# Calculation of heavy ion $e^+e^-$ pair production to all orders in $Z\alpha$

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The heavy ion cross section for continuum $e^+e^-$ pair production has been calculated to all orders in $Z\alpha$. The formula resulting from an exact solution of the semiclassical Dirac equation in the ultrarelativistic limit is evaluated numerically. An energy-dependent spatial cutoff of the heavy ion potential is used, leading to an exact formula agreeing with the known perturbative formula in the ultrarelativistic, perturbative limit. Cross sections and sample momentum distributions are evaluated for heavy ion beams at SPS, RHIC, and LHC energies. $e^+e^-$ pair production probabilities are found to be reduced from perturbation theory with increasing charge of the colliding heavy ions and for all energy and momentum regions investigated.

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## I. INTRODUCTION

The calculation of heavy ion-induced continuum $e^+e^-$ pair production to all orders in $Z\alpha$ is of continuing interest because until now it has only been carried out for total cross sections and in a limiting approximation [1–3]. Recent progress on this topic began with the realization that in an appropriate gauge [4], the electromagnetic field of a relativistic heavy ion is to a very good approximation a $\delta$ function in the direction of motion of the heavy ion times the two-dimensional solution of Maxwell's equations in the transverse direction [5]. This realization led to an exact solution of the appropriate Dirac equation for excitation of bound-electron positron pairs and a predicted reduction from perturbation theory of a little less than 10% for Au + Au at RHIC [6]. This reduction can be identified as a Coulomb correction to bound-electron pair production.

It soon followed that an analytical solution of the Dirac equation was obtained independently and practically simultaneously by two different collaborations [7–9] for the analagous case of continuum $e^+e^-$ pair production induced by the corresponding countermoving delta function potentials produced by ultrarelativistic heavy ions in a collider such as RHIC. An extended discussion and reanalyis of this solution, with comments on early parallel work in the literature, shortly followed [10]. Baltz and McLerran [8] noted the apparent agreement of the obtained amplitude with that of perturbation theory even for large $Z$. Segev and Wells [9] further noted the perturbative scaling with $Z_1^2 Z_2^2$ seen in CERN SPS data [11]. These data were obtained from reactions of 160 GeV/nucleon Pb ions on C, Al, Pa, and Au targets as well as from 200 GeV/nucleon S ions on the same C, Al, Pa, and Au targets. The group presenting the CERN data, Vane *et al.*, stated their findings in summary: "Cross sections scale as the product of the squares of the projectile and target nuclear charges." Conversely, it is well known that photoproduction of $e^+e^-$ pairs on a heavy target shows a negative (Coulomb) correction proportional to $Z^2$ that is well described by the Bethe-Maximon theory [12].

Several authors subsequently argued [1–3] that a correct regularization of the exact Dirac equation amplitude should lead to a reduction of the total cross section for pair production from perturbation theory, the so-called Coulomb corrections. The first analysis was done in a Weizsacker-Williams approximation [1]. Subsequently, Lee and Milstein computed [2,3] the total cross section for $e^+e^-$ pairs using approximations to the exact amplitude that led to a higher order correction to the well-known Landau-Lifshitz expression [13]. In a previous article [14] I have tried to explicate the Lee and Milstein approximate results and argued their qualitative correctness.

In the present article I undertake the full numerical calculation of electromagnetically induced ultrarelativistic heavy ion electron-positron pair production. I utilize a cross section formula derived from the exact solution of the Dirac equation with an appropriate energy dependent cutoff of the transversely eikonalized potential employed.

## II. CROSS SECTIONS WITH HIGHER ORDER COULOMB CORRECTIONS

For production of continuum pairs in an ultrarelativistic heavy ion reaction one may work in a frame of two countermoving heavy ions with the same relativistic $\gamma$, and the electromagnetic interaction arising from them goes to the limit of two $\delta$ function potentials as follows:

$$V(\boldsymbol{\rho}, z, t) = \delta(z-t)(1-\alpha_z)\Lambda^-(\boldsymbol{\rho}) + \delta(z+t)(1+\alpha_z)\Lambda^+(\boldsymbol{\rho}) \tag{1}$$

where

$$\Lambda^\pm(\boldsymbol{\rho}) = -Z\alpha \ln \frac{(\boldsymbol{\rho} \pm \mathbf{b}/2)^2}{(b/2)^2}. \tag{2}$$

