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Regularities with random interactions in energy centroids defined by group symmetries

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Regular structures generated by random interactions in energy centroids defined over irreducible representations (irreps) of some of the group symmetries of the interacting boson models *sd*IBM, *sdg*IBM, *sd*IBM-*T*, and *sd*IBM-*ST* are studied by deriving trace propagation equations for the centroids. It is found that, with random interactions, the lowest and highest group irreps in general carry most of the probability for the corresponding centroids to be lowest in energy. This generalizes the result known earlier, via numerical diagonalization, for the more complicated fixed spin J centroids where simple trace propagation is not possible.

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Johnson, Bertsch, and Dean in 1998 [1], using the nuclear shell model, found that random two-body interactions lead to ground states for even-even nuclei, having spin 0^+ with very high probability. Similarly, Bijker and Frank [2], using the interacting boson model (IBM) with s and d bosons (*sd*IBM), showed that random interactions generate vibrational and rotational structures with high probability. Further studies using the shell model, fermions in one or two j orbits, sd, sp, and sdg IBM's, bosons in a single ℓ orbit, etc., revealed statistical predominance of odd-even staggering in binding energies, $0^+, 2^+, 4^+, \ldots$ yrast sequence, regularities in ground states in parity distributions, occupation numbers, and so on; see [3–10] and references therein. Notably, Zelevinsky *et al.* [4] introduced the idea of geometric chaos as a basis for the regularities observed in shell model studies. Similarly, Zhao *et al.* [8] developed a prescription based on sampling of the corners of the parameter space, and Bijker, Frank and Kota [7,9] employed mean-field methods. The unexpected results for regularities with random interactions are reviewed in [6,10]. As Zhao *et al.* stated [10], "a more fundamental understanding of the robustness of $0^+_{\text{g.s.}}$ dominance is still out of reach." Therefore, going beyond the ground states and near yrast levels, energy centroids, spectral widths, and correlations among them are also being investigated by several groups [4,5,11–13] as they are expected to give new insights into regularities generated by random interactions. For example, Zhao *et al.* [11,12] initiated the study of energy centroids and analyzed fixed-L (fixed-J, JT) centroids in IBM (in shell model) spaces. They found that $L_{\min}$ (or $J_{\min}$) and $L_{\max}$ (or $J_{\max}$) will be lowest with largest probabilities, and others appear with negligible probability. Similarly Papenbrock and Weidenmüller [13] recently analyzed the structure of fixed-J spectral widths for fermions in a single-j shell.

An interesting and important question is the extension of the spin-zero ground state dominance (and also other regular structures seen in shell model and IBM studies) to group theoretical models with Hamiltonians preserving a symmetry higher than J (or L). Similarly, one may consider centroids and variances defined over good or broken symmetry subspaces. They open a new window to the regularities of many-body systems in the presence of random forces. Initiating work in this direction [9], we recently used random one- plus two-body Hamiltonians invariant with respect to $O(\mathcal{N}_1) \oplus O(\mathcal{N}_2)$ symmetry of a variety of interacting boson models to investigate the probability of occurrence of a given $(\omega_1\omega_2)$ irreducible representation (irrep) to be the ground state in even-even nuclei; $[\omega_1]$ and $[\omega_2]$ are symmetric irreps of $O(\mathcal{N}_1)$ and $O(\mathcal{N}_2)$ respectively. We found that the 0^+ dominance observed in ground states of even-even nuclei extends to group irreps. The purpose of this paper and others to follow is to go beyond this and study regularities generated by random interactions in energy centroids, variances, etc. defined over group irreps. Reported in this rapid communication are the results of a first analysis of energy centroids with examples from *sd*IBM, *sdg*IBM, *sd*IBM-*T* with the bosons carrying isospin (T), and *sd*IBM-*ST* with the bosons carrying spin-isospin (ST) degrees of freedom. Before proceeding further, it is important to stress that energy centroids (also variances) can be calculated as a function of particle number m and the quantum numbers labeling the group irreps, without recourse to the construction of the Hamiltonian matrix. The principle used here is trace propagation, a subject introduced in the context of statistical nuclear spectroscopy by French [14,15]. Readers not interested in the details of group algebra and derivation of trace propagation equations for the energy centroids [given by Eqs. (5)–(8)], may jump ahead to the discussion of results starting just after Eq. (8).

