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Expressions for the number of $J = 0$ pairs in even-even Ti isotopes

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We count the number of pairs in the single-j-shell model of ^{44}Ti for various interactions. For a state of total angular momentum I, the wave function can be written as $\Psi = \sum_{J_P\, J_N} D(J_P\, J_N)[(j^2)_{J_P}(j^2)_{J_N}]_I$, where $D(J_P\, J_N)$ is the probability amplitude that the protons couple to J_P and the neutrons to J_N. For $I = 0$ there are three states with $(I = 0,\ T = 0)$ and one with $(I = 0,\ T = 2)$. The latter is the double analog of ^{44}Ca. In that case $(T = 2)$, the magnitude of $D(J\,J)$ is the same as that of a corresponding two-particle coefficient of fractional parentage. In counting the number of pairs with an even angular momentum J, we find a new relationship is obtained by diagonalizing a unitary nine-j symbol. We are also able to get results for the "no-interaction" case for $T = 0$ states, for which it is found, e.g., that there are fewer $(J = 1,\ T = 0)$ pairs than on the average. Relative to this no-interaction case, we find that for the most realistic interaction used there is an enhancement of pairs with angular momentum $J = 0$, 2, 1, and 7, and a depletion for the others. Also considered are interactions in which only the $(J = 0,\ T = 1)$ pair state is at lower energy, interactions where only the $(J = 1,\ T = 0)$ pair state is lowered, interactions where both are equally lowered, and the $Q \cdot Q$ interaction. We are also able to obtain simplified formulas for the number of $J = 0$ pairs for the $I = 0$ states in ^{46}Ti and ^{48}Ti by noting that the unique state with isospin $|T_z| + 2$ is orthogonal to all the states with isospin $|T_z|$.

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I. INTRODUCTION

The plan in this work is to obtain, wherever possible, simplified expressions for the number of pairs of particles of a given angular momentum J_{12} in the Ti isotopes in a single-j-shell model. We shall show that such simplified expressions can be obtained for all even-J_{12} pairs in ^{44}Ti and for all $J_{12} = 0$ pairs in ^{44}Ti, ^{46}Ti, and ^{48}Ti. In a previous work [1], we calculated the number of pairs but did not derive simple expressions for a given J_{12}. Also in this work, unlike the previous one, we compare our results with what we call the no-interaction case, which will be described later.

In the single-j-shell model, ^{44}Ti consists of two valence protons and two valence neutrons in the $f_{7/2}$ shell. The allowed states for two identical particles have angular momenta $J =$ 0, 2, 4, and 6 and isospin $T = 1$. For a neutron-proton pair, we can have these and also states of isospin $T = 0$ with angular momenta $J = 1$, 3, 5, and 7. In other words, for even J the isospin is 1 and for odd J the isospin is 0.

The wave function of a given state of total angular momentum I can be written as

$$\Psi = \sum_{J_P J_N} D^I(J_P J_N)\big[\big(j_\pi^2\big)^{J_P}\big(j_\nu^2\big)^{J_N}\big]^I. \tag{1.1}$$

In the above, $D(J_P J_N)$ is the probability amplitude that the protons couple to angular momentum J_P and the neutrons to J_N. The normalization condition is

$$\sum_{J_P J_N} [D^I(J_P J_N)]^2 = 1, \tag{1.2}$$

and the orthonormality condition is

$$\sum_\alpha D^\alpha(J_P J_N) D^\alpha(J'_P J'_N) = \delta_{J_P\, J'_P}\delta_{J_N\, J'_N}, \tag{1.3}$$

where the sum is over the different eigenfunctions.

For states of angular momentum $I = 0$, J_P must be equal to J_N ($J_P = J_N \equiv J$) such that

$$\Psi(I = 0) = \sum_J D(J\,J)\big[\big(j_\pi^2\big)^J\big(j_\nu^2\big)^J\big]^0. \tag{1.4}$$

In the single-j-shell configuration of ^{44}Ti, there are three $I = 0$ states of isospin $T = 0$ and one of isospin $T = 2$. The latter is the double analog of a state in ^{44}Ca, i.e., of a state of four neutrons in the $f_{7/2}$ shell. For the unique $(I = 0,\ T = 2)$ state in ^{44}Ti, the magnitudes of the $D(J\,J)$'s are the same as those of two-particle coefficients of fractional parentage,

$$D(J\,J)_{(I=0,\,T=2)} = [j^2 J j^2 J|\}j^4 0]. \tag{1.5}$$

We thus have for $(I = 0,\ T = 2)$,

$$\begin{aligned} D(00) &= 0.5, \quad D(22) = -0.3727,\\ D(44) &= -0.5, \quad D(66) = -0.600. \end{aligned} \tag{1.6}$$

For the $(I = 0,\ T = 0)$ states, however, the D's do depend on the interaction. We show in Table I the values of the $D(J\,J)$'s for the lowest energy state for the following interactions:

A. $(J = 0,\ T = 1)$ pairing. All two-particle states are degenerate except $(J = 0,\ T = 1)$, which is lowered relative to the others.
B. $(J = 1,\ T = 0)$ pairing. Only $(J = 1,\ T = 0)$ is lowered.

TABLE I. Wave functions $D(JJ)$ of ^{44}Ti for various interactions: A. ($J = 0,\ T = 1$) pairing; B. ($J = 1,\ T = 0$) pairing; C. equal $J = 0$, $J = 1$ pairing; D. $Q \cdot Q$ interaction; E. spectrum of ^{42}Sc.

