Renormalization of *NN* scattering: Contact potential

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The renormalization of the T matrix for NN scattering with a contact potential is reexamined in a nonperturbative regime through rigorous nonperturbative solutions. Based on the underlying theory, it is shown that the ultraviolet divergences in the nonperturbative solutions of the T matrix should be subtracted through "endogenous" counterterms, which in turn leads to a nontrivial prescription dependence. Moreover, employing the effective range expansion, the importance of imposing physical boundary conditions to remove the nontrivial prescription dependence, especially before making any physical claims, is discussed and highlighted. As by-products, some relations between the effective range expansion parameters are derived. We also discuss the power counting of the couplings for the nucleon-nucleon interactions and other subtle points related to the EFT framework beyond perturbative treatment.

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I. INTRODUCTION

The effective field theory (EFT) method or strategy [1] has become a primary tool for studying a variety of low-energy problems in particle and nuclear physics; important examples include chiral perturbation theory (χPT) [2], heavy quark effective theory (HQET) [3], and nonrelativistic quantum chromodynamics (NRQCD) [4]. As an EFT parametrizes the short distance physics in a simple way, severe ultraviolet (UV) divergences appear. Then one must carefully work out pertinent power counting rules and renormalization prescriptions [5]. In nonperturbative regimes (for example, in the application of χPT to low-energy nucleon systems, as is advocated by Weinberg [6]), the "interplay" between power counting schemes and renormalization prescriptions becomes quite complicated [7]. To establish a more reasonable and consistent framework, many proposals have been put forward [7–10], creating controversies still to be settled. There were once (and perhaps still are) even doubts about the applicability of the EFT method.

The main difficulties stem from a distinct feature of the nonperturbative formulation, which invalidates the naive use of the perturbative renormalization (through subtraction) programs [11,12]. In this report, we continue our investigations of the renormalization of the EFT for nucleon-nucleon scattering in the nonperturbative regimes, which we started in Ref. [11]. We work here with a contact potential that allows us to rigorously obtain a closed form of the T matrix through the use of the Lippmann-Schwinger equation (LSE) [13]. In this way, many new features arising in the nonperturbative regime can be explicitly illustrated. In our approach, we utilize the underlying theory to understand the renormalization of an EFT. In this work we examine the renormalization prescription dependence of the on-shell T matrix together with the observables or parameters coming from the low energy theorems for nucleon-nucleon scattering. The latter are obtained through the effective range expansion [14]. This paper is organized as follows: In Sec. II, we sketch the nonperturbative parametrization of the T matrix proposed in Ref. [11] and its implications for nonperturbative renormalization. In Sec. III, we employ the algebraic method described in Ref. [13] to obtain a rigorous closed-form solution of the LSE in the case of contact nucleon-nucleon interactions. Then the regularization and renormalization of the T matrix in the nonperturbative regime are analyzed using the closed form solutions at various chiral orders. Both the prescription dependence and its removal from the observables or parameters obtained via the effective range expansion (low-energy theorems) are investigated in Sec. IV. In Sec. V, we study the interplay between the renormalization prescription and the power counting schemes for EFT, and the renormalization group evolution in the nonperturbative regime and the naturalness of the T matrix are also explored. Finally, Sec. VI contains our summary. Our main conclusion is that one should be aware of the nontrivial renormalization prescription dependence in the nonperturbative regime, with emphasis on physical boundaries.

II. A COMPACT PARAMETRIZATION

Let us start with a standard parametrization for the on-shell partial wave T matrix [7] (we consider the diagonal channels for simplicity),

$$T_{l;\mathrm{os}}(p) = -\frac{4\pi}{M}\frac{1}{p\cot\delta_l(p) - ip}, \tag{1}$$

with M and p being, respectively, the mass and on-shell momentum of a nucleon, and l denoting the angular momentum number. Literally, the potential could be systematically constructed or calculated using χPT [6] through counting the powers in terms of p^2/Λ^2 or m_π^2/Λ^2, with Λ being the high scale or upper limit for the EFT under consideration ($\Lambda \sim 500\,\mathrm{MeV}$). Then the off-shell T matrix for partial wave l could be found through the solution of LSE,

$$T_l(p', p; E) = V_l(p', p) + \int \frac{kdk^2}{(2\pi)^2} \times V_l(p', k)G_0(k; E^+)T_l(k, p; E), \tag{2}$$

$$G_0(k; E^+) \equiv \frac{1}{E^+ - k^2/M}, \tag{3}$$

where $E^+ \equiv E + i\epsilon$, with E being the center of mass energy. It should be noted that Eq. (2) is ill defined in terms of UV, as the potential V_l calculated using χPT is usually singular.

To highlight the nonperturbative features present in this problem, we previously proposed a tentative nonperturbative parametrization for the T matrix based on LSE [11],

$$\frac{1}{T_l(p, p'; E)} = \frac{1}{V_l(p, p')} - \mathcal{G}_l(p, p'; E), \tag{4}$$

$$\mathcal{G}_l(p, p'; E) \equiv \frac{\int \frac{kdk^2}{(2\pi)^2} V_l(p', k) G_0(k; E^+) T_l(k, p; E)}{V_l(p, p') T_l(p, p'; E)}, \tag{5}$$

where $\mathcal{G}_l$ carries the nonperturbative information of all the quantum processes (all the loop amplitudes in the field theoretic terminology) generated by V_l. We have already shown in [11] that the nonperturbative quantity $\mathcal{G}_l$ could not be renormalized through the introduction of "exogenous" counterterms in the potential.[1] That means $\mathcal{G}$, or equivalently T, is regularization and renormalization (R/R) prescription dependent. There can be only one renormalization prescription (contrary to perturbative treatment [15]) consistent with physical data or boundary conditions. In the following sections we demonstrate this point with rigorous solutions of LSE.

III. CONTACT POTENTIAL: RIGOROUS SOLUTIONS

The specific case of contact potential allows us to transform the integral equation into the algebraic one following Ref. [13]. In other words, the nonlocal pion exchange contributions to the potential are neglected. However, the main conclusions also remain qualitatively valid when pion exchanges are included.

A. Factorized LSE and its algebraic solutions

Next, to illustrate the role played by the nonperturbative features, we employ the $^1S_0$ channel. To see how the situation evolves with the chiral orders, we consider the solution of LSE with the following local potentials at three different chiral orders ($\Delta = 0, 2, 4$) (leading order, next-to-leading order, next-to-next-to-leading order):

$$\Delta = 0 : V_{(0)}^{^1S_0} = C_0; \tag{6}$$

$$\Delta = 2 : V_{(2)}^{^1S_0} = C_0 + C_2(p^2 + p'^2); \tag{7}$$

$$\Delta = 4 : V_{(4)}^{^1S_0} = C_0 + C_2(p^2 + p'^2) + \tilde{C}_4 p^2 p'^2 + C_4(p^4 + p'^4). \tag{8}$$

[1]We have introduced new terminology to distinguish between the counterterms introduced *before* the nonperturbative summation ("endogenous") and those introduced *after* the infinite summation ("exogenous").

Following Ref. [13], we "factorize" the potentials into matrices: $V = U^T \lambda U'$, with U and U' being column vectors, and λ being an $n \times n$ matrix. At next-to-next-to-leading order ($\Delta = 4$), they are

$$\lambda \equiv \begin{pmatrix} C_0 & C_2 & C_4 \\ C_2 & \tilde{C}_4 & 0 \\ C_4 & 0 & 0 \end{pmatrix}, \; U \equiv (1,\ p^2,\ p^4), \;\; U' \equiv (1,\ p'^2,\ p'^4); \tag{9}$$

whereas at next-to-leading order ($\Delta = 2$), they read

$$\lambda \equiv \begin{pmatrix} C_0 & C_2 \\ C_2 & 0 \end{pmatrix}, \quad U \equiv (1,\ p^2), \quad U' \equiv (1,\ p'^2). \tag{10}$$

The off-shell T matrix factorizes exactly in the same manner: $T = U^T \tau U'$, where τ is an $n \times n$ matrix. Generally, the coupling constants $[C_n]$ come from the chiral expansion of an underlying theory (say, QCD) for nucleons and pions in terms of p^2/Λ^2, and hence they scale like $C_{2n}/C_0 \sim \Lambda^{-2n}$ in the naive power counting scheme.