Let us begin with the spectrum generating algebra (SGA), say G_1, of a group theoretical model with all the many particle states in the model belonging to the irrep Γ_1 of G_1. For example, the SGA G_1 for *sd*IBM is U(6). Now the average of an operator $\mathcal{O}(k)$ of maximum body rank k over the irreps Γ_2 of a subalgebra G_2 of G_1 (G_2 in general denotes a set of subalgebras contained in G_1 and Γ_2 denotes all their irreps) is defined by

$$\langle \mathcal{O}(k)\rangle^{\Gamma_1,\Gamma_2} = \sum_{\beta}\sum_{\alpha\in\Gamma_2} \langle \Gamma_1\beta\Gamma_2\alpha|\mathcal{O}(k)|\Gamma_1\beta\Gamma_2\alpha\rangle \Bigg/ \left[\sum_{\beta}\sum_{\alpha\in\Gamma_2}\langle \Gamma_1\beta\Gamma_2\alpha|1|\Gamma_1\beta\Gamma_2\alpha\rangle\right] \quad (1)$$

In Eq. (1), β labels the multiple occurrence (multiplicity) of Γ_2 in a given Γ_1 irrep (i.e., in the reduction of Γ_1 to Γ_2). Removing the denominator in Eq. (1) gives the trace over (Γ_1, Γ_2) space, i.e., $\text{tr}[\mathcal{O}(k)]^{\Gamma_1,\Gamma_2}$. General theory for propagation of traces of operators over irreps of group symmetries was developed in Refs. [16–19]. In particular, Quesne [16] showed that for $G_1 \supset G_2$, trace propagation over the irreps Γ_1 and Γ_2 of G_1 and G_2 algebras is related to the so-called integrity basis of G_2 in G_1 which gives the minimal set of G_2 scalars in G_1. As discussed in Refs. [17,18], it is seen that in general the multiplicity of Γ_2 in a given Γ_1 irrep results in the propagation of the matrix of traces $\mathrm{tr}[\mathcal{O}(k)]^{\Gamma_1\Gamma_2;\beta\beta'} = \sum_\alpha \langle\Gamma_1\beta\Gamma_2\alpha|\mathcal{O}(k)|\Gamma_1\beta'\Gamma_2\alpha\rangle$. However, quite often the trace of this trace matrix or its average, as given by Eq. (1), which is important in applications, may not propagate in a simple manner. There are approximate methods for propagating trace of the trace matrix, and they are significant, particularly when the integrity basis contains far too many operators [18]. A very important example here is fixed-L averages in IBM's or fixed-J (and JT) averages in the shell model. For these, it is not possible to write a simple propagation equation in terms of the defining space averages. On the other hand, traces over irreps of group symmetries (higher than J symmetry) can be propagated in many situations using Casimir invariants. French and Draayer [19] showed that by simple counting of irreps of G_2 in G_1 and the scalars one can construct in terms of the Casimir invariants of G_1 and G_2 will immediately confirm if propagation via Casimir invariants is possible; in this situation, the integrity basis reduces to Casimir operators of G_1 and G_2. In this paper we restrict ourselves to examples in IBM's where this result applies; Refs. [20,21] give the first IBM examples.

For IBM's the SGA, called G_1 above, is $U(\mathcal{N})$, with $\mathcal{N}=6$ for *sd*IBM, 15 for *sdg*IBM, etc, and its irreps Γ_1 are labeled uniquely by the boson number m as all m boson states are symmetric with respect to $U(\mathcal{N})$. Now, consider the average of an operator $\mathcal{O}(k)$ over the irreps (m,Γ_2) with Γ_2 being the irreps of a subalgebra G_2 of $U(\mathcal{N})$. Say the number of (m,Γ_2), called Γ^i hereafter, for $m \leqslant k$ is r. Also assume that there are r number of invariants $\hat{\mathcal{C}}_i$, $i=1,2,\ldots,r$, of maximum body rank k constructed out of the products of m and the Casimir invariants of G_2. Then, for any irrep Γ^0, clearly $\langle\mathcal{O}\rangle^{\Gamma^0} = \sum_{i=1}^r a_i\langle\hat{\mathcal{C}}_i\rangle^{\Gamma^0}$, where a_i are constants. The a_i can be determined by assuming that the averages $\langle\mathcal{O}\rangle^{\Gamma^j}$ are known for the irreps Γ^j, $j=1,2,\ldots,r$. For example, Γ_j can be chosen to be the irreps (m,Γ_2) for $m \leqslant k$. With this, defining the row matrices $[\mathcal{C}]$ and $[\mathcal{O}_{\mathrm{inp}}]$ and the $r\times r$ matrix $[X]$ as

$$\begin{aligned}[][\mathcal{C}] &\Leftrightarrow \mathcal{C}_i = \langle\hat{\mathcal{C}}_i\rangle^{\Gamma^0},\\ [\mathcal{O}_{\mathrm{inp}}] &\Leftrightarrow \mathcal{O}_{\mathrm{inp}:i} = \langle\mathcal{O}\rangle^{\Gamma^i},\\ [X] &\Leftrightarrow X_{ij} = \langle\hat{\mathcal{C}}_j\rangle^{\Gamma^i},\end{aligned} \tag{2}$$

the propagation equation is

$$\langle\mathcal{O}\rangle^{\Gamma^0} = [\mathcal{C}]\,[X]^{-1}\,[\widetilde{\mathcal{O}_{\mathrm{inp}}}]. \tag{3}$$