	$D(JJ)$, ground state $T = 0$					$D(JJ)$, $T = 2$
	A	B	C	D	E	Any interaction
$J = 0$	0.866	0.380	0.826	0.7069	0.7878	0.5
$J = 2$	0.213	0.688	0.405	0.6863	0.5617	−0.3727
$J = 4$	0.289	0.416	0.373	0.1694	0.2208	−0.5
$J = 6$	0.347	−0.457	0.126	0.0216	0.1234	−0.6009

C. Equal $J = 0$ and $J = 1$ pairing. Both ($J = 0,\ T = 1$) and ($J = 1,\ T = 0$) are lowered by the same amount.
D. $Q \cdot Q$ interaction.
E. Spectrum of ^{42}Sc.

Interaction E is the same as the McCullen, Bayman, and Zamick (MBZ) calculation [2], except that the correct spectrum of ^{42}Sc is used (some of the $T = 0$ states were not known in 1964). We equate the matrix elements

$$\left\langle (f_{7/2})^2_J \middle| V \middle| (f_{7/2})^2_J \right\rangle \tag{1.7}$$

with $E(J)$, the excitation energy of the lowest state of angular momentum J in ^{42}Sc. The experimental values for $J = 0$ to $J = 7$ are (in MeV) 0.0, 0.6111, 1.5863, 1.4904, 2.8153, 1.5101, 3.2420, and 0.6163, respectively. Note that the three lowest states have angular momenta $J = 0, 1, 7$. One can add to all those numbers a constant equal to the pairing energy $E(^{42}\mathrm{Sc}) + E(^{40}\mathrm{Ca}) - E(^{41}\mathrm{Sc}) - E(^{41}\mathrm{Ca})$. The value is -3.182 MeV. However, adding this constant will not affect the spectrum or wave functions of ^{44}Ti. Note that for the even-J states of ^{42}Sc, the isospin is 1, while for the odd-J states the isospin is 0. The eigenvalues and eigenfunctions of interaction E are given in Table II.

II. NUMBER OF PAIRS IN ^{44}Ti

In this work we will use notation A for the number of valence nucleons and n for the number of valence neutrons. For Ti isotopes, $A = n + 2$.

As previously noted [1] for a system of A valence nucleons with total isospin T, we have the following result for the number of pair states:

- Total number of pair states is $A(A-1)/2$.

TABLE II. Excitation energies in MeV and eigenvectors of the spectrum of ^{42}Sc interaction.

	Eigenvectors			
Exc. energies $D(JJ)$	0.0000	5.4861	8.2840	8.7875
$D(00)$	0.78776	−0.35240	−0.50000	−0.07248
$D(22)$	0.56165	0.73700	0.37268	−0.04988
$D(44)$	0.22082	−0.37028	0.50000	0.75109
$D(66)$	0.12341	−0.44219	0.60093	−0.65432

- Number with isospin $T_{12} = 0$ is $A^2/8 + A/4 - T(T+1)/2$.
- Number with isospin $T_{12} = 1$ is $3A^2/8 - 3A/4 + T(T+1)/2$.

Hence, for the $T = 0$ state of ^{44}Ti ($A = 4$) we have three $T_{12} = 0$ pairs and three $T_{12} = 1$ pairs. For the $T = 2$ state, however, we have six $T_{12} = 1$ pairs. The important thing to note is that the number of pairs does not depend on the two-body interaction, except for the fact that it conserves isospin.

In ^{44}Ti the number of pairs (nn, pp, and np) with total angular momentum J_{12} ($J_{12} = 0, 1, 2, 3, 4, 5, 6, 7$) is given by

$$2\,|D(J_{12}J_{12})|^2\,\delta_{J_{12},\,\mathrm{even}} + |f(J_{12})|^2, \tag{2.1}$$

with

$$f(J_{12}) = 2\sum_{J_P} U_{9j}(J_P,\ J_{12}) D(J_P J_P), \tag{2.2}$$

where we introduce the abbreviated symbol U_{9j} to represent the unitary nine-j symbol, which can also be written in terms of a six-j symbol,

$$\begin{aligned} U_{9j}(J_P J_{12}) &= \langle (j^2)J_P(j^2)J_P|(j^2)J_{12}(j^2)J_{12}\rangle^0 \\ &= (2J_P+1)(2J_{12}+1)\begin{Bmatrix} j & j & J_P \\ j & j & J_P \\ J_{12} & J_{12} & 0 \end{Bmatrix} \\ &= (-1)^{1+J_P+J_{12}}\sqrt{(2J_P+1)(2J_{12}+1)} \\ &\quad\times \begin{Bmatrix} j & j & J_P \\ j & j & J_{12} \end{Bmatrix}. \end{aligned} \tag{2.3}$$

A derivation of the results up to now in this section is given in the Appendix. Since the last publication [1], we have found a relationship which in some cases simplifies the expression. The relationship pertains only to even J_{12}

$$\begin{aligned} \sum_{J_P} U_{9j}(J_P J_{12}) D(J_P J_P) &= D(J_{12}J_{12})/2 \qquad \text{for} \quad T = 0, \\ &= -D(J_{12}J_{12}) \qquad \text{for} \quad T = 2. \end{aligned} \tag{2.4}$$

Some useful relationships that we exploit to get this result are

$$\sum_{J_{12}} U_{9j}(J_P J_{12}) U_{9j}(J'_P J_{12}) = \delta_{J_P,\,J'_P}, \tag{2.5}$$

$$\sum_{J_{12}} U_{9j}(J_P J_{12}) U_{9j}(J'_P J_{12})(-1)^{J_{12}} = -(-1)^{J_P - J'_P} U_{9j}(J_P J'_P). \tag{2.6}$$

Relationship (2.4) does not depend upon which isospin conserving interaction is used. Using this result for even values of J_{12}, we find

$$|f(J_{12})|^2 = |D(J_{12}J_{12})|^2, \tag{2.7}$$

and hence

$$\begin{aligned} \text{number of } nn \text{ pairs } &= \text{number of } pp \text{ pairs} \\ &= \text{number of } np \text{ pairs } = D(J_{12}\,J_{12})^2. \end{aligned} \tag{2.8}$$

One way of looking at Eq. (2.4) is to say that we can also write the wave function as

$$N \sum_{J,\,\text{even}} D(JJ)[(\pi(1)\nu(2))^J(\pi(3)\nu(4))^J]^{I=0}, \qquad (2.9)$$

where N is a normalization factor.