Using this notation, the LSE can be reduced to the algebraic equation [13]

$$\tau(E^+) = \lambda + \lambda \circ \mathcal{I}(E^+) \circ \tau(E^+), \tag{11}$$

with $\circ$ denoting the matrix multiplication. For the 3×3 case, the matrix $\mathcal{I}$ and the related parametrizations and definitions are listed in Appendix A. With this algebraic parametrization, all the ill-defined integrals can be isolated and parametrized in any regularization prescription. The solution to this algebraic equation is easy to obtain:

$$\tau(E^+) = [1 - \lambda \circ \mathcal{I}(E^+)]^{-1} \circ \lambda. \tag{12}$$

In a similar fashion, the solution of the T matrix can be obtained. For the three chiral orders considered so far, they read (on shell)

$$\Delta = 0 : \frac{1}{T_{\rm os}(p)} = \frac{1}{C_0} + J_0 + \frac{M}{4\pi} ip; \tag{13}$$

$$\Delta = 2 : \frac{1}{T_{\rm os}(p)} = \frac{(1 - C_2 J_3)^2}{C_0 + C_2^2 J_5 + C_2(2 - C_2 J_3)p^2} + J_0 + \frac{M}{4\pi} ip; \tag{14}$$

$$\Delta = 4 : \frac{1}{T_{\rm os}(p)} = \frac{N_0 + N_1 p^2 + N_2 p^4}{D_0 + D_1 p^2 + D_2 p^4 + D_3 p^6} + J_0 + \frac{M}{4\pi} ip, \tag{15}$$

with the ill-defined integrals $[J_n]$ and the coefficients $[N_n]$ and $[D_n]$ as defined in Appendix B. The next-to-leading order has been considered in Ref. [13], whereas the $\Delta = 4$ result has not been given before. We should note that J_0 always stands "alone" in the real part of the inverse on-shell T matrix. (A rigorous proof of this fact at any order is given in Appendix C.) Here we would like to stress the compact or closed form of the expressions for the T matrix in terms of the couplings and the integrals $[J_n]$. It is this crucial property that distinguishes the nonperturbative solutions from the perturbative ones and complicates the renormalization.

Obviously, the T matrix becomes more complicated as higher order interactions are included. Nevertheless, when the nonlocal pion exchanges are included, one can naturally anticipate that the nonperturbative solutions would still take *compact or closed* forms. Thus, we expect that our conclusions here will also hold in realistic potentials with nonlocal pion contributions, at least qualitatively.

B. Failure of "exogenous" counter-term renormalization

Now let us consider the renormalization of the T matrix. The leading order case is trivial; one can absorb the only divergence in J_0 into the inverse coupling, $1/C_0$, similar to the perturbative cases.

However, in the presence of higher order interactions, this operation may not work. For example, at next-to-leading order, in order to renormalize the on-shell T matrix in Eq. (14), or to make the fraction $\frac{(1-C_2J_3)^2}{C_0+C_2^2J_5+C_2(2-C_2J_3)p^2}+J_0$ finite, one should make each of the following compact functions finite at the same time:

$$\left(C_0+C_2^2J_5\right)/(1-C_2J_3)^2, \quad C_2(2-C_2J_3)/(1-C_2J_3)^2, \quad J_0. \tag{16}$$

Now it is clear that the main obstacle for performing "exogenous" subtraction for the T matrix is the compact or closed expressions in terms of $[C_n]$ and $[\bar{J}_n]$: it is hard to see how to make $(C_0+C_2^2J_5)/(1-C_2J_3)^2$, $C_2(2-C_2J_3)/(1-C_2J_3)^2$ and J_0 finite simultaneously, since each of them is a compact or closed expression given in terms of the two couplings, C_0, C_2, and the three divergent integrals, J_0, J_3, and J_5. The situation differs strikingly from the perturbative case where counterterms are introduced order by order with the higher order terms discarded, as no compact or "closed" form of expression is involved. Moreover, no matter what was done for $(C_0+C_2^2J_5)/(1-C_2J_3)^2$ and $C_2(2-C_2J_3)/(1-C_2J_3)^2$, one should make sure that J_0 stays "separately" finite at the same time.

C. Nonperturbative renormalization in EFT

It is known that an EFT is often established through certain reorganization of parts of a well defined underlying theory (UT, at least renormalizable). Unfortunately, such reorganization usually (1) brings about new UV divergences and (2) impedes the "exogenous" counterterms from working. To see the first point, consider the diagrams shown in Figs. 1 and 2, with the heavy meson exchange diagrams (with g and m_h being the coupling constant and the meson mass) underlying the ones with contact interactions. For convenience, let us introduce a projection operator $\breve{\mathcal{P}}_{\rm LE}$ to symbolize the influences of this heavy meson: the extraction of the EFT vertices or couplings from the UT diagrams.

At tree level (Fig. 1),

$$-i\bar{C}_0 \equiv \breve{\mathcal{P}}_{\rm LE}\Gamma^{(4)}_{\rm tree} = \breve{\mathcal{P}}_{\rm LE}\left\{\frac{-ig^2}{k^2-m_h^2}\right\} = i\frac{g^2}{m_h^2}, \tag{17}$$

no divergence appears. The complication comes at the loop diagram level. For example, in the case of the convergent box diagram in Fig. 2, if $\breve{\mathcal{P}}_{\rm LE}$ is applied after the loop integration ($\int\frac{d^4l}{(2\pi)^4}$) has been done (*the correct order*), one would get a well defined expansion in terms of $\frac{1}{m_h^2}$. When $\breve{\mathcal{P}}_{\rm LE}$ is applied before $\int\frac{d^4l}{(2\pi)^4}$ (*the incorrect order*), the divergent bubble diagram results. Thus, the new divergences generally arise from the incorrect order of computations, as the following commutator does not vanish identically:

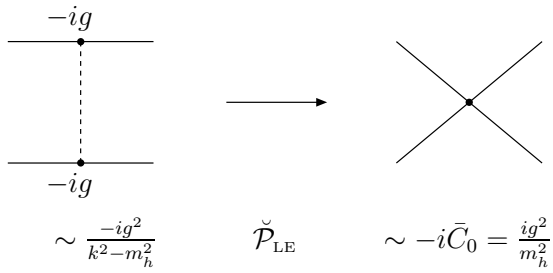

FIG. 1. Tree vertex for 4-nucleon in UT (left) and in EFT (right).

$$\hat{O}_{\rm c.t.} \equiv \left[\breve{\mathcal{P}}_{\rm LE}, \int\frac{d^4l}{(2\pi)^4}\right] \neq 0. \tag{18}$$

Embarrassingly, one has to use EFT either because UT is unavailable or because the calculations in UT are tedious. Combining this procedure with a nonperturbative context (infinite iteration or resummation) makes things even worse: the counterterms cannot be implemented "exogenously."

However, from the underlying theory the solution follows immediately: one should devise some procedures to effectively "recover" the correct order for $\breve{\mathcal{P}}_{\rm LE}$ and $\int\frac{d^4l}{(2\pi)^4}$ before anything else is done. The clue lies in Eq. (18). Through rearrangement, Eq. (18) is equivalent to the following equation for the integrand of a loop diagram (say, the box diagram integrand $f_{\rm box}$):

$$\begin{aligned}\breve{\mathcal{P}}_{\rm LE}\int\frac{d^4l}{(2\pi)^4}f_{\rm box} &= \int\frac{d^4l}{(2\pi)^4}\breve{\mathcal{P}}_{\rm LE}f_{\rm box} + \hat{O}_{\rm c.t.}f_{\rm box} \\ &= \int\frac{d^4l}{(2\pi)^4}f_{\rm bubble} + \hat{O}_{\rm c.t.}f_{\rm box}.\end{aligned} \tag{19}$$

That means that in order to recover the correct-order results in EFT, we must introduce a counterterm: $\hat{O}_{\rm c.t.}f_{\rm box}$. Therefore, the UT scenario provides a natural interpretation for the counterterms and, more importantly, a rationality for the subtraction at the level of the loop integral *without* any reference to a Lagrangian. That is, the counterterms must be endogenous: the divergent integrals must be subtracted *before*

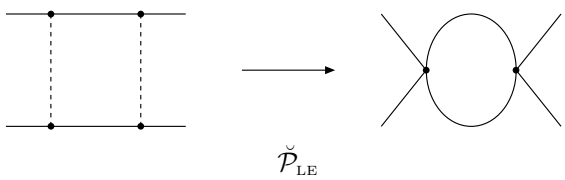

FIG. 2. Box diagram in UT (left) and bubble diagram in EFT (right).

the nonperturbative reorganization [11]. Finally, the subtracted integrals (finite) appear in the compact nonperturbative expressions, which are no longer compatible with exogenous counterterms. In this logic, the formal consistency issue [7] of the Weinberg power counting simply dissolves: in the nonperturbative regime of EFT, there is no point searching for exogenous counterterms and their counting rules. Of course, there might be other approaches that directly renormalize the integrals without the explicit use of counterterms.