The previously derived semiclassical amplitude for electron-positron pair production [7–10] written in the notation of Lee and Milstein [2] takes the following form:

$$M(p,q) = \int \frac{d^2k}{(2\pi)^2} \exp[i\mathbf{k}\cdot\mathbf{b}]\mathcal{M}(\mathbf{k})F_B(\mathbf{k})F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}), \tag{3}$$

where $p$ and $q$ are the four-momenta of the produced electron and positron, respectively, $p_\pm = p_0 \pm p_z$, $q_\pm = q_0 \pm p_z$, $\gamma_\pm = \gamma_0 \pm \gamma_z$, and $\boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma}$; $\mathbf{k}$ is an intermediate transverse momentum transfer from the ion to be integrated over the following:

$$\mathcal{M}(\mathbf{k}) = \bar{u}(p)\frac{\boldsymbol{\alpha}\cdot(\mathbf{k}-\mathbf{p}_\perp)+\gamma_0 m}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2 + i\epsilon}\gamma_- u(-q) + \bar{u}(p)\frac{-\boldsymbol{\alpha}\cdot(\mathbf{k}-\mathbf{q}_\perp)+\gamma_0 m}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2 + i\epsilon}\gamma_+ u(-q); \tag{4}$$

and the effect of the potential described in Eqs. (1) and (2) is contained in integrals $F_B$ and $F_A$ over the following transverse spatial coordinates [7–10]:

$$F(\mathbf{k}) = \int d^2\rho\, \exp[-i\,\mathbf{k}\cdot\boldsymbol{\rho}]\{\exp[-i2Z\alpha \ln\rho]-1\}. \tag{5}$$

$F(\mathbf{k})$ has to be regularized or cut off at large $\rho$. How it is regularized is the key to understanding Coulomb corrections.

Although, as has been pointed out [15], the derived exact semiclassical Dirac amplitude is not simply the exact amplitude for the excitation of a particular (correlated) electron-positron pair, there are observables, such as the total pair production cross section, that can be constructed straightforwardly from this derived amplitude [16–19]. This point has a long history of discussion in the literature [20–23]. The exact amplitude for a correlated electron-positron pair will not be treated here. The point is that the exact solution of the semiclassical Dirac equation may be used to compute the inclusive average number of pairs—not an exclusive amplitude for a particular pair. Calculating the exact exclusive amplitude to all orders in $Z\alpha$ is not easily tractable because of the need for Feynman propagators [15]. The possibility of solutions of the semiclassical Dirac equation is connected to the retarded propagators involved. In this article we do not consider the exclusive (Feynman propagator) amplitude at all. We concentrate on observables that *can* be constructed from the above amplitude and investigate the Coulomb corrections contained in them.

In a previous article [14] I have discussed these matters in more detail. There the uncorrelated cross-section expressions for $d\sigma(p)$, $d\sigma(q)$, and $\sigma_T$ were presented as follows:

$$d\sigma(p) = \int \frac{m\,d^3q}{(2\pi)^3\epsilon_q}\int\frac{d^2k}{(2\pi)^2}|\mathcal{M}(\mathbf{k})|^2 \times |F_A(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k})|^2|F_B(\mathbf{k})|^2, \tag{6}$$

$$d\sigma(q) = \int \frac{m\,d^3p}{(2\pi)^3\epsilon_p}\int\frac{d^2k}{(2\pi)^2}|\mathcal{M}(\mathbf{k})|^2 \times |F_A(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k})|^2|F_B(\mathbf{k})|^2, \tag{7}$$

$$\sigma_T = \int \frac{m^2\,d^3p\,d^3q}{(2\pi)^6\epsilon_p\epsilon_q}\int\frac{d^2k}{(2\pi)^2}|\mathcal{M}(\mathbf{k})|^2 \times |F_A(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k})|^2|F_B(\mathbf{k})|^2. \tag{8}$$

$d\sigma(p)$ is the cross section for an electron of momentum $(p)$ where the state of the positron is unspecified. Likewise, $d\sigma(q)$ is the cross section for a positron of momentum $(q)$ with the state of the electron unspecified. Note that $\sigma_T$ corresponds to a peculiar type of inclusive cross section that we should call the "number weighted total cross section":

$$\sigma_T = \int d^2b\,N = \int d^2b\sum_{n=1}^{\infty} nP_n(b), \tag{9}$$

in contrast to the usual definition of an inclusive total cross section $\sigma_I$ for pair production,

$$\sigma_I = \int d^2b\sum_{n=1}^{\infty} P_n(b). \tag{10}$$

In Eqs. (6)–(8), it is assumed that the sums have been taken over the electron and positron polarizations in $|\mathcal{M}(\mathbf{k})|^2$. Taking traces with the aid of the computer program FORM [24] one obtains

$$|\mathcal{M}(\mathbf{k})|^2 = \frac{2p_+q_-[(\mathbf{k}-\mathbf{p}_\perp)^2+m^2]}{m^2[p_+q_-+(\mathbf{k}-\mathbf{p}_\perp)^2+m^2]^2} + \frac{2p_-q_+[(\mathbf{k}-\mathbf{q}_\perp)^2+m^2]}{m^2[p_-q_++(\mathbf{k}-\mathbf{q}_\perp)^2+m^2]^2} + \frac{4[\mathbf{k}\cdot\mathbf{p}_\perp q_+q_- + \mathbf{k}\cdot\mathbf{q}_\perp p_+p_- - 2\mathbf{k}\cdot\mathbf{p}_\perp\mathbf{k}\cdot\mathbf{q}_\perp + k^2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp - m^2) - p_+p_-q_+q_-]}{m^2[p_+q_-+(\mathbf{k}-\mathbf{p}_\perp)^2+m^2][p_-q_++(\mathbf{k}-\mathbf{q}_\perp)^2+m^2]}. \tag{11}$$