We do not have a corresponding simple expression for odd J_{12}. However, the total number of odd-J_{12} pairs must be equal to 3, the same as the total number of even-J_{12} pairs.

We can prove relationship (2.4) by regarding the unitary nine-j symbol as a four-by-four matrix where J_P and J_{12} assume only even values (0, 2, 4, 6), despite the fact that J_{12} can also assume odd values. The eigenvalues of this matrix are -1 (singly degenerate) and 0.5 (triply degenerate). The eigenvalue -1 corresponds to the ($J = 0$, $T = 2$) state of ^{44}Ti, and indeed the values of $D(JJ)$ are identical to those obtained with a charge-independent Hamiltonian and are given in the last column of Table I. As previously mentioned, these are the two-particle coefficients of fractional parentage [1].

The triple degeneracy with eigenvalue 0.5 corresponds to the three $T = 0$ states being degenerate with this unitary nine-j Hamiltonian. This means that any linear combination of the three $T = 0$ states is an eigenvector. We can obtain the eigenvalues above without an explicit diagonalization. This is shown in Sec. IV.

A. Results for the ($I = 0$, $T = 2$) state

Because the ($I = 0$, $T = 2$) state is unique, we will give the results for this case first. Since the ^{44}Ti $T = 2$ state is the double analog of ^{44}Ca [3], a system of four identical particles, each pair must have even J_{12}. The number of pairs is $6|[(j^2)J_{12}(j^2)J_{12}|\}j^4 0]|^2$, i.e., proportional to the square of the two-particle coefficient of fractional parentage. The number of pairs is 1.5 for $J_{12} = 0$ and $(2J_{12} + 1)/6$ for $J_{12} = 2, 4, 6$. This is also the result for ^{44}Ca. Hence, even though the $I = 0$ ground state of ^{44}Ca has angular momentum zero and seniority zero, there are more $J_{12} = 6$ pairs in ^{44}Ca than there are $J_{12} = 0$ pairs. This should not be surprising. As noted by Talmi [4] for the simpler case of a closed neutron shell, i.e. ^{48}Ca, the number of pairs with angular momentum J is equal to $2J + 1$. There is only one $J = 0$ pair in ^{48}Ca.

B. Number of pairs for all states

We can count the number of pairs for all four $I = 0$ states (three with isospin $T = 0$ and one with $T = 2$). Using the orthonormality condition (1.3), we eliminate the D's and find

$$(\text{number of pairs})/4 = \tfrac{1}{2}\delta_{J_{12},\,\text{even}} + \tfrac{1}{2}[1 - U_{9j}(J_{12}J_{12})]. \qquad (2.10)$$

The values for $T_{12} = 1$ are 0.9375 for $J_{12} = 0$; 0.8542 for $J_{12} = 2$; 0.9375 for $J_{12} = 4$; and 1.0208 for $J_{12} = 6$. The total sum is 3.75.

The values for $T_{12} = 0$ are 0.3244 for $J_{12} = 1$; 0.6761 for $J_{12} = 3$; 0.7494 for $J_{12} = 5$; and 0.5001 for $J_{12} = 7$. The total sum is 2.25.

TABLE III. Number of pairs for the $T = 0$ state of ^{44}Ti with various interactions: A. ($J = 0$, $T = 1$) pairing; B. ($J = 1$, $T = 0$) pairing; C. equal $J = 0$, $J = 1$ pairing; D. Q · Q interaction; E. spectrum of ^{42}Sc; F. no interaction. In G, we give the number of pairs for the $T = 2$ state.

	A	B	C	D	E	F	G
$J_{12} = 0$	2.250	0.433	2.045	1.499	1.862	0.750	1.500
$J_{12} = 2$	0.139	1.420	0.492	1.413	0.946	0.861	0.833
$J_{12} = 4$	0.250	0.320	0.416	0.086	0.146	0.750	1.500
$J_{12} = 6$	0.361	0.626	0.048	0.001	0.046	0.639	2.167
$J_{12} = 1$	0.250	1.297	0.618	0.834	0.675	0.432	–
$J_{12} = 3$	0.583	0.388	0.165	0.156	0.271	0.902	–
$J_{12} = 5$	0.916	0.003	0.564	0.013	0.159	1.000	–
$J_{12} = 7$	1.250	1.311	1.654	1.996	1.895	0.667	–

C. Results for the $T = 0$ ground state of ^{44}Ti including the no-interaction case

In Table III we give results for the number of pairs for the five interactions A–E defined in Sec. I. We also consider the no-interaction case. This value is obtained by finding the total number of pairs for all three $T = 0$ states and dividing by three.

It should be noted that for odd angular momentum J_{12}, the pair must consist of one proton and one neutron. For even J_{12}, one third of the pairs consists of two protons, one third of two neutrons, and one third of one neutron and one proton. We start with the no-interaction result in column F. Since there are six pairs and eight J_{12}'s, if there were an equal distribution, then we could assign 0.75 pairs to each angular momentum. This serves us as a good basis for comparison. We find that even in the no-interaction case, the results depend on J_{12}. The minimum number of pairs comes with the ($J_{12} = 1$, $T_{12} = 0$) case and is only 0.432. This is of interest because there has been a lot of discussion in recent times about ($J_{12} = 1$, $T_{12} = 0$) pairing. We start out at least with a bias against it. The maximum number of pairs in the no-interaction case is for ($J_{12} = 5$, $T_{12} = 0$), a mode that has largely been ignored.