Thus, the nonperturbative renormalization must be implemented through either endogenous counterterms or other means that effectively "subtract" the EFT integrals (see similar prescriptions in Ref. [16]). For the T matrix considered in this paper, this procedure would be formulated as a simple replacement of the divergent $[J_n]$ with the subtracted $[\bar{J}_n]$, which are finite constants (prescription dependent and hence arbitrary), in the compact expressions:

$$\Delta = 0 : \bar{T}_{\rm os}^{-1}(p) = \frac{1}{C_0} + \bar{J}_0 + \frac{M}{4\pi} ip; \tag{20}$$

$$\Delta = 2 : \bar{T}_{\rm os}^{-1}(p) = \frac{(1 - C_2\bar{J}_3)^2}{C_0 + C_2^2\bar{J}_5 + C_2(2 - C_2\bar{J}_3)p^2} + \bar{J}_0 + \frac{M}{4\pi} ip; \tag{21}$$

$$\Delta = 4 : \bar{T}_{\rm os}^{-1}(p) = \frac{\bar{N}_0 + \bar{N}_1 p^2 + \bar{N}_2 p^4}{\bar{D}_0 + \bar{D}_1 p^2 + \bar{D}_2 p^4 + \bar{D}_3 p^6} + \bar{J}_0 + \frac{M}{4\pi} ip. \tag{22}$$

Here we wish to note that from a UT perspective, both $[C_n]$ and $[\bar{J}_n]$ come from the projection acting on the convergent vertices in UT. (Those that are divergent in UT will be renormalized before applying the projection and do not directly contribute to EFT renormalization due to scale hierarchy. We return to this point in Sec. VI.) In this sense, $[\bar{J}_n]$ are also fundamental parameters in EFT, so the nonperturbative T matrix is parametrized by both $[C_n]$ and $[\bar{J}_n]$. To illustrate this point, let us apply the projection on the box diagram after the loop integration is carried out. After some calculations we get

$$\breve{\mathcal{P}}_{\rm LE}\Gamma^{(4)}_{\rm box}\big|_{\rm leadingterm} = -i\frac{g^4}{m_h^4} I_0^{\rm (UT)}(M, m_h, p) = i\bar{C}_0^2\left[J_0^{\rm (UT)}(M, m_h) + \frac{M}{4\pi} ip\right], \tag{23}$$

where the definite parameter $J_0^{\rm (UT)}(M, m_h)$ (see Appendix B), in place of the divergent integral J_0, can be extracted in the following way:

$$J_0^{\rm (UT)}(M, m_h) = -\left.\frac{{\rm Re}\big(i\breve{\mathcal{P}}_{\rm LE}\Gamma^{(4)}_{\rm box}\big)}{\big(i\breve{\mathcal{P}}_{\rm LE}\Gamma^{(4)}_{\rm tree}\big)^2}\right|_{p=0} = -\breve{\mathcal{P}}_{\rm LE}\left\{\frac{{\rm Re}\big(i\Gamma^{(4)}_{\rm box}\big)}{\big(i\Gamma^{(4)}_{\rm tree}\big)^2}\right\}\Bigg|_{p=0}. \tag{24}$$

We should note that here we used the mesonic interaction for illustration. Of course, the true contact nucleon interactions should be computed from QCD. But the mechanism explained above still holds in general.

Finally, we note that the renormalized T matrix suffers from severe prescription dependence in the nonperturbative regime, which is incompatible with the exogenous counterterms. That means, given specific couplings, only one prescription could yield the physical T matrix; others have to be dropped even though they are finite. So the final resolution boils down to the flexible regularization methods that could facilitate convenient access to physical predictions [11], as already noted in other nonperturbative contexts [17]. This argument leads us to the following strategy: One first parametrizes the ill-defined integrals in terms of ambiguous constants and then imposes physical boundary conditions. A similar strategy also based on the underlying theory, has already been described in Ref. [18] for renormalizing any EFT.

IV. LOW-ENERGY THEOREMS (LET) AND PRESCRIPTION DEPENDENCE

A. Effective range expansion

Now let us consider effective range expansion (ERE) defined as

$${\rm Re}\left\{-\frac{4\pi}{M}T_{\rm os}^{-1}(p)\right\} = p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}r_e p^2 + \sum_{k=2}^{\infty} v_k p^{2k}, \tag{25}$$

with the parameters a and r_e being the scattering length and the effective range, which (including $[v_k]$) could be extracted from the scattering data. In this sense, we could impose their values as the boundary conditions for the T matrix. Performing the expansion for the T matrix obtained above, we get

$$\Delta = 0 : p\cot\delta(p) = -\frac{4\pi}{M}\left\{C_0^{-1} + \bar{J}_0\right\}; \tag{26}$$

$$\Delta = 2 : p\cot\delta(p) = -\frac{4\pi}{M}\left\{\bar{\nu}_0\bar{\delta}_0^{-1} + \bar{J}_0 - \bar{\nu}_0\bar{\delta}_1\bar{\delta}_0^{-2}p^2 + \sum_{k=2}^{\infty}\bar{\nu}_0\bar{\delta}_1^k\bar{\delta}_0^{-k-1}(-p^2)^k\right\},$$

$$\bar{\nu}_0 \equiv (1 - C_2\bar{J}_3)^2, \quad \bar{\delta}_0 \equiv C_0 + C_2^2\bar{J}_5, \quad \bar{\delta}_1 \equiv C_2(2 - C_2\bar{J}_3); \tag{27}$$

$$\Delta = 4 : p\cot\delta(p) = -\frac{4\pi}{M}\left\{\bar{N}_0\bar{D}_0^{-1} + (\bar{N}_1\bar{D}_0 - \bar{N}_0\bar{D}_1)\bar{D}_0^{-2}p^2 + \left[\bar{N}_2\bar{D}_0^2 - \bar{N}_1\bar{D}_1\bar{D}_0 + \bar{N}_0\big(\bar{D}_1^2 - \bar{D}_0\bar{D}_2\big)\right]\bar{D}_0^{-3}p^4 + \left[\bar{N}_0\big(2\bar{D}_1\bar{D}_2\bar{D}_0 - \bar{D}_3\bar{D}_0^2 - \bar{D}_1^3\big) + \bar{N}_1\bar{D}_0\big(\bar{D}_1^2 - \bar{D}_0\bar{D}_2\big) - \bar{N}_2\bar{D}_1\bar{D}_0^2\right]\bar{D}_0^{-4}p^6 + \cdots\right\}. \tag{28}$$

The scattering length, effective range, and v_k can be read from the results above. For the three orders considered so far, we have

$$\Delta = 0 : a^{-1} = \frac{4\pi}{M}\big(C_0^{-1} + \bar{J}_0\big), \quad r_e = 0, \quad v_k = 0, k \geqslant 2; \tag{29}$$

$$\Delta = 2 : a^{-1} = \frac{4\pi}{M}\left(\bar{\nu}_0\bar{\delta}_0^{-1} + \bar{J}_0\right), \quad r_e = \frac{8\pi}{M}\bar{\nu}_0\bar{\delta}_1\bar{\delta}_0^{-2},$$
$$v_k = \frac{4\pi}{M}\bar{\nu}_0\bar{\delta}_1^k(-\bar{\delta}_0)^{-k-1}, k \geqslant 2; \tag{30}$$

$$\Delta = 4 : a^{-1} = \frac{4\pi}{M}\left(\bar{N}_0\bar{D}_0^{-1} + \bar{J}_0\right),$$
$$r_e = \frac{8\pi}{M}(\bar{N}_0\bar{D}_1 - \bar{N}_1\bar{D}_0)\bar{D}_0^{-2},$$

$$v_2 = \frac{4\pi}{M}\left[\bar{N}_0\left(\bar{D}_0\bar{D}_2 - \bar{D}_1^2\right) + \bar{N}_1\bar{D}_1\bar{D}_0 - \bar{N}_2\bar{D}_0^2\right]\bar{D}_0^{-3},$$
$$v_3 = \frac{4\pi}{M}\left[\bar{N}_0\left(\bar{D}_3\bar{D}_0^2 - 2\bar{D}_1\bar{D}_2\bar{D}_0 + \bar{D}_1^3\right)\right.$$
$$\left. - \bar{N}_1\bar{D}_0\left(\bar{D}_1^2 - \bar{D}_0\bar{D}_2\right) + \bar{N}_2\bar{D}_1\bar{D}_0^2\right]\bar{D}_0^{-4}, \tag{31}$$
$$\ldots.$$

Note again that, at each order, $\bar{J}_0$ only enters the expression for the scattering length but is "decoupled" with all the other ERE parameters. The reason is clear: $\bar{J}_0$ stands alone in T^{-1}. As we make clear below, this point has very important implications.

Keeping in mind that the parameters $[\bar{J}_n]$ are in principle independent of each other, two distinct approaches can be adopted in order to impose physical or reasonable boundary conditions: (1) taking the couplings and the prescription parameters as the fundamental variables, and the ERE parameters as the functions of these variables; or (2) conversely, taking some of the ERE parameters (which should be physical) as fundamental and the others as the functions of them. For convenience, we could also parametrize $[\bar{J}_n]$ in terms of a dimensional scale $\widetilde{\mu}$ and dimensionless numbers $[q_{\ldots}]$:

$$J_0 \equiv q_0 M\widetilde{\mu}, \quad J_3 \equiv q_3 M\widetilde{\mu}^3, \quad J_5 \equiv q_5 M\widetilde{\mu}^5,$$
$$J_7 \equiv q_7 M\widetilde{\mu}^7, \quad J_9 \equiv q_9 M\widetilde{\mu}^9. \tag{32}$$

The appearance of M is easy to see from Appendix A. Generally, the magnitude of $\widetilde{\mu}$ could vary from a value on the EFT expansion scale, Λ, to the value of the pion decay constant (much smaller than M): $\widetilde{\mu} \in (f_\pi, \Lambda)$. One can also alter the integrals by letting the dimensionless numbers $[q_n]$ vary. Thus, a nonperturbative renormalization prescription is parametrized by $[q_n; \widetilde{\mu}]$. However, the magnitude of $[\bar{J}_n]$, which also comes from the low energy projection in UT, should not be larger than the naive powers of the chiral symmetry breaking scale $\Lambda_{\chi\mathrm{SB}} \simeq M$. In other words, we can safely assume that $|\bar{J}_n| \leqslant M^{n+1}$, $n \neq 0$, $|\bar{J}_0| \leqslant M^2$.