This expression exhibits the expected property that $|\mathcal{M}(k)|^2$ vanishes as $\mathbf{k}$ goes to zero; the positive squares of the direct and crossed amplitudes (terms one and two) are canceled by the negative product of direct and crossed amplitudes of term three. These background terms can be subtracted off analytically from the expression for $|\mathcal{M}(\mathbf{k})|^2$ to obtain an expression exhibiting only terms dependent on $\mathbf{k}$ in the numerators:

$$|\mathcal{M}(\mathbf{k})|^2 = \frac{2D_1^2\eta_{11} - 2A_{11}(2D_1\beta_1+\beta_1^2)}{m^2D_1^2(D_1+\beta_1)^2} + \frac{2D_2^2\eta_{22} - 2A_{22}(2D_2\beta_2+\beta_2^2)}{m^2D_2^2(D_2+\beta_2)^2} + 4\frac{D_1D_2\eta_{12} - A_{12}(D_2\beta_1 + D_1\beta_2 + \beta_1\beta_2)}{m^2D_1D_2(D_1+\beta_1)(D_2+\beta_2)} \tag{12}$$

where

$$A_{11} = p_+q_-({\mathbf{p}_\perp}^2+m^2) \qquad A_{22} = p_-q_+({\mathbf{q}_\perp}^2+m^2)$$
$$A_{12} = -({\mathbf{p}_\perp}^2+m^2)({\mathbf{q}_\perp}^2+m^2)$$
$$D_1 = p_+q_- + {\mathbf{p}_\perp}^2 + m^2 \qquad D_2 = p_-q_+ + {\mathbf{q}_\perp}^2 + m^2$$

$$\beta_1 = -2\mathbf{k}\cdot\mathbf{p}_\perp + k^2 \qquad \beta_2 = -2\mathbf{k}\cdot\mathbf{q}_\perp + k^2$$
$$\eta_{11} = p_+q_- - \beta_1 \qquad \eta_{22} = p_-q_+\beta_2$$
$$\eta_{12} = \mathbf{k}\cdot\mathbf{p}_\perp q_+q_- + \mathbf{k}\cdot\mathbf{q}_\perp p_+p_- - 2\mathbf{k}\cdot\mathbf{p}_\perp\mathbf{k}\cdot\mathbf{q}_\perp + k^2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp - m^2). \quad (13)$$

Every term in the numerators now has at least a linear dependence on $k$. This subtraction turned out to be necessary to limit roundoff error in calculations at the highest beam energies such as LHC.

If one merely regularizes the integral itself at large $\rho$ one obtains [8–10], apart from a trivial phase, the following:

$$F(\mathbf{k}) = \frac{4\pi\alpha Z}{k^{2-2i\alpha Z}}. \quad (14)$$

Then all the higher order $Z\alpha$ effects in $M(p,q)$ are contained only in the phase of the denominator of Eq. (14). Then, because the cross sections described in Eqs. (6)–(8) go as $|F(\mathbf{k})|^2$, the phase falls out of the problem and it directly follows that calculable observables are equal to perturbative results. However, in this approach a lower $k$ cutoff at some $\omega/\gamma$ has to be put in by hand to obtain dependence on the beam energy and to agree with the known perturbative result in that limit.

Our present strategy is to apply a spatial cutoff to the transverse potential $\chi(\boldsymbol{\rho})$ (which has been up to now set to $2Z\alpha\ln\rho$) to obtain an expression consistent with the perturbation theory formula [25,26] in the ultrarelativistic limit. Instead of regularizing the transverse integral itself [Eq. (5)] and letting the cutoff radius go to infinity, as was originally done [7–10], we will rather apply an appropriate physical cutoff to the interaction potential. In the Weizsacker-Williams or equivalent photon treatment of electromagnetic interactions the effect of the potential is cut off at impact parameter $b \simeq \gamma/\omega$, where $\gamma$ is the relativistic boost of the ion producing the photon and $\omega$ is the energy of the photon. If

$$\chi(\boldsymbol{\rho}) = \int_{-\infty}^{\infty} dz\, V\left(\sqrt{z^2+\rho^2}\right) \quad (15)$$

and $V(r)$ is cut off in such a physically motivated way, then [3]

$$V(r) = \frac{-Z\alpha\exp[-r\omega_{A,B}/\gamma]}{r} \quad (16)$$

where

$$\omega_A = \frac{p_+ + q_+}{2}; \quad \omega_B = \frac{p_- + q_-}{2} \quad (17)$$

with $\omega_A$ the energy of the photon from ion $A$ moving in the positive $z$ direction and $\omega_B$ the energy of the photon from ion $B$ moving in the negative $z$ direction. Note that we work in a different gauge than that used to obtain the original perturbation theory formula, and thus our potential picture is somewhat different. The transverse potential will be smoothly cut off at a distance where the the longitudinal potential $\delta$ function approximation is no longer valid.