As the eigenvalues of the Casimir invariants of the algebras $U(\mathcal{N})$, $O(\mathcal{N})$, etc. are known, construction of $[\mathcal{C}]$ and $[X]$ is easy. In the reminder of this paper, the H is assumed to be $(1+2)$-body. As an example, let us consider SU(3) centroids in *sd*IBM. Here $G_1 = U(6)$ and $G_2 = SU(3)$. Simple counting of scalar in terms of the number operator $\hat{n}$ and the quadratic Casimir operator $\hat{\mathcal{C}}_2$ and the cubic Casimir operator $\hat{\mathcal{C}}_3$ of SU(3) confirms that they exhaust all the scalars needed for propagating $\langle\mathcal{O}(k)\rangle^{m,(\lambda\mu)}$ for any k [18,21]. Note that $(\lambda\mu)$ denotes SU(3) irreps. Propagation equation for the energy centroids over SU(3) irreps can be written as $\langle H\rangle^{m,(\lambda\mu)} = a_0 + a_1 m + a_2 m^2 + a_3\mathcal{C}_2(\lambda\mu)$, where

$$\mathcal{C}_2(\lambda\mu) = \langle(\lambda\mu)\alpha|\hat{\mathcal{C}}_2|(\lambda\mu)\alpha\rangle = [\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)]. \tag{4}$$

Using Eqs. (3) and (4), the propagation equation, in terms of the energy centroids for $m \leqslant 2$, is [21]

$$\begin{aligned}\langle H\rangle^{m,(\lambda\mu)} &= \tfrac{1}{2}(2-3m+m^2)\langle H\rangle^{0,(00)} + (2m-m^2)\langle H\rangle^{1,(20)}\\ &\quad+\left[-\tfrac{5}{6}m+\tfrac{5}{18}m^2+\tfrac{1}{18}\mathcal{C}_2(\lambda\mu)\right]\langle H\rangle^{2,(40)}\\ &\quad+\left[\tfrac{1}{3}m+\tfrac{2}{9}m^2-\tfrac{1}{18}\mathcal{C}_2(\lambda\mu)\right]\langle H\rangle^{2,(02)}.\end{aligned} \tag{5}$$

Equation (5) extends easily to the SU(3) limit of *pf*IBM with U(10) SGA but not to *sdg*, *sdgpf*, etc., IBM's. Now we will derive three new propagation equations for energy centroids.

In the $U(\mathcal{N}) \supset \sum_i\,[U(\mathcal{N}_i) \supset O(\mathcal{N}_i)] \oplus$ symmetry limits of IBM's, with the bosons carrying angular momenta $\ell_1, \ell_2, \ldots$ so that $\mathcal{N}_i = (2\ell_i+1)$ and $\mathcal{N} = \sum_i \mathcal{N}_i$, for a given ith orbit, $U(\mathcal{N}_i)$ generates number of particles m_i in the orbit and $O(\mathcal{N}_i)$ generates the corresponding seniority quantum number ω_i. The number operators $\hat{n}_i$ of $U(\mathcal{N}_i)$ and the quadratic Casimir operators of $O(\mathcal{N}_i)$ or the corresponding pairing operators $\hat{P}_2(O(\mathcal{N}_i))$ suffice to give fixed $\widetilde{m}\widetilde{\omega} = (m_1\omega_1, m_2\omega_2, \ldots)$ averages of H. Appendix A in Ref. [22] gives the explicit form of $\hat{P}_2(O(\mathcal{N}_i))$ for a general situation. Fixed-$\widetilde{m}\widetilde{\omega}$ centroids of H can be written as $\langle H\rangle^{\widetilde{m}\widetilde{\omega}} = \sum_i m_i\epsilon_i + \sum_{i\geqslant j} a_{ij} m_i(m_j - \delta_{ij}) + \sum_i c_i\langle\hat{P}_2(O(\mathcal{N}_i))\rangle^{m_i\omega_i}$. Solving for a_{ij} and c_i in terms of the centroids for $m \leqslant 2$, the final propagation equation, for IBM's with no internal degrees of freedom, is