However, of greater relevance is what happens to the ground-state wave function when the interaction is turned on. Therefore, we compare the no-interaction case with case E, the Spectrum of ^{42}Sc interaction. We see striking differences. Relative to the no-interaction case, case E has an increase in the following number of pairs: (a) $J_{12} = 0$ from 0.75 to 1.8617; (b) $J_{12} = 2$ from 0.861 to 0.9458; (c) $J_{12} = 1$ from 0.432 to 0.6752; and (d) $J_{12} = 7$ from 0.667 to 1.8945. Since the sum of all pairs in both cases is 6, there must be a decrease in the number of pairs with the other angular momenta, and there is. For example, the number of $J_{12} = 6$ pairs decreases from 0.639 to 0.0457, and the number of $J_{12} = 5$ pairs from 1.00 to 0.1587. There is also a large decrease in the number of pairs for $J_{12} = 4$ and $J_{12} = 3$.

The results with the $Q \cdot Q$ interaction concerning the pairs distribution with odd angular momenta are qualitatively similar to the correct spectrum of ^{42}Sc. It is remarkable that the number of pairs with $J_{12} = 0$ is almost equal to that of pairs with $J_{12} = 2$. The number of pairs of other angular momenta is almost negligible. Thus the total number of $J = 0, 2$ pairs is 2.91 out of the total of 3 even pairs.

Looking at the wave functions for $Q \cdot Q$ (case D) and the realistic interaction (case E) in Table I, we see the dominance of $J = 0$ and 2 couplings for neutrons and protons. The percentages of the higher angular momentum couplings ($J =$ 4 and 6) are only 2.9% for $Q \cdot Q$ and 6.4% for the realistic case E. This is in accord with the interacting boson model IBM2 [5] where only *s* and *d* bosons are considered. With the simpler schematic interactions A and B, the percentages are much higher.

We next look at the schematic interactions A–C in the first three columns of Table III. For the ($J = 0,\ T = 1$) pairing interaction, there are a lot of $J_{12} = 0$ pairs (2.25) but very few $J_{12} = 1$ pairs (0.250). The number of $J_{12} = 7$ pairs is fairly large (1.250).

For case B, the ($J = 1,\ T = 0$) pairing interaction, there are, as expected, a lot of $J_{12} = 1$ pairs (1.297) and relatively few $J_{12} = 0$ pairs (0.433). But still there is a substantial number of $J_{12} = 7$ pairs (1.311). However, if we examine the wave function for this case it is very different from that of the correct spectrum of ^{42}Sc, and this case represents a rather unrealistic ground-state wave function.

For case C, corresponding to equal ($J = 0,\ T = 1$) and ($J = 1,\ T = 0$) pairing, we get much better agreement in the wave function compared with case E. The number of $J_{12} = 0$ pairs is 2.043 as compared with 1.862 for case E. For $J_{12} = 1$ the values are 0.618 and 0.675, and for $J_{12} = 7$ they are 1.654 and 1.895. Amusingly, when we lower the $J = 0$ and 1 matrix elements together we get more $J_{12} = 7$ pairs than we do when we lower each one separately (1.250 and 1.311). There is one main deficiency in the ($J = 0 + J = 1$) case: the number of ($J = 2,\ T = 0$) pairs is only 0.492 as compared with 0.9458 for the spectrum of ^{42}Sc case. The enhancement is undoubtedly due to the quadrupole correlations in the nucleus, an important ingredient that is sometimes forgotten when all the emphasis is on ($J = 0,\ T = 1$) and ($J = 1,\ T = 0$) pairing. However, if one restricts oneself to ($J = 0,\ T = 1$) and ($J = 1,\ T = 0$), then equal admixtures in the interaction yield much more realistic results than either of them yields singly.

For the $T = 2$ state of ^{44}Ti, the double analog of ^{44}Ca (a system of particles of one kind), the number of $J = 6$ pairs is the largest. This suggests that the more deformed the state (and the $T = 0$ ground state of ^{44}Ti is certainly more deformed than the ground state of ^{44}Ca), the lower the number of high angular momentum pairs with the exception of $J(\text{maximum}) = 7$. Or to put it in another way, the more spherical the state, the higher the number of high angular momentum pairs. It is interesting that for the $T = 2$ excited state, only the $T = 1$ pairs can contribute to excite the system. Therefore, the six pairs distributed in the ground state among eight angular momenta ($J_{12} = 0, 1, 2, 3, 4, 5, 6, 7$) are distributed in the excited state among four even angular momenta ($J_{12} = 0, 2, 4, 6$). Note that they are not distributed uniformly but with weights depending on the angular momentum carried by the pair. Thus, the number of pairs in the states of angular momenta equal to 0 and 4 increases by a factor of 2 when compared with those corresponding to the no-interaction case, while for $J_{12} = 6$ the factor is 3.391. Note that the number of pairs with angular momentum equal to 2 is decreased. For the $T = 2$ case, the number of pairs is independent of the interaction in the single j shell.

From a previous study of two-to-one relationships between the excitation energies for double and simple analog states in the shell $f_{7/2}$ [2,3,6], it is noted that if we write the wave function for ^{43}Ti (^{43}Sc) as

$$\psi^I = \sum_{J_P} C(J_P)[(j^2)^{J_P} j_\nu]^I,$$

then the $C(J_P)$'s for the $I = j$ states are identical to the $D(J_P J_P)$'s for corresponding $I = 0$ states in ^{44}Ti and the eigenvalues are one half of those. From this it follows that the number of pairs of a given J_{12} for ^{43}Ti in a single-j-shell calculation is one half of those in Table III.