The integral Eq. (15) can be carried out to obtain the following:

$$\chi(\rho) = -2Z\alpha K_0(\rho\omega_{A,B}/\gamma), \quad (18)$$

and Eq. (5) is modified to the following:

$$F_{A,B}(\mathbf{k}) = 2\pi\int d\rho\,\rho J_0(k\rho)\{\exp[2iZ_{A,B}\alpha K_0(\rho\omega_{A,B}/\gamma)]-1\}, \quad (19)$$

where $F_A(\mathbf{k})$ and $F_B(\mathbf{k})$ are functions of virtual photon $\omega_A$ and $\omega_B$, respectively. The modified Bessel function $K_0(\rho\omega/\gamma) = -\ln(\rho)$ plus constants for small $\rho$ and cuts off exponentially at $\rho \sim \gamma/\omega$. This is the physical cutoff to the transverse potential.

One may define $\xi = k\rho$ and rewrite Eq. (19) in terms of a normalized integral $I_{A,B}(\gamma k/\omega)$ as follows:

$$F_{A,B}(\mathbf{k}) = \frac{4\pi Z_{A,B}\alpha}{k^2} I_{A,B}(\gamma k/\omega) \quad (20)$$

where

$$I_{A,B}(\gamma k/\omega) = \frac{1}{2iZ_{A,B}\alpha}\int d\xi\,\xi J_0(\xi) \times\{\exp[2iZ_{A,B}\alpha K_0(\xi\omega/\gamma k)]-1\}. \quad (21)$$

It is now clear that $F_{A,B}$ is a function of $4\pi Z_{A,B}/k^2$ times a function of $(\gamma k/\omega)$. The limit as $Z\to 0$ of $I^0_{A,B}(\gamma k/\omega)$ is analytically soluble as follows:

$$I^0_{A,B}(\gamma k/\omega) = \frac{1}{1+\omega^2/k^2\gamma^2}, \quad (22)$$

and one has $F^0_{A,B}(\mathbf{k})$, the familiar perturbation theory form

$$F^0_{A,B}(\mathbf{k}) = \frac{4\pi Z_{A,B}\alpha}{k^2+\omega^2/\gamma^2}. \quad (23)$$

As I have shown in a previous article [14], one might use some other physical cutoff and still obtain the Lee-Milstein Coulomb correction as long as one was expanding the Coulomb cross-section correction only to lowest order in $k^2$. However, such an alternate physical cutoff would not lead to this correct perturbation theory form for $F^0_{A,B}(\mathbf{k})$ and would lead to modified results for the Coulomb corrections in a full integration over $\mathbf{k}$.

Figure 1 displays the results of numerical calculation of $|I(k\gamma/\omega)|^2$ for $Z = 82$ and in the perturbative limit. Note that the upper cutoff of $\rho$ at $\gamma/\omega$ has the effect of regularizing $F(\mathbf{k})$ at small $k$. $F(\mathbf{k})$ goes to the constant $4\pi\gamma^2/\omega^2$ as $k$ goes to zero in the perturbative case; it goes to a reduced constant value as $k$ goes to zero for $Z = 82$. The form of the original solution Eq. (14)

$$F(\mathbf{k}) = \frac{4\pi\alpha Z}{k^{2-2i\alpha Z}} \quad (24)$$

is simply wrong because it is unphysical. Because it lacks a proper physical cutoff in $\rho$, it not only blows up at $k = 0$ but also fails to exhibit the correct reduction in magnitude that occurs when $k\gamma/\omega$ is not too large.

It is clear from Fig. 1 that for large $Z = 82$ Coulomb corrections reduce $|F(\mathbf{k})|^2$ from the perturbative result for $k\gamma/\omega \ll 100$. Only for $k >\ \sim 100\ \omega/\gamma$ does the magnitude of $F(\mathbf{k})$ go over into the perturbative result.

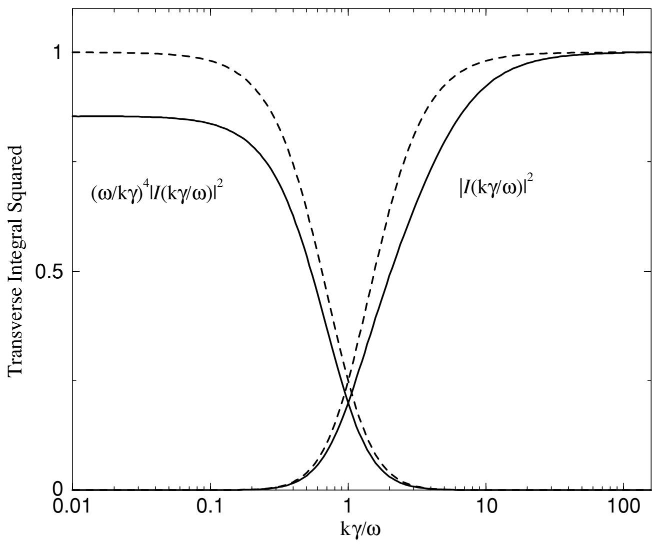


FIG. 1. The sold curve is the normalized integral squared for $Z = 82$. The dashed line is the corresponding perturbation theory result.