$$\begin{aligned}\langle H\rangle^{\widetilde{m}\widetilde{\omega}} &= \sum_i m_i\epsilon_i + \sum_{i>j}\overline{V_{ij}}\,m_i m_j\\ &\quad+\sum_i \frac{m_i(m_i-1)}{2}\langle V\rangle^{m_i=2,\omega_i=2}\\ &\quad+\sum_i\frac{\langle V\rangle^{m_i=2,\omega_i=0}-\langle V\rangle^{m_i=2,\omega_i=2}}{2\mathcal{N}_i}\\ &\quad\times(m_i-\omega_i)(m_i+\omega_i+\mathcal{N}_i-2);\\ \overline{V_{ij}} &= \{[\mathcal{N}_i(\mathcal{N}_j+\delta_{ij})]/(1+\delta_{ij})\}^{-1}\\ &\quad\times\sum_L V^L_{\ell_i\ell_j\ell_i\ell_j}(2L+1),\\ \langle V\rangle^{m_i=2,\omega_i=0} &= \langle(\ell_i\ell_i)L_i=0|V|(\ell_i\ell_i)L_i=0\rangle,\\ \langle V\rangle^{m_i=2,\omega_i=2} &= \left[\frac{\mathcal{N}_i(\mathcal{N}_i+1)}{2}\,\overline{V_{ii}} - \langle V\rangle^{m_i=2,\omega_i=0}\right]\\ &\quad\bigg/\left[\frac{\mathcal{N}_i(\mathcal{N}_i+1)}{2}-1\right].\end{aligned} \tag{6}$$

Note that in (6), ϵ_i are energies of the single- particle levels with angular momentum ℓ_i and $V^L_{\ell_i\ell_j\ell_i\ell_j} = \langle(\ell_i\ell_j)L|V|(\ell_i\ell_j)L\rangle$ are two-particle matrix elements of the two-body part of H. Also in Eq. (6), for s orbit, $m_s = 2$ and $\omega_s = 2$ and there will be no two-boson state with $\omega_s = 0$. Equation (6) for *sdg*IBM is given first in [23], i.e., for averages over the irreps of the algebras in the chain $U_{sdg}(15) \supset U_s(1) \oplus [U_d(5) \supset O_d(5)] \oplus [U_g(9) \supset$

$O_g(9)$]. Similarly Eq. (6) gives H averages over the irreps of $U_{sd}(6) \supset U_d(5) \supset O_d(5)$ of *sd*IBM, $U_{sdpf}(16) \supset [U_d(5) \supset O_d(5)] \oplus [U_p(3) \supset O_p(3)] \oplus [U_f(7) \supset O_f(7)]$ of *sdpf* IBM, etc. Moreover, this extends easily (this will be discussed elsewhere) to IBM's with internal degrees of freedom. Let us add that it is also possible to write propagation equations for the variances $\langle [H - \langle H\rangle^{\widetilde{m}\widetilde{\omega}}]^2\rangle^{\widetilde{m}\widetilde{\omega}}$ using the results in [20,24].

In IBM-*T* with $U(3\mathcal{N}) \supset U(\mathcal{N}) \otimes [SU_T(3) \supset O_T(3)]$ where $U(\mathcal{N})$ gives the spatial part (for sd, sdg, $sdpf$, etc.) and $O_T(3)$ generating isospin [25], it is possible to propagate the centroids $\langle H\rangle^{m,\{f\},T} \equiv \langle H\rangle^{m,(\lambda\mu),T}$. Note that the $U(\mathcal{N})$ irreps are labeled by $\{f\} = \{f_1, f_2, f_3\}$ where $f_1 \geqslant f_2 \geqslant f_3 \geqslant 0$ and $m = f_1 + f_2 + f_3$. The corresponding $SU_T(3)$ irreps are $(\lambda, \mu) = (f_1 - f_2, f_2 - f_3)$. The $SU_T(3)$ to $O_T(3)$ reductions follow from the formulas given by Elliott [26,27]. The scalars $1, \hat{n}, \hat{n}^2, \hat{C}_2(SU_T(3))$ and $\hat{T}^2$ and the energy centroids for $m \leqslant 2$, via Eqs. (2)–(4), give

$$\begin{aligned}\langle H\rangle^{m,(\lambda\mu),T} &= \left[1 - \tfrac{3}{2}m + \tfrac{m^2}{2}\right]\langle H\rangle^{0,(00),0} + [2m - m^2]\\ &\times \langle H\rangle^{1,(10),1} + \left[-\tfrac{1}{6}m + \tfrac{1}{18}m^2 + \tfrac{1}{9}\mathcal{C}_2(\lambda\mu)\right.\\ &\left. - \tfrac{1}{6}T(T+1)\right]\langle H\rangle^{2,(20),0} + \left[-\tfrac{5}{6}m + \tfrac{5}{18}m^2\right.\\ &\left. + \tfrac{1}{18}\mathcal{C}_2(\lambda\mu) + \tfrac{1}{6}T(T+1)\right]\langle H\rangle^{2,(20),2}\\ &+ \left[\tfrac{1}{2}m + \tfrac{1}{6}m^2 - \tfrac{1}{6}\mathcal{C}_2(\lambda\mu)\right]\langle H\rangle^{2,(01),1}. \qquad (7)\end{aligned}$$