III. NUMBER OF $J_{12} = 0$ PAIRS IN ^{44}Ti, ^{46}Ti, AND ^{48}Ti

In this section we show that with further development, we can obtain the number of ($J_{12} = 0,\ T_{12} = 1$) pairs not only in ^{44}Ti, but also in ^{46}Ti and ^{48}Ti.

A. *np* pairs

We let n be the number of valence neutrons in a given Ti isotope. For ^{44}Ti, ^{46}Ti, and ^{48}Ti, $n = 2$, 4, and 6, respectively. Of course, the number of valence nucleons is $A = n + 2$.

The wave function for a state with total angular momentum $I = 0$ is

$$\psi^{I=0} = \sum D(J\,Jv)[(j^2)^J (j^n)^{Jv}]^{I=0}. \tag{3.1}$$

Here v is the seniority quantum number. In the Ca isotopes there is only one state for each Jv pair. In ^{42}Ca we have Jv pairs (00), (22), (42), and (62). In ^{44}Ca they are (00), (22), (42), (62), (24), (44), (54), and (84). In ^{46}Ca they are (00), (22), (42), and (62).

For states with total angular momentum $I = 0$ in the even-even Ti isotopes, the possible isospins in the single-j-shell model space are $T_{\min} = |N - Z|/2$ and $T_{\max} = T_{\min} + 2$. There are no $I = 0$ states with $T = T_{\min} + 1$.

Thus, for ^{44}Ti we have three $I = 0$ states with isospin $T = 0$ and one with isospin $T = 2$; for ^{46}Ti, five with isospin $T = 1$ and one with isospin $T = 3$; for ^{48}Ti, three with isospin $T = 2$ and one with isospin $T = 4$.

The formula for the number of *np* pairs with angular momentum J_{12} in a state of total angular momentum $I = 0$ is

$$N_{np}(J_{12}) = 2n\left|\sum_J D(J\,Jv)[j^{n-1}(J_0)j|\}j^n Jv] \times\sqrt{(2J+1)(2J_{12}+1)}\begin{Bmatrix} j & j & J \\ j & J_0 & J_{12}\end{Bmatrix}\right|^2. \tag{3.2}$$

In (3.2), we have a coefficient of fractional parentage (cfp) which is needed to separate one neutron from the others. The six-j symbol is needed, of course, to combine this neutron with a proton in order to form an *np* pair with angular momentum J_{12}. For ^{44}Ti the cfp is unity, so the expression is simpler.

Note that for each of the even-even Ti isotopes that we are considering, there is only one state with isospin $T_{\max} = T_{\min} + 2$. Because such a state is a double analog of a corresponding state for a system of identical particles, i.e., neutrons only in the Ca isotopes, the amplitudes $D(JJv)$ of the $T_{\max}$ column vector are known. They are two-particle coefficients of fractional parentage

$$D^{I=0,T_{\max}}(JJv) = [j^n J j^2 J|\}j^{n+2}0]. \tag{3.3}$$

The two-particle cfp for the Ca isotope separates a system of $(n+2)$ neutrons into one of two neutrons and one of n neutrons. In the Ti isotopes, we separate $(n+2)$ nucleons into two protons and n neutrons.

We have also found in the past an identity that relates the above two-particle cfp to a one-particle cfp

$$[j^n(J)j|\}j^{n+1}j] = [j^n J j^2 J|\}j^{n+2}0]. \tag{3.4}$$

Now the $T_{\min}$ states must be orthogonal to the $T_{\max}$ states. Thus, we get two conditions:

$$\text{Orthogonality:} \quad \sum_J D^{I=0,T_{\min}}(JJ)[j^n(J)j|\}j^{n+1}j] = 0, \tag{3.5}$$

$$\text{Normalization:} \quad \sum_J D^{I=0,T_{\max}}(JJ)[j^n(J)j|\}j^{n+1}j] = 1. \tag{3.6}$$

We can use these conditions to get the number of ($J_{12} = 0$, $T_{12} = 1$) pairs in one of the above Ti isopotes. To do so, we find useful the following explicit formulas for cfp from de Shalit and Talmi [7] and Talmi [4]:

$$[j^{n-1}(jv=1)j|\}j^n J=0 v=0] = 1, \tag{3.7}$$

$$[j^{n-1}(jv=1)j|\}j^n J v=2] = \sqrt{\frac{2(2j+1-n)}{n(2j-1)}}, \tag{3.8}$$

$$[j^n(J=0 v=0)j|\}j^{n+1} jv=1] = \sqrt{\frac{(2j+1-n)}{(n+1)(2j+1)}}, \tag{3.9}$$

$$[j^n(Jv=2)j|\}j^{n+1} jv=1] = -\sqrt{\frac{2n(2J+1)}{(n+1)(2j+1)(2j-1)}}. \tag{3.10}$$

Alternatively we can use explicit expressions for two-particle cfp's given by Lawson (his A5.57 and A5.58 equations in [8]). They are

$$[j^n(J=0, v=0)j^2 0|\}j^{n+2}0 v=0] = \left\{\frac{2j+1-n}{(n+1)(2j+1)}\right\}^{1/2}, \tag{3.11}$$

$$[j^n(J, v=2)j^2 J|\}j^{n+2}J v=0] = -\left\{\frac{2n(2J+1)}{(n+1)(2j+1)(2j-1)}\right\}^{1/2}\left\{\frac{1+(-1)^J}{2}\right\}. \tag{3.12}$$