## III. CALCULATIONS: NUMERICAL TECHNIQUE AND RESULTS

The expression for the total cross section, Eq. (8), involves an eight-dimensional integral over the positron and electron momenta as well as the virtual photon transverse momentum. This integral reduces to seven dimensions in the usual way by symmetry, for example, let the positron transverse momentum define the $x$ axis. The usual method of evaluation, for example, in perturbation theory, is via Monte Carlo. However, I have chosen to do the seven-dimensional integral directly on meshes uniform on a logarithmic scale in each radial momentum dimension. It was possible to carry out the calculation without using Monte Carlo because the integrand is very smooth and smoothly goes to zero at both the high end and low end of the momentum ranges. No artificial cutoffs were applied.

Calculations labeled exact and perturbative differ only in the expressions used for $F_{A,B}(\mathbf{k})$. The analytical expression Eq. (23) is used for perturbative calculations. The exact calculations makes use the expression Eq. (19), which must be evaluated numerically, but only once for each $Z_{A,B}$ of interest.

Results of the numerical calculations will be compared with previously derived closed formulas for total cross sections. It is useful here to review those formulas. The Racah formula for the total $e^+e^-$ cross section in perturbation theory is [27] as follows:

$$\sigma_R = \frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\left[\frac{28}{27}\mathcal{L}^3 - \frac{178}{27}\mathcal{L}^2 + \frac{370+7\pi^2}{27}\mathcal{L} - \frac{116}{9} - \frac{13\pi^2}{54} + \frac{7}{9}\zeta(3)\right] \tag{25}$$

where

$$\mathcal{L} = \log\left[2\frac{P_1\cdot P_2}{M_1 M_2}\right] = [\log 2(2\gamma^2-1)], \tag{26}$$

the relativistic $\gamma$ is that of each colliding ion in an equal and opposite ion velocity frame, and the Riemann $\zeta$ function is as follows:

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1.2020569. \tag{27}$$

The $\log^3(\gamma^2)$ term is the same as the original Landau-Lifshitz formula [13], but the other additional terms are an improvement that allows this very early formula to attain a remarkable degree of accuracy as demonstrated by comparison with recent Monte Carlo evaluations.

The Lee-Milstein formula [3] includes higher order $\alpha Z$ effects in addition to the following $\log^3(\gamma^2)$ term:

$$\sigma_{\rm LM} = \frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\frac{28}{27}[\log^3(\gamma^2) - 3[f(Z_A\alpha) + f(Z_B\alpha)]\log^2(\gamma^2) + 6f(Z_A\alpha)f(Z_B\alpha)\log(\gamma^2)] \tag{28}$$

where

$$f(Z_{A,B}\alpha) = Z_{A,B}^2\alpha^2\sum_{n=1}^{\infty}\frac{1}{n\left(n^2+Z_{A,B}^2\alpha^2\right)}. \tag{29}$$

The dominant (negative) Coulomb correction in this formula is the $\log^2(\gamma^2)$ term, which was originally obtained by Ivanov, Schiller, and Serbo [1] with the Weizsacker-Williams approximation. The last (positive) $\log(\gamma^2)$ term in the formula can be thought of as representing the Coulomb correction corresponding to multiple photon emission of both ions [28] and as we show below is relatively small.

Table I shows the results of numerical calculations. The present perturbative computer calculations are in good agreement with the Racah formula at RHIC and LHC energies, as expected, and with the published Monte Carlo RHIC calculations of Hencken, Trautmann, and Baur [29,30]. At SPS energies the present perturbative computer calculation results are a bit higher (7%) than the Racah formula and the Hencken, Trautmann, and Baur calculation, perhaps indicating divergence in those results from the ultrareletivistic limit of the present treatment. The full numerical evaluation of the exact semiclassical total cross section for $e^+e^-$ production with gold or lead ions shows reductions from perturbation theory of 28% (SPS), 17% (RHIC), and 11% (LHC). Clearly with increasing beam energy (and a larger value for the spatial cutoff of the transverse integral in the formula) higher order corrections to perturbation theory are relatively smaller. The S + Au calculation at SPS energy shows an expected smaller reduction from perturbation theory (15%) than the 28% reduction of Pb + Au at the same energy.