For *sd*IBM-*T*, starting with the general Hamiltonian given in Appendix A of [27] which contains the *s* and *d* boson energies and 17 two-particle matrix elements $V^{L,t}_{\ell_1\ell_2\ell_3\ell_4}$, it is easy to write $\langle H\rangle^{m,(\lambda\mu),T}$ for $m \leqslant 2$; for $m = 2$, the two-boson isospins t uniquely define the corresponding $SU_T(3)$ irreps. Thus Eq. (7) for $\langle H\rangle^{m,(\lambda\mu),T}$ is easy to apply for any m.

In IBM-*ST*, a group chain of interest is [28] $U(6\mathcal{N}) \supset U(\mathcal{N}) \otimes [SU_{ST}(6) \supset O_{ST}(6)]$, with $U(\mathcal{N})$ generating the spatial part and $SU_{ST}(6)$ [or $U_{ST}(6)$] generating the spin-isospin part; note that the Wigner's spin-isospin super-multiplet algebra $SU_{ST}(4)$ is isomorphic to $O_{ST}(6)$. Just as before, it is possible to propagate the centroids $\langle H\rangle^{m,\{f\},[\sigma]}$. Here $\{f\}$'s are the irreps of $U(\mathcal{N})$ or equivalently $U_{ST}(6)$ and $\{f\} = \{f_1, f_2, f_3, f_4, f_5, f_6\}$, where $\sum_i f_i = m$ and $f_i \geqslant f_{i+1} \geqslant 0$. The $O_{ST}(6)$ irreps are labeled by $[\sigma] = [\sigma_1, \sigma_2, \sigma_3]$, and the $\{f\}$ to $[\sigma]$ reductions, needed for the results discussed ahead, follow from the analytical formulas given in [27] and the tabulations in [29]. Equations (2) and (3) give, using the quadratic Casimir invariants $\hat{C}_2$ of $U_{ST}(6)$ and $O_{ST}(6)$,

$$\begin{aligned}\langle H\rangle^{m,\{f\},[\sigma]} &= \left[1 - \tfrac{3}{2}m + \tfrac{m^2}{2}\right]\langle H\rangle^{0,\{0\},[0]} + [2m - m^2]\\ &\times \langle H\rangle^{1,\{1\},[1]} + \left[-\tfrac{5}{3}m + \tfrac{1}{4}m^2 + \tfrac{1}{6}\mathcal{C}_2(\{f\})\right.\\ &\left. + \tfrac{1}{12}\mathcal{C}_2([\sigma])\right]\langle H\rangle^{2,\{2\},[2]} + \left[-\tfrac{1}{12}m\right.\\ &\left. + \tfrac{1}{12}\mathcal{C}_2(\{f\}) - \tfrac{1}{12}\mathcal{C}_2([\sigma])\right]\langle H\rangle^{2,\{2\},[0]}\\ &+ \left[\tfrac{5}{4}m + \tfrac{1}{4}m^2 - \tfrac{1}{4}\mathcal{C}_2(\{f\})\right]\langle H\rangle^{2,\{1^2\},[1^2]}, \qquad (8)\end{aligned}$$

where $\mathcal{C}_2(\{f\}) = \langle \hat{C}_2[U_{ST}(6)]\rangle^{\{f\}} = \sum_{i=1}^{6} f_i(f_i + 7 - 2i)$ and $\mathcal{C}_2([\sigma]) = \langle \hat{C}_2[O_{ST}(6)]\rangle^{[\sigma]} = \sum_{i=1}^{3} \sigma_i(\sigma_i + 6 - 2i)$. Diagonalizing $\hat{C}_2(O_{ST}(6))$ in the $|(\ell_1\ell_2)LST\rangle$ basis and applying the resulting unitary transformation to the H matrix in this basis will give the input averages in Eq. (8).