Having made these preparations, we can start to examine the low-energy expansions listed above order by order in chiral expansion. The leading order case ($\Delta = 0$) is trivial: we have only one condition, i.e., imposing that $\frac{1}{C_0} + J_0 = a$, with a being experimentally measured, is enough since $r_e = v_k = 0, k \geqslant 2$, which is obviously a bad theoretical prediction, although it is not prescription dependent. Thus, the situation at this order is physically uninteresting, and most importantly, the distinctive nonperturbative features we wish to expose are not obvious here. Therefore, we examine in detail the higher order cases.

B. LET at next-to-leading order: $\Delta = 2$

Let us start with the next-to-leading order: $\Delta = 2$. As mentioned above, we discuss the problem from two perspectives.

1. First perspective

For convenience, let us list the explicit expressions of a, r_e, etc., in terms of $[C_n]$ and $[\bar{J}_n]$, which read

$$a = \frac{M}{4\pi}\frac{C_0 + C_2^2\bar{J}_5}{(1 - C_2\bar{J}_3)^2 + \bar{J}_0\left(C_0 + C_2^2\bar{J}_5\right)},$$
$$r_e = \frac{8\pi}{M}\frac{\left(2C_2 - C_2^2\bar{J}_3\right)(1 - C_2\bar{J}_3)^2}{\left(C_0 + C_2^2\bar{J}_5\right)^2},$$
$$v_k = \frac{4\pi}{M}(-)^{k+1}\frac{(1 - C_2J_3)^2C_2^k(2 - C_2\bar{J}_3)^k}{\left(C_0 + C_2^2\bar{J}_5\right)^{k+1}}, k \geqslant 2. \tag{33}$$

Before imposing reasonable boundary conditions, we could not make any physical predictions, as $\bar{J}_0$, $\bar{J}_3$, and $\bar{J}_5$ each independently varies. The independent variations of $\bar{J}_0$, $\bar{J}_3$, and $\bar{J}_5$ could not be easily absorbed into the couplings, for the reasons explained in Sec. III. Thus, unlike the leading-order case, we need (more) exogenous constraints to fix the values of $\bar{J}_0$, $\bar{J}_3$, and $\bar{J}_5$.

From Eq. (33), $\bar{J}_3$ and $\bar{J}_5$ could be solved in terms of r_e, v_2 and C_0, C_2; the solution might be unique after accounting for a reasonable size. After the insertion of the obtained numbers back into the formula for a, $\bar{J}_0$ could be expressed in terms of the physical value of the scattering length. In this sense, imposing two additional boundary conditions could fix the prescription or make the next-to-leading order result unambiguous. Now, the $[v_k]$ with $k \geqslant 3$ are taken to be theoretical predictions, which now shall be better than the leading-order ones, as we have more degrees of freedom to work with: $\bar{J}_3$ and $\bar{J}_5$, which come together with the new interactions. The fact, that some of the predictions are still poor can be attributed to the inadequacy of the next-to-leading-order potential: even higher order terms should be put in and accordingly more physical boundary conditions are needed. Thus, in spite of the fact, that the procedure for fixing the prescription becomes more nontrivial and laborious, the predictions for the ERE parameters improve when the higher order interactions are included (because of the "freedom" brought by the augmented interactions). We must repeat here that the predictions are made using the prescription that is most compatible with the physical boundary conditions.

Employing Eg. (32), we could also write the above equations as

$$a = \frac{M}{4\pi}\frac{C_0 + C_2^2q_5M\widetilde{\mu}^5}{(1 - C_2q_3M\widetilde{\mu}^3)^2 + q_0M\widetilde{\mu}\left(C_0 + C_2^2q_5M\widetilde{\mu}^5\right)},$$
$$r_e = \frac{8\pi}{M}\frac{\left(2C_2 - C_2^2q_3M\widetilde{\mu}^3\right)(1 - C_2q_3M\widetilde{\mu}^3)^2}{\left(C_0 + C_2^2q_5M\widetilde{\mu}^5\right)^2},$$
$$v_k = \frac{4\pi}{M}(-)^{k+1}\frac{(1-C_2q_3M\widetilde{\mu}^3)^2C_2^k(2-C_2q_3M\widetilde{\mu}^3)^k}{\left(C_0 + C_2^2q_5M\widetilde{\mu}^5\right)^{k+1}}, k \geqslant 2. \tag{34}$$

Note that even though the $\widetilde{\mu}$ scale dependence can be removed (fixed), the prescription dependence remains in terms of the dimensionless parameters $[q_n]$ that are independent of each other, a subtle point that often seems to be overlooked. For example, in a cutoff scheme, the renormalization is usually performed in such a way that the cutoff dependence is removed by letting the couplings develop a certain cutoff dependence. However, residual prescription dependence remains due to the cutoff-independent but prescription-dependent numbers $[q_n]$. Without fully appreciating this point, any fitting procedure that uses tuning of the couplings or even tuning of the cutoff only amounts to fitting along a special orbit in the space $[q_n]$, not in the whole space. The result thus obtained is still prescription dependent.

2. Second perspective

Taking a, r_e, and $\{v_2\}$ as the elementary parameters, we can express all the higher order constants $[v_k, k \geqslant 3]$ as

$$v_k = \frac{M - 4\pi a \bar{J}_0}{Ma}(-2v_2/r_e)^k, k \geqslant 3. \tag{35}$$

Note that from this perspective, the prescription-dependent parameter $\bar{J}_0$ seems to be an independent constant in addition to the three elementary parameters. At first sight, this ambiguity calls for one more condition: the value for v_3. But from the discussion above, we know that this seemingly independent parameter is in fact determined together with $\bar{J}_3$ and $\bar{J}_5$ by Eqs. (33). Of course, the nature of the problem remains the same even when it is taken as independent. The most striking point here is again that the prescription is "removed" through fixing, i.e., through boundary conditions, which is a nontrivial procedure, as articulated above.

One can also put Eq. (35) into the form that contains no explicit prescription dependence:

$$v_k = v_3(-2v_2/r_e)^{k-3}, k \geqslant 4. \tag{36}$$

This relation should hold for any problems (in certain atomic or molecular contexts) with contact potential such as $V(\mathbf{x}) \sim C_{(0)}\delta^{(3)}(\mathbf{x}) + C_{(2)}\nabla^2\delta^{(3)}(\mathbf{x})$.

So, we may conclude that, no matter what point of view one adopts, the key point is that the compact nonperturbative formulations make the prescription dependence and its removal a very nontrivial problem or procedure. However, the predictions also improve with the use of these formulations, in spite of their technical complexity. This nontrivial procedure becomes more involved as more higher order corrections to the potential are included. To verify this, let us turn to the next-to-next-to-leading order.

C. LET at next-to-next-to-leading order: $\Delta = 4$

Again we begin with the first perspective.

1. First perspective

Now we have four couplings $(C_0, C_2, C_4, \tilde{C}_4)$ and five prescription-dependent parameters $(\bar{J}_0, \bar{J}_3, \bar{J}_5, \bar{J}_7, \bar{J}_9)$ in eight compact expressions: $\bar{N}_0, \bar{N}_1, \bar{N}_2, \bar{D}_0, \bar{D}_1, \bar{D}_2, \bar{D}_3$, and $\bar{J}_0$, which stays alone. From Eq. (31) it is clear that we need at least five conditions to fix $\bar{J}_0, \bar{J}_3, \bar{J}_5, \bar{J}_7$, and $\bar{J}_9$, say, a, r_e, v_2, v_3, and v_4. But the compact expressions such as that for the scattering length, $a = \frac{M}{4\pi}\frac{\bar{D}_0}{\bar{N}_0 + \bar{J}_0\bar{D}_0}$ with $\bar{D}_0$ and $\bar{N}_0$ given in Appendix B, become more involved. This means that the boundary conditions might be more stringent for $\bar{J}_0$, $\bar{J}_3$, $\bar{J}_5$, $\bar{J}_7$, and $\bar{J}_9$ and the analytical work more difficult. In the meantime, the predictions for v_k at this order should be better than those at the leading and next-to-leading orders, as we have more parameters.

Here, some remarks are in order. At next-to-leading order, we ignored the possible multiple solutions for the fixing procedure. Here, with more compact expressions being involved, we should be more careful about this multiplicity of solutions. To this end, we note that the multiplicity can be effectively reduced with the limitations on the reasonable magnitudes of $[\bar{J}_n]$ together with the experimental values of the higher ERE parameters (say, $v_k, k \geqslant 3$). However, no matter how the multiplicity is removed, the solution is still an approximate (though nonperturbative) one: Eqs. (29)–(31) are obtained from a truncated potential and cannot be exact ones. Then, the theoretical predictions based on such equations will be less credible, especially for the ERE parameters that dominate higher and higher energy regions. In other words, the boundary conditions should be given by a procedure similar to fitting the shape of the phase shift within the corresponding ranges at each chiral order. This is actually what most authors have done, though the regularization schemes used vary significantly. Of course, our remarks in Sec. IV B 1 concerning the residual prescription dependence still apply for all the higher order calculations.