The Lee-Milstein higher order overall correction to perturbation theory (difference column) is negative but somewhat larger than the difference evaluated here numerically. The small positive contribution of multiple photon emission from both ions to the overall negative Coulomb correction is shown in parentheses in the difference column. Because of the way the numerical calculations were organized it was straightforward to extract this contribution from the exact

TABLE I. Computer calculations compared with analytical formula results. $\gamma$ is defined for one of the ions in the frame of equal magnitude and opposite direction velocities. Total cross sections are expressed in barns. The positive contribution of multiple photon emission from both ions to the overall difference betwen exact and perturbative results is shown in parentheses.

| | | Exact | Perturbative | Difference |
|---|---|---|---|---|
| Pb + Au | Computer evaluation | 2670 | 3720 | −1050 (+80) |
| $\gamma = 9.2$ | Racah formula | | 3470 | |
| | Lee-Milstein | 3050 | 5120 | −2070 (+160) |
| S + Au | Computer evaluation | 119.7 | 141.6 | −21.9 (+0.15) |
| $\gamma = 9.2$ | Racah formula | | 132.0 | |
| | Lee-Milstein | 152.0 | 195.0 | −43.0 (+0.30) |
| Pb + Pb | Computer evaluation | 3210 | 4500 | −1290 (+100) |
| $\gamma = 10$ | Racah formula | | 4210 | |
| | Hencken, Trautmann, Baur | | 4210 | |
| | Lee-Milstein | 3690 | 6160 | −2470 (+190) |
| Au + Au | Computer evaluation | 28,600 | 34,600 | −6,000 (+220) |
| $\gamma = 100$ | Racah formula | | 34,200 | |
| | Hencken, Trautmann, Baur | | 34,000 | |
| | Lee-Milstein | 34,100 | 42,500 | −8,400 (+290) |
| Pb + Pb | Computer evaluation | 201,000 | 227,000 | −26,000 (+600) |
| $\gamma = 2960$ | Racah formula | | 226,000 | |
| | Lee-Milstein | 226,000 | 258,000 | −32,000 (+700) |

computer evaluation. Again the Lee-Milstein formula overestimates this small positive contribution, especially for the SPS case.

There is the question of whether Coulomb corrections might become vanishingly small in some momentum regions. Let us take the Au + Au RHIC case as an example. If one looks at the uncorrelated positron cross section [Eq. (7)] as a function of momentum, then one finds that throughout the transverse and longitudinal momentum space of the final positron, the smallest reduction from perturbation theory is 12.5% and the largest reduction is 25% in comparison to the mean or integrated total cross-section reduction of 17% of the table. Thus the argument given in Ref. [1] that Coulomb corrections contribute mostly for $\mathbf{q}_\perp = \mathbf{m_e}$ but should disappear for larger and smaller $\mathbf{q}_\perp$ is not verified.

Figure 2 shows the transverse momentum spectrum integrated over all longitudidal momenta. The overall contribution does peak at about $\mathbf{q}_\perp = \mathbf{m_e}$. However, Coulomb corrections persist to the highest and lowest values of $\mathbf{q}_\perp$, scaling roughly with the perturbative cross section. Figure 3 shows the logitudinal momentum spectrum integrated over all transverse momenta, and likewise Coulomb corrections persist to the highest and lowest values of $q_z$.

Given the decrease of Coulomb corrections with increasing beam energy one might ask, “At what $\gamma$ of colliding Pb beams would Coulomb corrections be relatively unimportant, say, less than 1% for the total cross section?” If for the purposes of *reductio ad absurdum* one takes the Lee-Milstein formula as a reasonable order of magnitude approximation, then the answer is $\gamma = 10^{43}$. The point is that for any conceivable accelerator beyond LHC the Coulomb corrections to $e^+e^-$ pair production will still be significant.

One can calculate momentum spectra to compare with the CERN SPS data. Because the CERN data comprise positrons uncorrelated with electrons, comparison with a full calculation of the positron momentum spectrum $d\sigma(q)$ is appropriate. Figure 4 shows the data for a Pb projectile on a Au target.

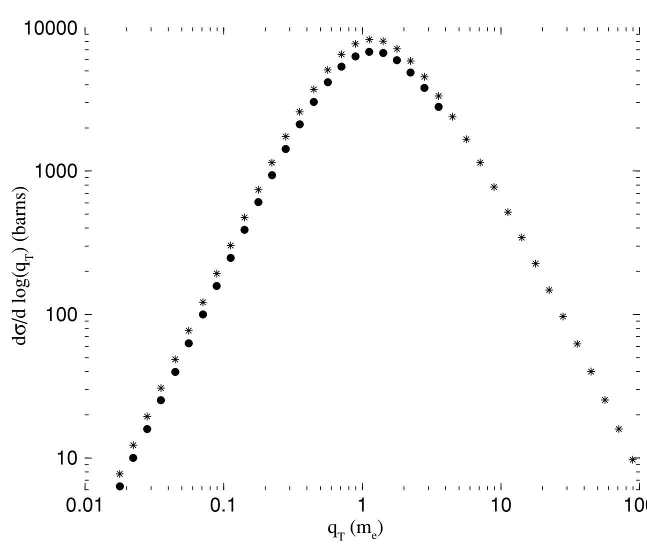


FIG. 2. Positron transverse momentum spectrum for Au + Au at RHIC with $\gamma = 100$. The filled circles are the exact calculation and the stars the perturbation theory.