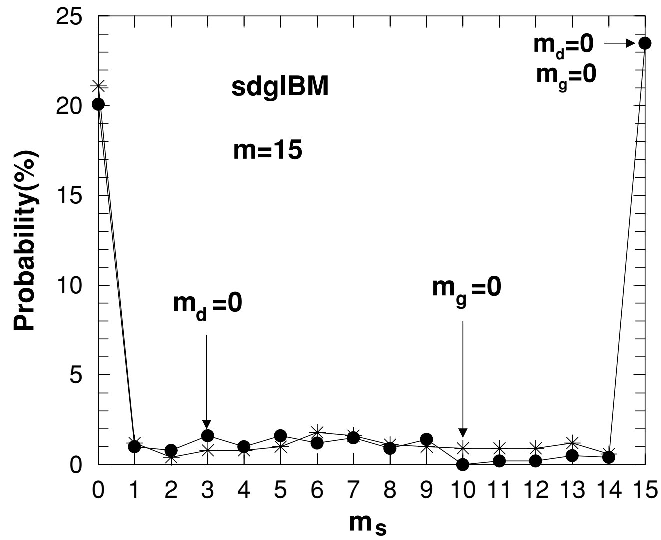

FIG. 1. Probabilities for *sdg*IBM fixed-$(m_s, m_d, v_d, m_g, v_g)$ centroid energies to be lowest in energy vs m_s for a system of 15 bosons ($m = 15$). For each m_s, the probability shown is the sum of the probabilities for the irreps with the seniority quantum number lowest [$v_\ell = \pi(m_\ell)$] and highest ($v_\ell = m_\ell$). Filled circles and stars are for configurations with $m_d = 0$ and $m_g = 0$, respectively; they are joined by lines to guide the eye. Note that for $m_s = 15$, both $m_d = 0$ and $m_g = 0$.

Now we will apply Eqs. (5)–(8) to study regularities generated by random interactions in energy centroids. In all the calculations used are independent Gaussian random variables with zero center and unit variance and a 1000-member ensemble. We begin with the simplest example of *sd*IBM centroids. The highest SU(3) irrep for a given m is $(2m, 0)$ and Eq. (5) gives $\langle H\rangle^{m,(\lambda\mu)} - \langle H\rangle^{m,(2m,0)} = [\mathcal{C}_2(\lambda\mu) - \mathcal{C}_2(2m, 0)]\Delta/18$ with $\Delta = \langle H\rangle^{2,(40)} - \langle H\rangle^{2,(02)}$. Therefore, the probability of finding Δ to be positive or negative will simply give the probability for finding the highest or lowest m particle SU(3) irrep to be lowest in energy. With the two-particle matrix elements chosen to be Gaussian variables (with zero center and unit variance), Δ itself will be a Gaussian variable with zero center. For $m = 3k, 3k + 1$, and $3k + 2$, k being a positive integer, the lowest SU(3) irreps are (00), (20), and (02), respectively. They will be lowest in energy with 50% and the $(2m, 0)$ irrep will be lowest in energy with 50% probability. Thus, it is easy to understand the regularities in centroids defined over fixed SU(3) irreps in *sd*IBM with one-plus two-body Hamiltonians, without constructing the many boson Hamiltonian matrix but just by using the propagation equation (5).

In *sdg*IBM, regularities in fixed-$(m_s, m_d, v_d, m_g, v_g)$ centroids are studied using the propagation equation (6). Choosing the three single-particle energies $(\epsilon_s, \epsilon_d, \epsilon_g)$ and the 16 diagonal two-particle matrix elements $V^L_{\ell_1\ell_2\ell_1\ell_2}$, with $\ell_i = 0, 2$, and 4 to be Gaussian variables, the probability for the centroid of a given $(m_s, m_d, v_d, m_g, v_g)$ configuration to be lowest is calculated for $m = 6 - 25$, and the results are shown in Fig. 1 for $m = 15$. To maintain proper scaling, the ϵ are divided by m and the V^L by $m(m-1)$ just as in [2]. For the discussion

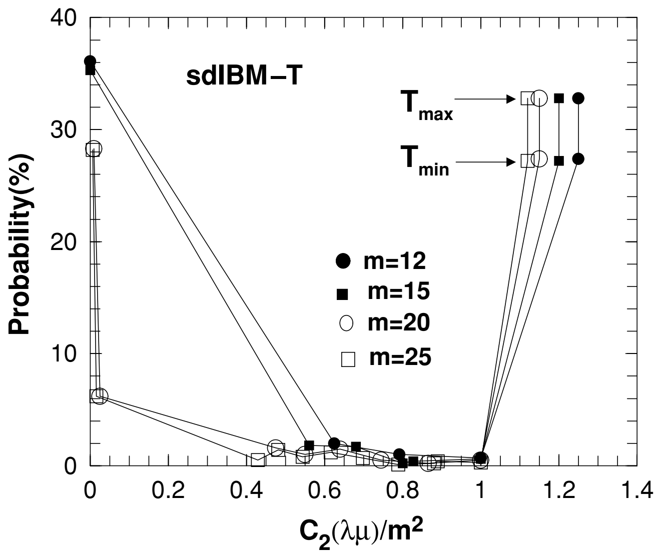

FIG. 2. Probabilities for the *sd*IBM-*T*'s $(\lambda\mu)T$ centroid energies to be lowest in energy vs $\mathcal{C}_2(\lambda\mu)/m^2$ for boson systems with $m = 12, 15, 20$, and 25. Except for the highest $(\lambda\mu)$, for all other $(\lambda\mu)$ shown in the figure, the probabilities are for $T_{\rm max}$ if $\lambda \neq 0$ and $\mu \neq 0$ and they are for $T_{\rm min}$ if $\lambda = 0$ or $\mu = 0$. For the irreps not shown in the figure, the probability is $<0.1\%$. All the points for a given m are joined by lines to guide the eye.