Mathematically, the multiplicity of solutions might be generic for nonperturbative renormalization because of the compact expressions involved. Therefore, the limit cycles encountered in the Schrödinger approach of renormalizing singular potentials [19] might just be examples of such multiplicity in certain regularization schemes.

2. Second perspective

To discuss the problem from the second perspective, we need to express everything in terms of the first five ERE parameters. Then the arguments go as in the preceding subsection. We shall not, however, repeat such complicated technical details here. One can also find relations such as Eq. (35) or (36), which hold true independently of the prescriptions, by repeatedly using the recursive relations

$$\tilde{v}_n = -\sum_{k=1}^{3}\frac{\bar{D}_k}{\bar{D}_0}\tilde{v}_{n-k}, n \geqslant 5; \quad \tilde{v}_3 = \frac{\bar{D}_3}{\bar{D}_0}\tilde{a}^{-1} - \frac{\bar{D}_2}{\bar{D}_0}\tilde{r}_e - \frac{\bar{D}_1}{\bar{D}_0}\tilde{v}_2; \tag{37}$$

$$\tilde{v}_4 = -\frac{\bar{D}_3}{\bar{D}_0}\tilde{r}_e - \frac{\bar{D}_2}{\bar{D}_0}\tilde{v}_2 - \frac{\bar{D}_1}{\bar{D}_0}\tilde{v}_3;$$

$$\tilde{a}^{-1} \equiv \frac{M}{4\pi a} - \bar{J}_0, \quad \tilde{r}_e \equiv \frac{M}{8\pi}r_e, \quad \tilde{v}_n \equiv \frac{M}{4\pi}v_n, n \geqslant 2, \tag{38}$$

where the coefficients $[\frac{\bar{D}_k}{\bar{D}_0}]$ can be solved in terms of M, a, r_e, v_2, v_3, and v_4.

D. Lessons from nonperturbative solutions

Now it is clear that things get more complicated as more higher order terms are included in the potential. Given this, we should not take the lower order results too seriously. For instance, the low energy theorems at leading order are too simple to be true in practice: $r_e = v_k = 0, k \geqslant 2$. This implies the necessity of including higher order terms, which, however, will bring us both favorable and unfavorable consequences. On one hand, more severe prescription dependence will appear and make this analysis more difficult. On the other hand, more prescription ambiguities also provide us with more chances to access the measured values of the ERE parameters. Although our calculations were done for the case of contact potentials, the core feature of our analysis—more ambiguities or more divergences at higher orders—holds true also for realistic potentials. At this stage, we shall mention that the freedoms in the prescription are in fact limited: $[\bar{J}_n]$ must satisfy certain requirements as presented in the discussion following Eq. (32). Moreover, the coupling constants should generally follow certain rules of EFT power counting. Then, after putting all these theoretical aspects into consideration, the EFT predictions must lie in a certain region of the "space" of observables.

Now we provide another way to see the virtue of the fitting procedure. Let us examine the variation of the functional form of the scattering length a in terms of the couplings and $[\bar{J}_n]$ for different chiral orders:

$$\Delta = 0: \quad a = \frac{M}{4\pi}\frac{C_0}{1 + C_0\bar{J}_0},$$

$$\Delta = 2: \quad a = \frac{M}{4\pi}\frac{C_0 + C_2^2\bar{J}_5}{(1 - C_2\bar{J}_3)^2 + \bar{J}_0\left(C_0 + C_2^2\bar{J}_5\right)},$$

$$\Delta = 4: \quad a = \frac{M}{4\pi}\frac{\bar{D}_0(C_0, C_2, C_4, \tilde{C}_4; \bar{J}_5, \bar{J}_7, \bar{J}_9)}{\bar{N}_0(C_0, C_2, C_4, \tilde{C}_4; \bar{J}_3, \bar{J}_5, \bar{J}_7, \bar{J}_9) + \bar{J}_0\bar{D}_0(C_0, C_2, C_4, \tilde{C}_4; \bar{J}_5, \bar{J}_7, \bar{J}_9)}.$$

It is obvious that the theoretical form of the scattering length varies with the chiral order quite significantly! So the scattering length calculated at lower orders should not be directly identified with the experimental value in order to accommodate the higher order terms. Thus, a more reasonable way to fix the renormalization prescription is to avoid the direct identification of physical parameters in order to accommodate higher order contributions in a consistent way. To this end, again, a procedure such as fitting the empirical curve over appropriate low energy regions might be more plausible.

V. POWER COUNTING AND RENORMALIZATION IN THE NONPERTURBATIVE REGIME

In all the discussions above, we have left out the power counting of the couplings. Since they constitute the basis for the EFT methods, it is necessary to see what the nonperturbative renormalization procedure described above means for the power counting rules. In fact, as was stressed in Ref. [11], the parametrization in the nonperturbative regime given in Eq. (1) implies that, in order for a power counting scheme for couplings to be meaningful, the corresponding prescription for the constants $[\bar{J}_n]$ must be appropriately chosen. Otherwise, one could not obtain the physical T matrix.

From the standpoint of UT, both $[C_n]$ and $[\bar{J}_n]$ come from the well defined low-energy projection ($\breve{\mathcal{P}}_{\rm LE}$) applied to UT amplitudes. So both $[C_n]$ and $[\bar{J}_n]$ serve as the elementary parameters for parametrizing the T matrix for the low-energy nucleon-nucleon scattering. In the EFT treatment without knowledge of the details from UT, we are forced to employ $[C_n]$ as the elementary couplings according to certain counting rules, while the constants $[\bar{J}_n]$ appear as the divergent pieces in the EFT loops constructed with the use of $[C_n]$. Thus, it is the EFT treatment that makes $[C_n]$ and $[\bar{J}_n]$ look disparate. In UT they are organized and derived together according to more elementary rules. Therefore, changing any of them (each single parameter in $[C_n] \cup [\bar{J}_n]$) alone would alter the physical behavior of T. Thus, they must be considered together.

One can also understand it from the Wilsonian definition of EFT through successive decimation of the higher scales, where different EFT expansion points lead to both different couplings and different $[J_n]$, as long as the expansions are compatible with the chiral power counting.

To be specific, the variations of $[C_n]$ and $[\bar{J}_n]$ (from now on $\bar{J}_0$ is excluded from $[\bar{J}_{\ldots}]$ for the reasons given below) must not alter the functional form (shape) of the T matrix:

$$\mathrm{Re}\left\{\frac{1}{T_{\rm os}(p)} - \bar{J}_0\right\} = \frac{\sum \bar{N}_i([C'_{\ldots}]; [\bar{J}'_{\ldots}])p^{2i}}{\sum \bar{D}_j([C'_{\ldots}]; [\bar{J}'_{\ldots}])p^{2j}} = \frac{\sum \bar{N}_i([C_{\ldots}]; [\bar{J}_{\ldots}])p^{2i}}{\sum \bar{D}_j([C_{\ldots}]; [\bar{J}_{\ldots}])p^{2j}} = \frac{\sum N_i^{(\rm phys)}p^{2i}}{\sum D_j^{(\rm phys)}p^{2j}}. \quad (39)$$

Here, we use the superscript "phys" to indicate that the parameters in the last fraction are physically determined, for example, from a genuine UT. To see why $\bar{J}_0$ is excluded from $[\bar{J}_n]$, consider the physical parametrization of the T matrix

(independent of the variations of $[C_{...}]$ and $[\bar{J}_n]$), which has the form

$$\frac{1}{T^{(\mathrm{phys})}} = \frac{\sum N_i^{(\mathrm{phys})} p^{2i}}{\sum D_j^{(\mathrm{phys})} p^{2j}} + M\gamma + \frac{M}{4\pi} ip, \tag{40}$$

where γ must be a physical scale, just like the nucleon mass M and the on-shell momentum p. Now, it is clear that $\bar{J}_0$ alone corresponds to the physical parameter $M\gamma$, which should therefore be independent of prescriptions. If this were not the case, or if $\bar{J}_0$ could vary with prescriptions, we would have to alter $\bar{N}_0$ and $\bar{D}_0$ to compensate for such variation in $\bar{J}_0$. Now, to keep the proportionality between $\bar{N}_0$, $\bar{D}_0$ and $\bar{N}_i p^{2i}$, $\bar{D}_j p^{2j}$ invariant, all the rest of $[\bar{N}_{...}, \bar{D}_{...}]$ must be accordingly altered, which in turn leads to an overall factor for $(\sum N_i^{(\mathrm{phys})} p^{2i})/(\sum D_j^{(\mathrm{phys})} p^{2j})$. Then, the functional dependence of the T matrix upon p would be altered, since its imaginary part, $\frac{M}{4\pi} ip$, remains intact. Hence, $\bar{J}_0$ must stay independent of prescriptions, i.e., physical. One could also verify this by examining the consequences of this procedure on the ERE parameters.