We define $M = \sum_{J\geqslant 2} D(JJ)\sqrt{(2J+1)}$. From Eqs. (3.5), (3.6), (3.9), and (3.10) we find

$$\begin{aligned}\sqrt{\frac{(2j+1-n)}{(n+1)(2j+1)}}D(00) - M\sqrt{\frac{2n}{(n+1)(2j+1)(2j-1)}} &= 0, \qquad T = T_{\min},\\ &= 1, \qquad T = T_{\max}.\end{aligned} \tag{3.13}$$

We find $D(00) = M/3$ for ^{44}Ti, $M/\sqrt{3}$ for ^{46}Ti, and M for ^{48}Ti. Now the number of pairs with angular momentum $J_{12} = 0$ can be obtained from Eq. (3.2).

$$\text{Number of pairs } (J_{12}=0) = \frac{2n}{(2j+1)^2} \times \left|\sum_J D(JJv)[j^{n-1}(jv=1)j|\}j^n Jv]\sqrt{(2J+1)}\right|^2. \tag{3.14}$$

By using the explicit form of the cfp's in Eqs. (3.9) and (3.10), we can obtain the following relation from Eq. (3.13). For $T = T_{\min}$,

$$D(00) = \frac{n}{(2j+1)}\sum_J D(JJv)[j^{n-1}(jv=1)j|\}j^n J] \times \sqrt{(2J+1)}. \tag{3.15}$$

This fits in very nicely into Eq. (3.2) to give us one of our main results,

$$\text{Case 1: } T = T_{\min} \quad \text{number of } np \text{ pairs } (J_{12}=0) = \frac{2|D(00)|^2}{n}, \tag{3.16}$$

i.e., $|D(00)|^2$, $|D(00)|^2/2$, and $|D(00)|^2/3$ for ^{44}Ti, ^{46}Ti, and ^{48}Ti, respectively. For $T = T_{\max}$, we use the second part of Eq. (3.13) to obtain

$$\text{Case 2: } T = T_{\max} \quad \text{number of } np \text{ pairs } (J_{12}=0) = 2n|D(00)|^2 = \frac{2n(2j+1-n)}{(2j+1)(n+1)}. \tag{3.17}$$

Note that for ^{48}Ti ($T = 4$), the number of all (nn, pp, and np) $J = 0$ pairs is 1, the same as for ^{48}Ca. A similar problem of counting the pairs in a single j shell was earlier addressed in Ref. [9]. Before closing this section, we would like to compare the results obtained in the present paper with those in [9].

In 1996 Engel, Langanke, and Vogel [9] worked out the number of J_A pairs in a single j shell for an isovector ($J_A = 0, T = 1$) pairing interaction. Their main results for nuclei with ground state with isospin $T = |T_3| = |(N-Z)/2|$ are

$$\begin{aligned}\langle\mathcal{N}_{np}\rangle &= \frac{\mathcal{N}-T_3}{2T_3+3}\left(1-\frac{\mathcal{N}-T_3-3}{2\Omega}\right),\\ \langle\mathcal{N}_{pp}\rangle &= (T_3+1)\langle\mathcal{N}_{np}\rangle,\\ \langle\mathcal{N}_{nn}\rangle &= \langle\mathcal{N}_{pp}\rangle + T_3\left(1-\frac{\mathcal{N}-1}{\Omega}\right).\end{aligned} \tag{3.18}$$

In (3.18), $\mathcal{N}$ is half of the total number of particles [in our notation, $\mathcal{N} = (n+2)/2$] and $\Omega = (2j+1)/2$. Their result [9] is exact for the ($J_{12} = 0$, $T = 1$) pairing interactions, and the formulas clearly show what whould happen if we dropped terms in $1/\Omega$. However, we obtain results for any isospin conserving interaction. For ^{44}Ti we obtain simple results for any even-J_{12} pair, not just for $J_{12}=0$. For ^{46}Ti and ^{48}Ti, we obtain simple results for $J_{12}=0$ only. We can show that our results, where they overlap, agree with those of Engel *et al.* [9]. Indeed, as can be seen in our Table I, the value of $D(J_{12}\,J_{12})$ for $J_{12}=0$ is 0.866 for the isovector pairing interaction. This number is more precisely $\sqrt{3/4}$. Thus we claim that the number of $J_{12}=0$ pairs with this interaction is 3/4. This is what their formula for $\langle\mathcal{N}_{np}\rangle$ yields for $N=Z$, $T=0$, and $\Omega=\frac{2j+1}{2}=4$. For $T=2$ the number of even-J_A pairs that we get is 1.5 for $J_{12}=0$ and $(2J_{12}+1)/6$ for $J_{12}=2$, 4, and 6. However, Ref. [9] does not give formulas for the $T=T_3+2$ case.

B. *nn* pairs

For the case of ^{46,48}Ti, the number of *nn* pairs with angular momentum J_B can be obtained by means of the following expression:

$$\sum_{J_A\,v_A}\sum_{J v v'} D^0(J\,J v)D^0(J\,J v')\frac{n(n-1)}{2}[j^{n-2}J_A\,v_A\,j^2J_B|\}j^n\,J v] \times [j^{n-2}J_A v_A\,j^2J_B|\}j^n\,J v']. \quad (3.19)$$

For $J_B=0$, we can get a simpler formula with the help of an equality that we can find in de Shalit and Talmi [7],

$$[j^{n-2}J v j^2 0|\}j^n\,J v]^2=\frac{(n-v)(2j+3-n-v)}{n(n-1)(2j+1)}, \quad (3.20)$$

with $2j+1-v\geqslant n\geqslant v+2$. Thus,

$$\text{number of } nn \text{ pairs } (J_B=0)=\frac{1}{2(2j+1)} \times\sum_{J v}|D^0(J\,J v)|^2(n-v) \times(2j+3-n-v). \quad (3.21)$$