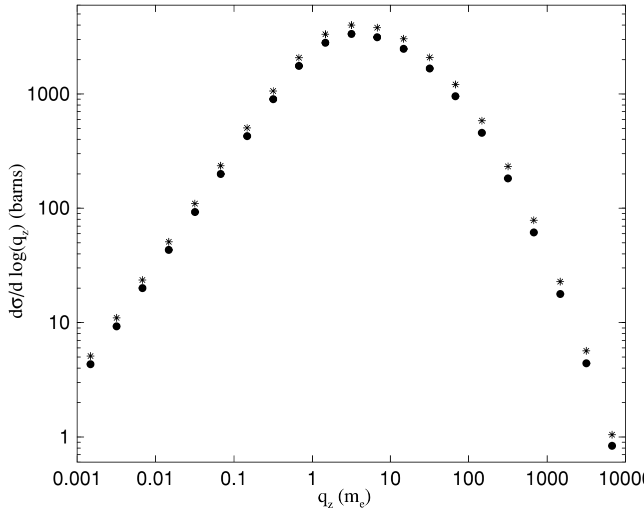


FIG. 3. As in Fig. 2 but for the positron longitudinal momentum spectrum for Au + Au at RHIC with $\gamma = 100$.

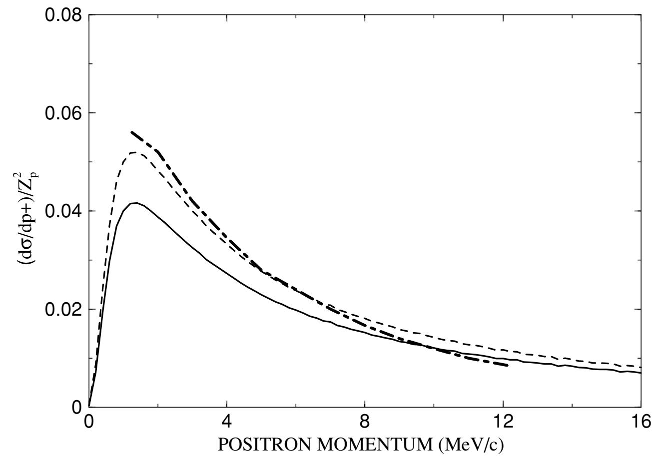


FIG. 5. As in Fig. 4 but with an S projectile. The dot-dashed line follows the authors' representation of the CERN SPS data.

On the whole the perturbation theory curve (dashed line) perhaps seems closer to the data than to the solid full exact calculation.

Figure 5 show an analogous comparison for an S projectile on a Au target. Again, the perturbation theory curve seems closer to the data, represented by the dot-dashed line. Figures 4 and 5 provide an illustration of the statement of the experimental authors that the cross sections follow perturbative scaling. However, especially given the difficulty of the SPS experiment as described by the authors, the apparent lack of Coulomb corrections seen here needs to be verified in other ultrarelativistic heavy ion experiments.

The first experimental observation of $e^+e^-$ pairs at RHIC has been published by STAR [31]. Events were recorded where pairs were accompanied by nuclear dissociation. Comparison with perturbative QED calculations allowed a limit to be set "on higher-order corrections to the cross section, $-0.5\sigma_{\rm QED} < \Delta\sigma < 0.2\sigma_{\rm QED}$ at a 90% confidence level."

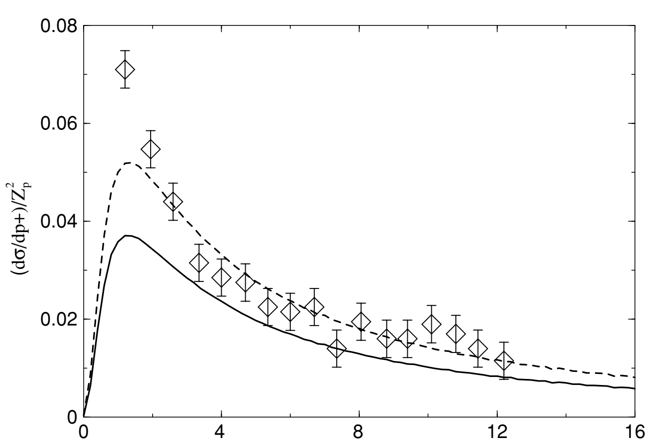


FIG. 4. Calculated positron momentum spectrum compared with the CERN SPS data for Pb + Au. The solid line is the exact calculation and the dashed line perturbation theory.