As p is arbitrary in the supposed range, the "invariance" discussed above leads to the following nontrivial equations for $[C_n]$ and $[\bar{J}_n]$ with the crucial presence of the physical parameters $[N_i^{(\mathrm{phys})}]$ and $[D_j^{(\mathrm{phys})}]$ for the on-shell T matrix,

$$\bar{N}_i([C_{...}]; [\bar{J}_{...}]) = N_i^{(\mathrm{phys})}, \quad \bar{D}_j([C_{...}]; [\bar{J}_{...}]) = D_j^{(\mathrm{phys})}, \quad \forall i, j. \tag{41}$$

These equations have dual implications: they can be used either (1) to fix the prescription ($[\bar{J}_{...}]$) in terms of the couplings ($[C_{...}]$) and the physical parameters ($[N_{...}^{(\mathrm{phys})}; D_{...}^{(\mathrm{phys})}]$) or conversely (2) to examine the influence of prescription upon the couplings with the help of the physical parameters. The first use parallels what we have done in Secs. IV B and C.

A. Interplay between power counting and prescription: next-to-leading order

Let us illustrate the interplay between power counting and prescription at next-to-leading order; that is, we try to solve the following equations for couplings:

$$\frac{C_0 + C_2^2 \bar{J}_5}{(1 - C_2 \bar{J}_3)^2} = \alpha_0 \equiv \frac{D_0^{(\mathrm{phys})}}{N_0^{(\mathrm{phys})}}, \quad \frac{2C_2 - C_2^2 \bar{J}_3}{(1 - C_2 \bar{J}_3)^2} = \alpha_2 \equiv \frac{D_2^{(\mathrm{phys})}}{N_0^{(\mathrm{phys})}}. \tag{42}$$

The solutions are easy to find as

$$C_2^{(\pm)} = \bar{J}_3^{-1} \left\{ 1 \pm (1 + \alpha_0 \bar{J}_3)^{-\frac{1}{2}} \right\}, \tag{43}$$

$$C_0^{(\pm)} = \frac{\alpha_0}{1 + \alpha_2 \bar{J}_3} - \frac{2\bar{J}_5}{\bar{J}_3^2} \left\{ 1 \pm (1 + \alpha_0 \bar{J}_3)^{-\frac{1}{2}} \right\}^2. \tag{44}$$

Taking into account the natural boundary condition for C_2: $C_2|_{J \to 0} \Longrightarrow \alpha_2/2$, we are left with the unique solution: $C_2^{(-)}$ (and $C_0^{(-)}$). Thus, assigning a power counting to C_0 and C_2 means assigning the sophisticated scaling for $\bar{J}_3$ and $\bar{J}_5$. Conversely, one can come up with an alternative interpretation: The power counting for the couplings can only be preserved in some particular prescription in order to obtain the expected physical behavior from the T matrix. Note that here we have deliberately not mentioned $\bar{J}_0$; it will be exclusively discussed below. Equations (41) or (42) now formalize our discussions concerning the interplay between power counting and prescription.

More interestingly, these equations have a further utility: they can be used to describe the evolution of the couplings in terms of a sliding scale (μ) in $[\bar{J}_n](= [q_n M \mu^n, n \neq 0])$. Since the exogenous counterterms are incompatible with the closed form of the T matrix, the conventional route to the evolution described by the renormalization group equation does not exist. But we can take the evolution implied by Eqs. (41) or (42) as a nonperturbative "renormalization group" evolution. We discuss this point in the next subsection.

B. Nonperturbative "renormalization group" (RG) evolution

To proceed, let us choose the prescription with $[\bar{J}_n \equiv q_n M \mu^n]$ to examine the evolution of the couplings enforced by (42). Let us assume that there exist enough boundary conditions to obtain the "physical" solutions for the couplings from the equations in (41):

$$C_i = F_i([N_{...}^{(\mathrm{phys})}, D_{...}^{(\mathrm{phys})}, M]; [q_{...}]; \mu), \forall i. \tag{45}$$

With such nonperturbative solutions, the complete evolution of the couplings is determined and both the IR and the UV fixed points can be identified. For example, at next-to-leading order, we have from Eqs. (44) and (43):

$$C_0(\alpha_0, \alpha_2, M, q_3, q_5; \mu) = \frac{\alpha_0}{1 + q_3 \alpha_2 M \mu^3} - \frac{2q_5}{q_3^2 M \mu} \left\{ 1 - (1 + q_3 \alpha_0 M \mu^3)^{-\frac{1}{2}} \right\}^2, \tag{46}$$

$$C_2(\alpha_0, \alpha_2, M, q_3, q_5; \mu) = (q_3 M \mu^3)^{-1} \left\{ 1 - (1 + q_3 \alpha_0 M \mu^3)^{-\frac{1}{2}} \right\}. \tag{47}$$

It is easy to see that they have both IR and UV fixed points:

$$\text{IR fixed point } (\mu \Rightarrow 0): \quad C_0^{(\mathrm{IR})} = \alpha_0, \quad C_2^{(\mathrm{IR})} = \alpha_2/2; \tag{48}$$

$$\text{UV fixed point } (\mu \Rightarrow \infty): \quad C_0^{(UV)} = C_2^{(UV)} = 0. \tag{49}$$

Note that the prescription dependence is obvious in Eq. (45) with the presence of $[q_{...}]$ but the UV and IR fixed points are prescription independent. While the IR fixed points are realistic, as the couplings were defined in the low-energy limit, the UV fixed points seem not to be realistic. But such UV behavior of the EFT couplings is compatible with the fact that the EFT couplings would be dominated by the UT couplings at high energy, and therefore "vanish." Of course we should bear in mind that what we obtained are only approximate answers, though nonperturbative.

Note that Eqs. (41), (45) contain the full dependence upon the prescription parameters. So one could also derive the equations a la Stückelberg and Petermann [20] that describe the laws for transitions from one prescription to another, which are not related by running the renormalization scale:

$$\frac{d}{d[\bar{J}_{...}]} \{\bar{N}_i, \bar{D}_j\} = 0, \forall i, j. \tag{50}$$

TABLE I. Determinant for naturalness/unnaturalness of T matrix.

	Natural $[C_{\cdots}]$	Unnatural $[C_{\cdots}]$
Natural $[\bar{J}_{\cdots}]$	Natural T matrix	Unnatural T matrix
Unnatural $[\bar{J}_{\cdots}]$	Unnatural T matrix	Natural/Unnatural T?

In terms of $[q_{\cdots};\mu]$, they become

$$\frac{d}{d[q_{\cdots}]}\{\bar{N}_i,\bar{D}_j\}=0,\forall i,j. \tag{51}$$

In the foregoing discussion, physical requirements are imposed on the functional shape of the T matrix. Alternatively, we could also employ the physically determined ERE parameters (scattering length, effective range, etc.), instead of $[N^{(\mathrm{phys})}_{\cdots};D^{(\mathrm{phys})}_{\cdots}]$, to solve the couplings in terms of $[\bar{J}_n]$. In principle, the two approaches should lead to the same evolution behavior, but the ERE approach is more involved than the shape approach as is clear from the comparison between Eqs. (30) and (32) and Eqs. (41).

C. Determinant for the natural or unnatural scattering length: $\bar{J}_0$

Now we discuss the determinant(s) of the size of physical parameters $[N^{(\mathrm{phys})}_i]$ and $[D^{(\mathrm{phys})}_j]$ or $a,r_e,v_k,\forall k\geqslant 2$. As argued above, a complete parametrization of the T matrix is given by $[C_{\cdots}]$ supplemented with $[\bar{J}_{\cdots}]$; then $[C_{\cdots}]$ and $[\bar{J}_{\cdots}]$ together determine whether the physical parameters are of natural size or not. We can have four rough scenarios, listed in Table I, where by a natural C_n we mean that the scale Λ in its parametrization $C_n\sim 1/(M\Lambda^{n+1})$ is of the size of the expansion scale (unnatural if $\Lambda\sim p,m_\pi$), whereas for $[\bar{J}_{\cdots}=q_{\cdots}M\mu^{\cdots}]$ the situation is reversed: the natural size of μ should be $\sim p,m_\pi$. A natural T matrix is parametrized by $[N^{(\mathrm{phys})}_i,D^{(\mathrm{phys})}_j]$ (or for $a,r_e,v_k,\forall k\geqslant 2$) such that the dimensional parameters are of the same magnitudes as the natural couplings.

