqslant \frac{1}{2}\left[P\left(\mu + \mu_3^I, 0\right) + P\left(\mu - \mu_3^I, 0\right)\right]. \tag{80}$$

The equation of state for nuclear matter with small isospin asymmetries can be measured indirectly in the laboratory by studying nuclear multifragmentation. Of the inequalities presented here, the simplest and perhaps most interesting to check is the isospin convexity inequality (80) in the Wigner symmetry limit. Since much is still unknown about asymmetric nuclear matter, this Wigner pressure inequality may be a useful consistency check for proposed phenomenological models for asymmetric nuclear matter.

While some of the inequalities are difficult to observe in nuclear physics experiments, each of our results could be tested in the cold Fermi gas system where parameters in the effective Lagrangian can be tuned. Such experiments can in principle test the inequalities over a range of physical parameters and probe universal results in the limit of infinite scattering length and zero range. Although four-fermion systems have not yet been produced, these may be possible in the near future.

On the computational side, the inequalities can also be checked by nonperturbative lattice simulations. There have been several recent simulations of effective theories on the lattice [1,3,5,6]. It will be particularly interesting to look at symmetric and asymmetric nuclear matter in the Wigner symmetry limit, which can be simulated without any sign problem.

It remains to be seen how well many-body nucleon systems can be described without explicit pions. Results from Ref. [5] for dilute neutron matter suggest that lowest-order effective field theory without pions works very well in describing the neutron equation of state. The situation for nearly symmetric nuclear matter, however, is less clear due to the effect of saturation which requires higher densities.

With pions included the effective theory action can in general become negative. This would in principle invalidate any inequality based on positivity of the action. However, it has been shown that this sign problem goes away in the static limit [32]. Furthermore the sign problem has been numerically observed to be small [3] in simulations with neutrons and

neutral pions for temperatures above 10 MeV and densities at or below normal nuclear matter density. If one neglects these sign changes, then the sign-quenched results for the effective theory with pions will also satisfy each of the inequalities proven here.

[1] H. M. Müller, S. E. Koonin, R. Seki, and U. van Kolck, Phys. Rev. C **61**, 044320 (2000).

[2] J.-W. Chen and D. B. Kaplan, Phys. Rev. Lett. **92**, 257002 (2004).

[3] D. Lee, B. Borasoy, and T. Schaefer, Phys. Rev. C **70**, 014007 (2004).

[4] T. Abe, R. Seki, and A. N. Kocharian, Phys. Rev. C **70**, 014315 (2004).

[5] D. Lee and T. Schaefer, nucl-th/0412002.

[6] M. Wingate, hep-lat/0409060.

[7] D. Lee, Phys. Rev. C **70**, 064002 (2004).

[8] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983).

[9] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).

[10] C. Vafa and E. Witten, Phys. Rev. Lett. **53**, 535 (1984).

[11] C. Vafa and E. Witten, Commun. Math. Phys. **95**, 257 (1984).

[12] E. Witten, Phys. Rev. Lett. **51**, 2351 (1983).

[13] S. Nussinov, Phys. Rev. Lett. **51**, 2081 (1983).

[14] S. Nussinov and M. A. Lampert, Phys. Rep. **362**, 193 (2002).

[15] S. Nussinov, hep-ph/0306187.

[16] T. D. Cohen, Phys. Rev. Lett. **91**, 032002 (2003).

[17] K. M. O'Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas, Science **298**, 2179 (2002).

[18] S. Gupta, Z. Hadzibabic, M. W. Zwierlein, C. A. Stan, K. Dieckmann, C. H. Schunck, E. G. M. van Kempen, B. J. Verhaar, and W. Ketterle, Science **300**, 1723 (2003).

[19] C. A. Regal and D. S. Jin, Phys. Rev. Lett. **90**, 230404 (2003).

[20] T. Bourdel, J. Cubizolles, L. Khaykovich, K. M. F. Magalhaes, S. J. J. M. F. Kokkelmans, G. V. Shlyapnikov, and C. Salomon, Phys. Rev. Lett. **91**, 020402 (2003).

[21] M. E. Gehm, S. L. Hemmer, S. R. Granade, K. M. O'Hara, and J. E. Thomas, Phys. Rev. A **68**, 011401(R) (2003).

[22] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).

[23] S. R. Beane, P. F. Bedaque, A. Parreno, and M. J. Savage, hep-lat/0312004.

[24] R. L. Stratonovich, Sov. Phys. Dokl. **2**, 416 (1958).

[25] J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959).

[26] S. Hands *et al.*, Eur. Phys. J. C **17**, 285 (2000).

[27] T. Mehen, I. W. Stewart, and M. B. Wise, Phys. Rev. Lett. **83**, 931 (1999).

[28] P. F. Bedaque, H. W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999).

[29] P. F. Bedaque, H. W. Hammer, and U. van Kolck, Nucl. Phys. **A676**, 357 (2000).

[30] E. Wigner, Phys. Rev. **51**, 106 (1939).

[31] J.-W. Chen, D. Lee, and T. Schaefer, Phys. Rev. Lett. **93**, 242302 (2004).

[32] S. Chandrasekharan, M. Pepe, F. D. Steffen, and U. J. Wiese, J. High Energy Phys. **12**, 035 (2003).