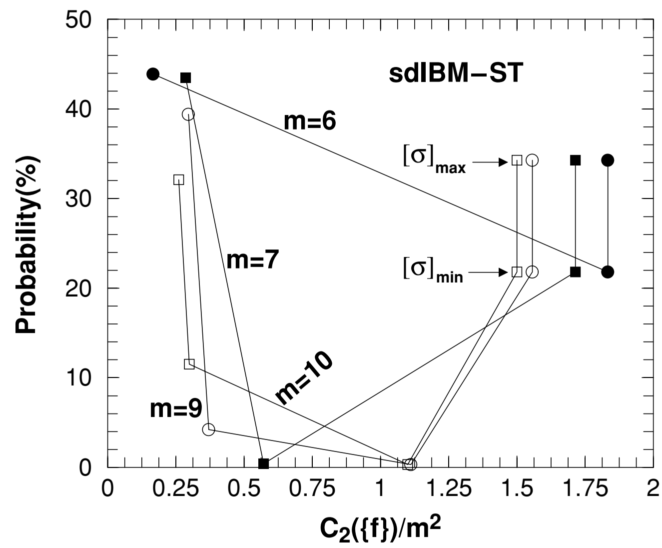

FIG. 3. Probabilities for the *sd*IBM-*ST* $\{f\}[\sigma]$ centroid energies to be lowest in energy vs $\mathcal{C}_2(\{f\})/m^2$ for boson systems with $m = 6, 7, 9$, and 10. For the structure of the irreps with probability $>2\%$, see text. For $m = 7, 9$, and 10 there is one additional irrep with $\sim 0.3\%$ probability. For the irreps not shown in the figure, the probability is $<0.1\%$. All the points for a given m are joined by lines to guide the eye.

of the results, we define $\pi(x)$ such that $\pi(x) = 0$ for x even and $\pi(x) = 1$ for x odd. It is seen from Fig. 1, and also valid for any m, that the configurations $(m_s, m_d = v_d = m - m_s, m_g = v_g = 0)$, $[m_s, m_d = m - m_s, v_d = \pi(m_d), m_g = v_g = 0]$, $(m_s, m_d = v_d = 0, m_g = v_g = m - m_s)$ and $[m_s, m_d = v_d = 0, m_g = m - m_s, v_g = \pi(m_g)]$ exhaust about 91% probability. Moreover, the configurations with $m_s = m_d = 0$ carry $\sim 20\%$, $m_s = m_g = 0$ carry $\sim 21\%$, $m_s = m$ carries $\sim 24\%$, and $m_s \neq 0$ but $m_d = 0$ or $m_g = 0$ carry $\sim 26\%$ probability. Thus the $m_s = m$ configuration and the four configurations with $m_s = 0$ are most probable to be lowest in energy. However, other configurations with $m_s \neq 0, m$ (they are 49 out of 1195 configurations in the $m = 15$ example) give nonnegligible probability for being lowest. Thus, about $\sim 4\%$ of the $(m_s, m_d, v_d, m_g, v_g)$ configurations will have probability to be lowest with random interactions.