Examining the concrete expressions of the T matrix, we find that whether the T matrix is natural or not is determined by both the sizes of the couplings and the magnitudes of the dimensionless combinations such as $\prod_{n,m}C_n^{\pm 1}\bar{J}_m$ ($\dim[\prod_{n,m}C_n^{\pm 1}\bar{J}_m]=0$). Now suppose we have natural couplings, i.e., $C_n\sim 1/(M\Lambda^{n+1}),\forall n$. If $[\bar{J}_{\cdots}]$ are also natural, we should have $|\prod_{n,m}C_n^{\pm 1}\bar{J}_m|\ll 1$ for all the dimensionless combinations. Then, given our experience at next-to-leading and next-to-next-to-leading orders, we can anticipate that

$$N_0^{(\mathrm{phys})}\sim 1,\ N_i^{(\mathrm{phys})}\sim\frac{1}{\Lambda^{2i}};\ D_j^{(\mathrm{phys})}\sim\frac{1}{M\Lambda^{j+1}}(\sim C_j),\forall i,j. \tag{52}$$

In this case, we obtain a natural T matrix, or ERE parameters $(a,r_e,v_k,\forall k\geqslant 2)$, of natural sizes. If $[\bar{J}_{\cdots}]$ are unnatural, then in general, we could have $|\prod_{n,m}C_n^{\pm 1}\bar{J}_m|\sim 1$ for the dimensionless combinations. Therefore, we have

$$N_0^{(\mathrm{phys})}\gg 1(\ll 1),\quad N_i^{(\mathrm{phys})}\gg(\ll)\frac{1}{\Lambda^{2i}};$$
$$D_j^{(\mathrm{phys})}\gg(\ll)\frac{1}{M\Lambda^{j+1}}(\sim C_j),\forall i,j. \tag{53}$$

In this case, we obtain an unnatural T matrix, or unnatural ERE parameters $(a,r_e,v_k,\forall k\geqslant 2)$ with natural couplings. For example, at next-to-leading order, we have

$$\begin{aligned}&\text{natural }[\bar{J}_{\cdots}]:\ |C_2\bar{J}_3|\ll 1,\left|C_2^2C_0^{-1}\bar{J}_5\right|\ll 1,\\ \Longrightarrow&\text{natural }T:\ \frac{(1-C_2\bar{J}_3)^2}{C_0+C_2^2\bar{J}_5+C_2(2-C_2\bar{J}_3)p^2}\\ &\qquad\simeq\frac{1}{C_0+2C_2p^2};\end{aligned} \tag{54}$$

$$\begin{aligned}&\text{unnatural }[\bar{J}_{\cdots}]:\ |C_2\bar{J}_3|\sim 1,\left|C_2^2C_0^{-1}\bar{J}_5\right|\sim 1,\\ \Longrightarrow&\text{unnatural }T:\ \frac{(1-C_2\bar{J}_3)^2}{C_0+C_2^2\bar{J}_5+C_2(2-C_2\bar{J}_3)p^2}\\ &\qquad=\frac{\zeta_1}{\zeta_2C_0+2\zeta_3C_2p^2},\end{aligned} \tag{55}$$

where each of the $\zeta_{\cdots}$ can be either fairly small or fairly large, and therefore the T matrix cannot be a natural one.

Now let us consider $\bar{J}_0$ and the scattering length in particular. As argued above, $\bar{J}_0$ should be viewed as an independent physical parameter, not as a common prescription parameter. From the parametrization of T and the formulas in the preceding sections, $\bar{J}_0$ will only contribute to the scattering length:

$$\begin{aligned}a^{-1}&=\frac{4\pi}{M}\left\{\bar{J}_0+\frac{\bar{N}_0([C_{\cdots},\bar{J}_{\cdots}])}{\bar{D}_0([C_{\cdots},\bar{J}_{\cdots}])}\right\},\\ r_e&=\frac{8\pi}{M}\left\{\frac{\bar{N}_0([C_{\cdots},\bar{J}_{\cdots}])\bar{D}_1([C_{\cdots},\bar{J}_{\cdots}])}{\bar{D}_0^2([C_{\cdots},\bar{J}_{\cdots}])}\right.\\ &\quad\left.-\frac{\bar{N}_1([C_{\cdots},\bar{J}_{\cdots}])}{\bar{D}_0([C_{\cdots},\bar{J}_{\cdots}])}\right\},\cdots\\ \Rightarrow&\ \frac{\partial a^{-1}}{\partial\bar{J}_0}=\frac{4\pi}{M},\ \frac{\partial r_e}{\partial\bar{J}_0}=\frac{\partial v_k}{\partial\bar{J}_0}=0,\forall k\geqslant 2.\end{aligned} \tag{56}$$

Now we can see that, even when both $[C_{\cdots}]$ and $[\bar{J}_{\cdots}]$ are of natural size, the scattering length could be unnaturally large once $\bar{J}_0$ is unnatural ($\sim M\Lambda$),

$$\text{natural }\bar{J}_0(\sim M\mu):a^{-1}\simeq-\mathcal{O}(\Lambda)+\mathcal{O}(\mu)\sim-\mathcal{O}(\Lambda); \tag{57}$$

$$\text{unnatural }\bar{J}_0(\sim M\Lambda):a^{-1}\simeq-\mathcal{O}(\Lambda)+\mathcal{O}(\Lambda)\sim-\mathcal{O}(\mu). \tag{58}$$

That is, in the ${}^1S_0$ channel, there theoretically exists a scenario such that the scattering length could be unnaturally large while all the rest ERE parameters are naturally sized. Then, the first situation in Table I should be amended as follows: even when all the couplings and all the rest of the $[\bar{J}_{\cdots}]$ are natural, we get an unnatural scattering length as long as $\bar{J}_0$ is unnatural.

For an unnatural power counting of the couplings, the discussion would be more difficult, and we refrain from exploring such situations here. As we have shown that both the natural and the unnatural physical parameters can be explained with the natural couplings (provided the nontrivial nonperturbative prescription dependence is fully explored), we feel that it is more reasonable to work with natural or conventional power counting of EFT couplings.

VI. DISCUSSION AND SUMMARY

Now it is time for us to address some theoretical aspects that have been omitted or not fully discussed so far. Let us start with the relation between the UT renormalization and the EFT renormalization involved in Sec. III C. Generally, in an EFT one deals with the new divergences in the diagrams that are induced by some low energy expansion (with the "wrong" order of operations; cf. Sec. III C) or a similar operation in UT, whereas the diagrams that need renormalization in UT are usually hidden in the EFT couplings. Note that the diagrams that renormalize UT dominate the quantum fluctuations at short distances, whereas the ones that are divergent in EFT dominate those at long distances. So the two renormalizations do not interfere with (affect) each other due to the large scale hierarchy between UT and EFT, i.e., they work at two widely separated scales. Thus, the renormalization in UT does not affect the renormalization in EFT. This supplements our remarks following Eq. (22) in Sec. III C.

Next, let us address the effect of the potential truncation on the nonperturbative renormalization group evolution. At any fixed chiral order, the nonperturbative evolution behaviors of the coupling of the highest chiral dimension should be less trustworthy. This is because once the next-order interactions are included, the coefficients for the term with the highest power of p would suffer the largest changes in the functional forms; the coefficients for the lower power terms receive smaller changes from the new couplings. That means that due to the truncation of the potential, the nonperturbative evolution behaviors of the EFT couplings with lower chiral dimensions should be more trustworthy than those with higher chiral dimensions. One can see this point by noting how the forms of $[N_i([C_{\cdots}];[\bar{J}_{\cdots}])$, $D_i([C_{\cdots}];[\bar{J}_{\cdots}])]$ (as functions of the couplings $[C_{\cdots}]$) change with the inclusion of higher order interactions.

In Sec. V C., we have shown that a natural (or conventional) chiral power counting of the EFT couplings does allow the T matrix to have unnatural parameters, or unnatural scattering length, etc. In particular, there is a possibility that only the scattering length is unnatural while the rest of the parameters are natural. This seems to be the realisty of the situation with the $^1S_0$ channel nucleon-nucleon scattering at low energy. This scenario is clearly different from the one discussed in the literature where unusual power counting of the couplings was employed [21]. Here, the key role is played by the nonperturbative renormalization prescription.

Although our conclusions have been reached with the use of contact interactions, we feel that they should remain qualitatively true even in a realistic situation because the crucial features of the nonperturbative renormalization remain unchanged: (1) More ill-defined pieces in the loop integrals appear at higher chiral orders; (2) The nonperturbative solution of the T matrix takes a closed form that can only be renormalized via endogenous counterterms. Alternatively, one could also take the rational function form as a Padé approximant to the realistic T matrix.

Now let us comment on the literature. In Ref. [22], a subtraction similar to the endogenous one described in the present paper is employed: the counterterm is introduced before the T matrix is calculated, a procedure that parallels the loop integrations. However, it is not clear if the subtraction described in some papers is equivalent to the endogenous one or not. For example, the subtraction procedure described in Ref. [23] does not appear to be an endogenous one. Thus, it may be flawed, as already noted in Ref. [24]. In Ref. [24], the whole investigation is done with the nonperturbative formulation (compact) of the T matrix, a positive aspect of this study. However, a special regularization (cutoff regularization) exclusively used in Ref. [24] unfortunately makes their analysis inevitably prescription dependent. In contrast, the strategy employed in Ref. [9] for parametrizing and fixing the nonperturbative renormalization prescription dependence is closer to the one used in the present paper. The importance of boundary conditions has already been stressed in Ref. [10], where the physical observables, such as phase shifts, were parametrized without involving explicit divergences.