So, for ^{46}Ti, we get

$$\text{number of } nn \text{ pairs } =\frac{1}{2}\left\{3|D^0(00)|^2 +\sum_{J=2,4,6}|D^0(J\,J v=2)|^2\right\}. \quad (3.22)$$

For ^{48}Ti, because of the normalization of the wave function, we get a simpler expression:

$$\text{number of } nn \text{ pairs } =|D^0(00)|^2+0.5. \quad (3.23)$$

And for ^{44}Ti, we already know the number of *nn* pairs coupled to $J=0$ from Eq. (2.8):

$$\text{number of } nn \text{ pairs } =|D^{I=0}(00)|^2. \quad (3.24)$$

C. *pp* pairs

For all three cases (^{44,46,48}Ti), the expression for the number of *pp* pairs coupled to angular momentum J_B is the same: $|D^{I=0}(J_B\,J_B)|^2$. Thus, for $J=0$ pairs, we have

$$\text{number of } pp \text{ pairs } (J=0)=|D^{I=0}(00)|^2. \quad (3.25)$$

IV. UNIFYING THE APPROACHES OF SEC. II, SEC. III, AND OTHER WORKS

In Sec. II we have in Eq. (2.4) an eigenvalue equation for the $D(J\,J)$'s where the linear operator is a nine-j symbol (or actually a six-j symbol), whereas in Sec. III we have somewhat similar-looking equations derived from the fact that $T_{\min}$ and $T_{\max}$ states are orthogonal, but the linear operator is a coefficient of fractional parentage.

We can unify these two approaches by using a recursion relation for cfp's derived by Redmond [10]:

$$m[j^{m-1}(\alpha_0 J_0)j|\}j^m\,J][j^{m-1}(\alpha J)j|\}j^m\,J] =\delta_{\alpha\alpha_0}\delta_{J J_0}+(m-1)\sqrt{(2J_0+1)(2J+1)} \times\sum_{J_2\alpha_2}\begin{Bmatrix}J_2 & j & J_1\\ J & j & J_0\end{Bmatrix}(-1)^{J_0+J_1}[j^{m-2}(\alpha_2 J_2)j|\}j^{m-1}\alpha_0 J_0] \times[j^{m-2}(\alpha_2 J_2)j|\}j^{m-1}\alpha_1 J_1]. \quad (4.1)$$

In particular, for $m=3$ we obtain the following expression for the unitary six-j symbol:

$$\sqrt{(2J+1)(2J'+1)}\begin{Bmatrix}j & j & J'\\ j & j & J\end{Bmatrix} =-\frac{\delta_{J J'}}{2}+\frac{3}{2}[j^2(J)j|\}j^3\,j v=1] \times[j^2(J')j|\}j^3\,j v=1]. \quad (4.2)$$

From the fact that the eigenvalues obtained from the orthogonality of $T_{\min}$ and $T_{\max}$ states and the orthonormality of $T_{\max}$ states as seen in Eqs. (3.5) and (3.6) are 0 and 1, we see that the eigenvalues of the unitary six-j symbol are $-1/2$ and 1. The latter results were, of course, shown in Sec. II.

It is very interesting to note that Rosensteel and Rowe [11] also found the need to diagonalize the above unitary six-j symbol [left-hand side of Eq. (4.2)]. They were addressing a problem different from ours. Whereas we are dealing with a system of both neutrons and protons, they were considering only particles of one kind, i.e., only neutrons. Whereas we are addressing the number of *np* pairs of a given angular momentum, they were addressing the problem of the number of seniority conserving interactions in a single j shell. But the diagonalization of the same unitary six-j comes into play in both problems.

V. CLOSING REMARKS

In this work, we studied the effects of the nucleon-nucleon interaction on the number of pairs of a given angular momentum in ^{44}Ti. We found that, as expected, the more attractive the nucleon-nucleon interaction is in a state with angular momentum J, the more pairs of that given J will be found in ^{44}Ti. As a basis of comparison, we defined the no-interaction case in which we averaged over all three $T = 0$ states (in the single-j-shell approximation) in ^{44}Ti. Even in this case, the number of J pairs is not independent of J, and there are, for example, less $J = 1$ pairs than the average (0.432 vs. $6/8 = 0.75$). When the realistic interaction is turned on, one gets, relative to this no-interaction case, an increase in the number of $J = 0, 1, 2$, and 7 pairs and a decrease in the others. This is in accord with the fact that in ^{42}Sc the states with angular momentum $J = 0, 1, 2$, and 7 are lower than the others.

The number of pairs obviously is relevant to two-nucleon transfer experiments, and we plan to address this more explicitly in the near future. For example, the pickup of an *np* pair in ^{44}Ti to the $J = 1$ state in ^{42}Sc will be enhanced, relative to the no-interaction case, by a factor of $(0.675/0.432)^2$. Although ^{44}Ti is radioactive, such experiments, as well as transfer reactions, are feasible. Indeed, there are approved proposals by a Berkeley group to perform the reaction ^{3}He(^{44}Ti, p)^{46}V.

In the course of performing this work, we found a relationship between the wave function coefficients $D(J\,J v)$ for the even-even Ti isotopes by exploiting the fact that, in the single j shell, the unique state with isospin $T_{\max}$ is orthogonal to all the states with isospin $T_{\min}$ [Eqs. (3.5) and (3.6)]. This enabled us to greatly simplify the expression for the number of *np* pairs with angular momentum $J_{12} = 0$ in the even-even Ti isotopes.

In the near future, we intend to extend this analysis for a single shell with large angular momentum and an arbitrary number $Z = N$ of protons and neutrons. This would allow us to study the competition between the $T = 1$ and 0 pairing interactions in the structure of the ground state in the medium mass nuclei.