The present technology of the properly regualarized exact computer code does not include an impact parameter representation and thus does not allow for evaluating a cross section where pair production is in coincidence with nuclear dissociation. Furthermore, the retarded propagators are not strictly appropriate when the range of both electrons and positrons are restricted such as the STAR data. However, a comparison of calculations in the STAR acceptance without nuclear dissociation is of interest as an indication of the relative difference between perturbation theory and the regularized exact result. In the STAR acceptance the exact result is calculated to be 17% lower than that with perturbation theory (as is coincidently true for the total RHIC $e^+e^-$ cross section in Table I). This rough estimate $\Delta\sigma = -0.17\sigma_{\rm QED}$ is not excluded by the above STAR limit.

## IV. SUMMARY AND CONCLUSIONS

A full numerical evaluation of the "exact" semiclassical total cross section for $e^+e^-$ production with gold or lead ions shows reductions from those from perturbation theory of 28% (SPS), 17% (RHIC), and 11%(LHC).

For large $Z$ no final momentum region was found in which there was no reduction or an insignificant reduction of the exact cross section from the perturbative cross section.

The CERN SPS data cover a large part of the momentum range of produced positrons, and the present theory predicts a reduction of cross section at high $Z$ from the perturbative result. That the CERN SPS data apparently do not show a reduction from perturbation theory is a puzzle. It would be of great interest to obtain more precise data on the variation of heavy ion pair production cross sections with ion charge at RHIC or LHC. If the present apparent lack of evidence for Coulomb corrections in ultrarelativistic heavy ion $e^+e^-$ pair production were to be reproduced in other experiments, it would provide a unique challenge to our theoretical understanding of strong-field QED.

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[1] D. Y. Ivanov, A. Schiller, and V. G. Serbo, Phys. Lett. **B454**, 155 (1999).
[2] R. N. Lee and A. I. Milstein, Phys. Rev. A **61**, 032103 (2000).
[3] R. N. Lee and A. I. Milstein, Phys. Rev. A **64**, 032106 (2001).
[4] A. J. Baltz, M. J. Rhoades-Brown, and J. Weneser, Phys. Rev. A **44**, 5569 (1991).
[5] A. J. Baltz, Phys. Rev. A **52**, 4970 (1995).
[6] A. J. Baltz, Phys. Rev. Lett. **78**, 1231 (1997).
[7] B. Segev and J. C. Wells, Phys. Rev. A **57**, 1849 (1998).
[8] A. J. Baltz and L. McLerran, Phys. Rev. C **58**, 1679 (1998).
[9] B. Segev and J. C. Wells, Phys. Rev. C **59**, 2753 (1999).
[10] U. Eichmann, J. Reinhardt, S. Schramm, and W. Greiner, Phys. Rev. A **59**, 1223 (1999).
[11] C. R. Vane, S. Datz, E. F. Deveney, P. F. Dittner, H. F. Krause, R. Schuch, H. Gao, and R. Hutton, Phys. Rev. A **56**, 3682 (1997).
[12] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954); H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954).
[13] L. D. Landau and E. M. Lifshitz, Phys. Z. Sowjetunion **6**, 244 (1934).
[14] A. J. Baltz, Phys. Rev. C **68**, 034906 (2003).
[15] A. J. Baltz, F. Gelis, L. McLerran, and A. Peshier, Nucl. Phys. **A 695**, 395 (2001).
[16] J. F. Reading, Phys. Rev. A **8**, 3262 (1973).
[17] J. Reinhardt, B. Müller, W. Greiner, and G. Soff, Phys. Rev. Lett. **43**, 1307 (1979).
[18] K. Rumrich, K. Momberger, G. Soff, W. Greiner, N. Grün, and W. Scheid, Phys. Rev. Lett. **66**, 2613 (1991).
[19] J. C. Wells, V. E. Oberacker, A. S. Umar, C. Bottcher, M. R. Strayer, J. S. Wu, G. Plunien, Phys. Rev. A **45**, 6296, (1992).
[20] G. Baur, Phys. Rev. A **42**, 5736 (1990).
[21] C. Best, W. Greiner, and G. Soff, Phys. Rev. A **46**, 261 (1992).
[22] K. Hencken, D. Trautmann, and G. Baur, Phys. Rev. A **51**, 998 (1995).
[23] A. Aste, G. Baur, K. Hencken, D. Trautmann, and G. Scharf, Eur. Phys. J. C **23**, 545 (2002).
[24] J. A. M. Vermaseren, arXiv:math-ph/0010025 (2000).
[25] C. Bottcher and M. R. Strayer, Phys. Rev. D **39**, 1330 (1989).
[26] K. Hencken, D. Trautmann, and G. Baur, Phys. Rev. A **51**, 1874 (1995).
[27] G. Racah, Nuovo Cimento **14**, 93 (1937).
[28] R. N. Lee, A. I. Milstein, and V. G. Serbo, Phys. Rev. A **65**, 022102 (2002).
[29] K. Hencken, D. Trautmann, and G. Baur, Phys. Rev. C **59**, 841 (1999).
[30] A. Alscher, K. Hencken, D. Trautmann, and G. Baur, Phys. Rev. A **55**, 396 (1997).
[31] STAR Collaboration, J. Adams *et al.*, Phys. Rev. C **70**, 031902(R) (2004).