For *sd*IBM-*T*, it is easily seen from Eq. (7) that the one-body part of H will not play any role in the study of fixed-$(\lambda, \mu)T$ centroids. Choosing $V^{L,t}_{\ell_1\ell_2\ell_1\ell_2}$ to be Gaussian variables, the centroids are generated, using Eq. (7), for $m = 10 - 25$ and for all allowed $(\lambda, \mu)T$. Some typical results for the regularities are shown in Fig. 2. First, for a given m the highest $\mathrm{SU}_T(3)$ irrep is $(m, 0)$ with $T_{\rm max} = m$ and $T_{\rm min} = \pi(m)$. For $m = 3k, 3k+1$, and $3k+2$, with k being a positive integer, the lowest $\mathrm{SU}_T(3)$ irreps are (00), (10), and (01) with $T = 0$, 1, and 1, respectively; for the latter two situations, the next lowest irreps are (02) and (20), respectively, with $T_{\rm min} = 0$. For $m = 3k$, Fig. 2 shows that the lowest $\mathrm{SU}_T(3)$ irrep centroid (here T is unique) is lowest with $\sim 35\%$ probability. Similarly the highest irrep centroid is lowest with $\sim 60\%$ probability and this splits into $\sim 30\%$ each for the lowest and highest T's. For $m = 3k+1$ and $3k+2$, the probability for the centroid of the highest irrep to be lowest in energy is the same as for $m = 3k$. However, for the centroid of the lowest irrep, the probability is $\sim 29\%$, and the next lowest irrep appears with $\sim 6\%$. Thus, in general, the centroids of the highest and the lowest (for $m = 3k+1$ and $3k+2$, the lowest two) $\mathrm{SU}_T(3)$ irreps exhaust about 95% of the probability for being lowest in energy. As the two-particle centroids $X^t = \langle H\rangle^{m=2,t}$ are linear combinations of V, it can be seen that they themselves are Gaussian variables. Note that $\langle H\rangle^{m,(\lambda\mu)T} - \langle H\rangle^{m,(m,0)m} = [\mathcal{C}_2(\lambda\mu) - \mathcal{C}_2(m,0)]\Delta_1 + [T(T+1) - m(m+1)]\Delta_2$, where $\Delta_1 = \frac{1}{9}X^0 + \frac{1}{18}X^2 - \frac{1}{6}X^1$ and $\Delta_2 = \frac{1}{6}(X^2 - X^0)$. Calculations with X^t taken as Gaussian variables with the same variance (actually, the variance of X^0 and X^2 are the same, and that of X^1 is $\sim 20\%$ higher) are carried out, and it is seen that they give almost the same results as in Fig. 2.

For *sd*IBM-*ST*, as seen from Eq. (8), the energy centroids $\langle H\rangle^{m,\{f\},[\sigma]}$ are determined by the two-particle averages $\langle H\rangle^{2,\{2\},[2]}$, $\langle H\rangle^{2,\{2\},[0]}$, and $\langle H\rangle^{2,\{1^2\},[1^2]}$, and they are linear combinations of the two-particle matrix elements V^{LST} in the $|(\ell_1\ell_2)LST\rangle$ basis. Instead of choosing V^{LST} to be Gaussian variables, we have chosen, using the result found in the *sd*IBM-*T* examples, the three two-particle averages to be Gaussian variables. Using this, the probabilities are calculated for various m values, and some of the results are shown in Fig. 3. First, for a given m, the highest $\{f\}$ is $\{m\}$. The corresponding highest and lowest $[\sigma]$ are $[m]$ and $[\pi(m)]$. For all m, the centroid of the highest $\mathrm{U}_{ST}(6)$ irrep is lowest with $\sim 56\%$ probability, and this splits into $\sim 34\%$ and $\sim 22\%$ for the highest and lowest $\mathrm{O}_{ST}(6)$ irreps. For $m = 6k, 6k \pm 1, 6k \pm 2$, and $6k+3$, with k a positive integer, the lowest $\mathrm{U}_{ST}(6)$ irreps are those that can be reduced to the irreps {0}, {1}, ({2}, $\{1^2\}$), and ($\{1^3\}$, {21}), respectively. These irreps with the corresponding lowest [σ] are lowest, with probability ∼43%.

In conclusion, with random interactions, the lowest and highest group irreps (i.e., irreps of G_2 in $G_1 \supset G_2$) carry most of the probability for the corresponding centroids to be lowest in energy. With the inclusion of a subalgebra ($G_1 \supset G_2 \supset G_3$), these probabilities split into the probabilities for the corresponding lowest and highest irreps of the subalgebra. This is indeed the situation for all the examples discussed in this paper. Continuing with the process of embedding subalgebras, the O(3) algebra generating L can be reached (with generalization for systems with LT, LST, or JT). Then, clearly, the energy centroids of highest and lowest L should be most probable, and this is found to be true numerically in [11,12]. An important aspect of the energy centroids is that they propagate via Casimir invariants in many situations. New propagation equations are derived in this paper [Eqs. (6)–(8)]. In fact, there are many other situations where such equations can be derived; an example is the centroids over the irreps $[m_{sd}(\lambda_{sd}\mu_{sd}); m_{pf}(\lambda_{pf}\mu_{pf})]$ of $[\mathrm{U}_{sd}(6) \supset \mathrm{SU}_{sd}(3)] \oplus [\mathrm{U}_{pf}(10) \supset \mathrm{SU}_{pf}(3)]$ algebra of *sdpf*IBM [30]. These will be discussed in a longer paper along with extensions of the present work to spectral variances and also to shell model symmetries. Finally, an important observation is that the propagators carry information about $G_1 \supset G_2$ geometry (i.e., $G_1 \supset G_2$ reduced Wigner coefficients and G_2 Racah coefficients); thus, it is plausible that propagation equations may be useful in quantifying geometric chaos. This is being investigated, and it should be remarked that only recently has the role of Wigner-Racah algebra in two-body random matrix ensembles been established [31].

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