The competition betwen $T = 0$ and 1 pairs in the ground state might be best studied with a particle and isospin projected generalized BCS function derived by two of us (A.A.R. and E.M.G.) in Ref. [12]. This project is under way, and we hope to have then a more fair comparison between various approaches. However, based on the present results, we may say that while in the ground state the number of $T = 1$ and 0 pairs are equal to each other, in the $T = 2$ state the number of $T = 1$ pairs prevails over that of the $T = 0$ pairs, which is equal to zero. Thus, in the present case the change in the wave function from a mixed state of $T = 0$ and 1 pairs to a pure state of $T = 1$ pairs is achieved by exciting the system from a $T = 0$ ground state to a $T = 2$ state.

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I. APPENDIX: THE NUMBER OF PAIRS OF A GIVEN ANGULAR MOMENTUM IN THE SINGLE-j SHELL IN ^{44}Ti

In the single-j shell, there are eight two-body interaction matrix elements

$$E(J) = \big\langle (f_{7/2})^2_J \big| V \big| (f_{7/2})^2_J \big\rangle, \tag{A1}$$

$J = 0, 1, \ldots, 7$. For even J, the isospin is $T = 1$; for odd J, $T = 0$. The energy of a ^{44}Ti state can be written as $\langle \psi H \psi \rangle$. This can also be written as a linear combination of the eight two-body matrix elements $E(J)$

$$E(^{44}\mathrm{Ti}) = \sum_{J=0}^{7} C_J E(J). \tag{A2}$$

We can identify C_J as the number of pairs in ^{44}Ti with a given angular momentum J,

$$\langle \psi H \psi \rangle = \sum D(J'_P J'_N) D(J_P J_N) \langle [J'_P\, J'_N]^I H [J_P\, J_N]^I \rangle, \tag{A3}$$

$$\begin{aligned} &\langle [J'_P J'_N]^I H [J_P J_N]^I \rangle \\ &\quad = [E(J_P) + E(J_N)]\, \delta_{J_P J'_P} \delta_{J_N J'_N} \\ &\qquad + 4 \sum_{J_A J_B} \langle (j^2) J'_P (j^2) J'_N | (j^2) J_A (j^2) J_B \rangle^I \\ &\qquad \times \langle (j^2) J_P (j^2) J_N | (j^2) J_A (j^2) J_B \rangle^I E(J_B). \end{aligned} \tag{A4}$$

In the above, the first two terms are the *pp* and *nn* interactions and the last one is the *np* interaction. The factor of 4 is used because there are 4 *np* pairs. The unitary nine-j symbol recombines a proton and a neutron. Note that J_P and J_N are even, but J_A and J_B can be even or odd.

By identifying the coefficient of $E(J_B)$ as the number of pairs with angular momentum J_B, we get the expression for $I = 0$ (for which $J_P = J_N$)

$$\begin{aligned} C_{J_B} &= \text{number of } J_B \text{ pairs} \\ &= 2[D(J_B J_B)]^2 \delta_{J_B,\,\mathrm{even}} \\ &\quad + 4 \sum_{J_P J_N} D(J_P J_N) \langle (j^2) J_P (j^2) J_N | (j^2) J_B (j^2) J_B \rangle^0 \\ &\quad \times \sum_{J'_P J'_N} D(J'_P J'_N) \langle (j^2) J'_P (j^2) J'_N | (j^2) J_B (j^2) J_B \rangle^0. \end{aligned} \tag{A5}$$

We can rewrite this as

$$\text{number of } J_B \text{ pairs } = 2\,[D(J_B J_B)]^2\, \delta_{J_B,\,\mathrm{even}} + |f(J_B)|^2\,, \tag{A6}$$

with

$$f(J_B) = 2 \sum_{J_P J_N} D(J_P J_N) \langle (j^2) J_P (j^2) J_N | (j^2) J_B (j^2) J_B \rangle^0, \tag{A7}$$

and

$$\begin{aligned} &\langle (j_1 j_2) J_P (j_3 j_4) J_N | (j_1 j_3) J_A (j_2 j_4) J_B \rangle^I \\ &\quad = \sqrt{(2J_P + 1)(2J_N + 1)(2J_A + 1)(2J_B + 1)} \\ &\quad \times \begin{Bmatrix} j_1 & j_2 & J_P \\ j_3 & j_4 & J_N \\ J_A & J_B & I \end{Bmatrix}. \end{aligned} \tag{A8}$$

We can get the number of pairs for a given total isospin T by using a simple interaction $a + bt(1) \cdot t(2)$, where a and b are constants. The value of this interaction for two particles is

$$\begin{aligned} a - 3b/4 \quad &\text{for} \quad T_{12} = 0, \quad \text{and} \\ a + b/4 \quad &\text{for} \quad T_{12} = 1. \end{aligned} \tag{A9}$$

For A valence nucleons ($A = n + 2$ for the Ti isotopes), we have

$$\begin{aligned} \sum_{i<j} (a + bt(i) \cdot t(j)) &= \frac{a}{2} A(A-1) + \frac{b}{2} \sum_{i,j} t(i) \cdot t(j) \\ &\quad - \frac{b}{2} \sum_i t(i)^2 \\ &= \frac{a}{2} A(A-1) + \frac{b}{2} T(T+1) - \frac{3}{8} Ab. \end{aligned} \tag{A10}$$

We can write this as

$$A_0(a - 3b/4) + A_1(a + b/4), \tag{A11}$$

and identify $A_{T_{12}}$ as the number of pairs with isospin T_{12}. We then get the result of Sec. II:

$$A_0 = \frac{A^2}{8} + \frac{A}{4} - \frac{T(T+1)}{2}, \tag{A12}$$

$$A_1 = \frac{3A^2}{8} - \frac{3A}{4} + \frac{T(T+1)}{2}. \tag{A13}$$

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