Obviously, we just explored some convenient scenarios of the nonperturbative solutions. In our arguments, we have been unable to exclude many other possible scenarios. The only point in favor of the scenarios discussed in this paper is that they are relatively simple, whereas the other possibilities seem rather sophisticated, and they often use fine tuning or similar arguments.

In our opinion, a better way to work with the renormalization in the nonperturbative regime is to appreciate the presence of a well defined theory underlying an EFT, as illustrated in this paper. In this sense, the renormalization of singular potentials in quantum mechanics [25], or equivalently the self-adjoint extension of singular operators in Hilbert space, should also be embedded in the underlying theory background. This is plausible since quantum mechanics is an effective theory of quantum field theory.

In summary, we reconsidered the renormalization of the EFT for nucleon-nucleon scattering in the nonperturbative regime using contact potentials that facilitate rigorous solutions of LSE. Detailed analysis reveals that the T matrix in the nonperturbative regime should be renormalized through the endogenous counterterms whose net effects are to remove the divergences in the loop integrals, or through means that could yield the same results. The rationality for the subtractions at the loop integral level is naturally provided by the underlying theory, with the UV divergences being shown to come from the "incorrect" order of operations in the construction of EFT. Then, using the effective range expansion, we demonstrated that the nontrivial renormalization prescription dependence in the nonperturbative regime must be "removed" by imposing appropriate boundary conditions. We also argued that when imposing boundary conditions, the full "space" for renormalization prescriptions should be explored in order to be able to remove any residual prescription dependence. It is also important to impose the boundary conditions in such a way that higher order terms in the potential can be consistently incorporated. Finally, the nontrivial relation between the power counting of the couplings and the renormalization prescription was highlighted in the nonperturbative regime. As by-products, (1) the nonperturbative "renormalization group" evolution was described; and (2) the naturalness of the scattering length, etc., was shown to be compatible with the natural or conventional

power counting of the couplings because of the nontrivial prescription dependence. That is, the nontrivial prescription dependence becomes a virtue in such a case. Obviously, much work remains to be done.

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APPENDIX A

$$\mathcal{I}(E^+) \equiv \begin{pmatrix} \int \frac{d^3k}{(2\pi)^3}\frac{1}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^4}{E^+-\frac{k^2}{M}} \\ \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^4}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^6}{E^+-\frac{k^2}{M}} \\ \int \frac{d^3k}{(2\pi)^3}\frac{k^4}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^6}{E^+-\frac{k^2}{M}}, & \int \frac{d^3k}{(2\pi)^3}\frac{k^8}{E^+-\frac{k^2}{M}} \end{pmatrix}. \tag{A1}$$

The entries of this matrix can be parametrized as follows ($p \equiv \sqrt{ME}$):

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{E^+-\frac{k^2}{M}} \equiv I_0 = -J_0 - i\frac{Mp}{4\pi}; \tag{A2}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{E^+-\frac{k^2}{M}} \equiv -J_3 + I_0 p^2; \tag{A3}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^4}{E^+-\frac{k^2}{M}} \equiv -J_5 - J_3 p^2 + I_0 p^4; \tag{A4}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^6}{E^+-\frac{k^2}{M}} \equiv -J_7 - J_5 p^2 - J_3 p^4 + I_0 p^6; \tag{A5}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^8}{E^+-\frac{k^2}{M}} \equiv -J_9 - J_7 p^2 - J_5 p^4 - J_3 p^6 + I_0 p^8. \tag{A6}$$

Here, $\{J_n\}$ with $n = 0, 3, 5, 7, 9$ are regularization and renormalization prescription-dependent constants.

APPENDIX B

$$\begin{aligned}
N_0 &= (1 - C_2 J_3 - C_4 J_5)^2 - C_0\tilde{C}_4 J_3^2 - \tilde{C}_4 J_5 \\
&\quad + 2\tilde{C}_4 C_4 J_5^2 - \tilde{C}_4 C_4^2 J_5^3 - 2\tilde{C}_4 C_4 J_3 J_7 \\
&\quad - \tilde{C}_4 C_4^2 J_3^2 J_9 + 2\tilde{C}_4 C_4^2 J_3 J_5 J_7; \\
N_1 &= -2C_4 J_3 - \tilde{C}_4 J_3 + 2C_2 C_4 J_3^2 + 2\tilde{C}_4 C_4 J_3 J_5 \\
&\quad + 2C_4^2 J_3 J_5 - \tilde{C}_4 C_4^2 J_3 J_5^2 + \tilde{C}_4 C_4^2 J_3^2 J_7; \\
N_2 &= C_4^2 J_3^2; \\
D_0 &= C_0 + C_2^2 J_5 + C_4^2 J_9 - C_0\tilde{C}_4 J_5 + C_4^2\tilde{C}_4 J_7^2 \\
&\quad + 2C_2 C_4 J_7 - C_4^2\tilde{C}_4 J_5 J_9; \\
D_1 &= 2C_2 - C_2^2 J_3 + C_0\tilde{C}_4 J_3 + C_4^2 J_7 + 2C_4\tilde{C}_4 J_7 \\
&\quad - C_4^2\tilde{C}_4 J_5 J_7 + \tilde{C}_4 C_4^2 J_3 J_9; \\
D_2 &= 2C_4 + \tilde{C}_4 - 2C_2 C_4 J_3 - 2C_4\tilde{C}_4 J_5 - C_4^2 J_5 \\
&\quad + \tilde{C}_4 C_4^2 J_5^2 - C_4^2\tilde{C}_4 J_3 J_7; \\
D_3 &= -C_4^2 J_3.
\end{aligned} \tag{B1}$$

$$J_0^{(\mathrm{UT})}(M, m_h) = \frac{m_h^4}{(4\pi)^2 M^2}\int_0^1 dx \int_0^x dy \times \frac{(y+3-2x)^2 + 8(x-1)}{\left[(y+1-2x)^2 + y\frac{m_h^2}{M^2}\right]^2}. \tag{B2}$$

APPENDIX C

Consider the contact potential given at any chiral order. In the matrix form defined in Sec. III A, we have $V = U^T\lambda U$, $T = U^T\tau U$, with $U(p) \equiv (1, p^2, p^4, p^6, \ldots)$ being a column vector and U^T being the transposed vector. Then, the convolution in LSE could be factorized as $VG_0T = U^T(p)\lambda\mathcal{I}\tau U(q)$, with the matrix $\mathcal{I}$ being defined as follows:

$$\mathcal{I} \equiv \int \frac{kdk^2}{(2\pi)^2}\frac{U(k)U^T(k)}{E - k^2/M + i\epsilon}. \tag{C1}$$

The 3×3 case of $\mathcal{I}$ is given in Appendix A. It is easy to see that we could rewrite $\mathcal{I}$ as

$$\mathcal{I} = I_0 U(\sqrt{ME})U^T(\sqrt{ME}) + \tilde{\mathcal{I}}([J_m]; ME), m \neq 0, \tag{C2}$$

where I_0 and J_m with $m \neq 0$ are defined in Appendix A. Here, $\tilde{\mathcal{I}}$ is a real matrix independent of I_0. From Eq. (C2), it follows that

$$VG_0T = U^T\lambda(I_0 UU^T + \tilde{\mathcal{I}})\tau U = I_0 VT + U^T\lambda\tilde{\mathcal{I}}\tau U. \tag{C3}$$

Then, using the parametrization in (4), we find that, for on-shell momentum,

$$T^{-1} = V^{-1} - \mathcal{G} = V^{-1} - I_0 - \tilde{\mathcal{G}}, \tag{C4}$$

with $\tilde{\mathcal{G}} \equiv \frac{U^T\tilde{\lambda}\tilde{\mathcal{I}}\tau U}{VT} = \mathcal{G} - I_0$. Now comparing this with the following representation of T derived in Ref. [11] using the relation between the on-shell T matrix and the on-shell K matrix, $T^{-1} = K^{-1} + \frac{M}{4\pi}ip$, we find that

$$\tilde{\mathcal{G}} = V^{-1} - K^{-1} + J_0, \tag{C5}$$

that is, $\tilde{\mathcal{G}}$ must be a real number. But this real quantity is constructed with a complex T that contains the infinite iterations of the complex number I_0 as given in (C4). That means I_0 must cancel out in the infinite iteration and hence must disappear in the real quantity $\tilde{\mathcal{G}}$. This in turn implies that J_0, as the real part of I_0, does not appear in $\tilde{\mathcal{G}}$. Finally, these facts will lead to the form of the T matrix constructed with local potential,

$$\begin{aligned}
T^{-1} &= \frac{\sum N_i([C_{\ldots}], [J_{\ldots}])p^{2i}}{\sum D_j([C_{\ldots}], [J_{\ldots}])p^{2j}} - I_0 \\
&= \frac{\sum N_i([C_{\ldots}], [J_{\ldots}])p^{2i}}{\sum D_j([C_{\ldots}], [J_{\ldots}])p^{2j}} + J_0 + \frac{M}{4\pi}ip,
\end{aligned} \tag{C6}$$

with $[N_i, D_j]$ being independent of J_0. *QED*.

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