PHYSICAL REVIEW C **71**, 014318 (2005)

Test of consistency of the so-called fully renormalized quasiparticle random phase approximation

O. Civitarese* and M. Reboiro†
Department of Physics, University of La Plata, c.c. 67 1900, La Plata, Argentina

J. G. Hirsch‡
Instituto de Ciencias Nucleares, UNAM, México, Apdo Postal 70-543, Mexico 04510 D.F.

(Received 29 June 2004; published 28 January 2005)

The validity of the so-called fully renormalized quasiparticle random phase approximation is tested, under schematic model conditions. It is shown that this approximation does not fulfill the consistency required by the linealization procedure. The results are illustrated by the analysis of Fermi-type transitions.

DOI: 10.1103/PhysRevC.71.014318 PACS number(s): 21.60.Jz, 21.60.Fw.

I. INTRODUCTION

The interest in the study of the validity of the quasiparticle random phase approximation (QRPA) has been continuously renewed because it was proposed, long ago, by M. Baranger [1]. The QRPA is, perhaps, the best known bosonization method [2]. It is a rather friendly formalism that exploits the correspondence between a fermionic hamiltonian and its harmonic representation [3]. The QRPA is a suitable formalism to describe spherical and deformed nuclei [4]. Basically, it is a theory of small vibrations around spherical or deformed mean field minima. Various extensions of the QRPA method have been developed in the past to account for correlations among unlike (proton-neutron) quasiparticles [5]. Nearly two decades ago the issue of particle-particle correlations was raised [6] to explain the strong cancellation of the matrix elements that govern certain exotic electroweak processes, such as the nuclear double beta decay [7,8]. The question about the validity of these extensions persists, and from time to time a new proposal emerges as a cure to some of the apparent failures of the QRPA approach. Among these extensions, the renormalized QRPA (RQRPA) of Refs. [9–11] was presented as a suitable alternative to the standard QRPA. However, the violation of the Ikeda sum rule found in the RQRPA [11] raised strong doubts about its validity [12]. The renormalization procedure is rather well established, but it introduces correlations that exceed the order of approximation required by the QRPA, as it has been shown in [12]. Moreover, the results of [13,14] show that the RQRPA is not able to reproduce the trend of the exact solution in a very schematic and solvable situation. This is a matter of concern, because the validity of the RQRPA in realistic situations may be hampered by the fact that it does not work in a simple, schematic, and solvable model [12]. Unfortunately, this point has been ignored by some authors, and different recipes have been imposed on top of the RQRPA [15]. Some of these recipes are just ad hoc procedures [16]. In this article, we focus our attention on a latest attempt known as *fully renormalized quasiparticle random phase approximation* (FRQRPA) [17,18]. Therein it is claimed that the difficulties of the RQRPA, concerning the conservation of the Ikeda sum rule, have been solved. As we show in this article, the claim of [17,18] may not be supported by the results of a test of the formalism. In performing this test we have followed the steps of the FRQRPA and searched for nontrivial solutions of it. As we show later, the solutions of the FRQRPA reduces trivially to the ones of the QRPA. Thus, the FRQRPA does not seem to be a real improvement as respect to the QRPA, contrary to the claims of [17,18].

II. FORMALISM

For the sake of completeness we briefly review the basic notions of the QRPA. Let us considerer a very schematic situation consisting of protons and neutrons in a single j shell. They are interacting via monopole pairing forces, separately for protons and neutrons, and charge dependent two body forces of the Fermi type. The Hamiltonian is written [19] as follows:

$$\begin{aligned} H = {} & e_p n_p - G_p S_p^\dagger S_p + e_n n_n - G_n S_n^\dagger S_n \\ & + 2\chi \beta^- \beta^+ - 2\kappa P^- P^+, \end{aligned} \tag{1}$$

where

$$\begin{aligned} n_i &= \sum_{m_i} a_{m_i}^\dagger a_{m_i}, \\ S_i &= \sum_{m_i} a_{m_i}^\dagger a_{\overline{m}_i}^\dagger, \qquad i = p, n, \\ \beta^- &= \sum_{m_p = m_n} a_{m_p}^\dagger a_{m_n}, \\ P^- &= \sum_{m_p = -mn} a_{m_p}^\dagger a_{\overline{m}_n}^\dagger, \end{aligned} \tag{2}$$

are the number operator, the monopole pair operator, the one-particle charge-exchange operator, and the two-particle charge-exchange operator, respectively. Proton and neutron single-particle orbits are denoted by the subindexes (p) and

*Electronic address: civitare@fisica.unlp.edu.ar.
†Electronic address: reboiro@fisica.unlp.edu.ar.
‡Electronic address: hirsch@nuclecu.unam.mx.

(n) and $a_p^\dagger = a_{j_p m_p}^\dagger$ is a particle creation operator and $a_{\bar{p}}^\dagger = (-)^{j_p - m_p} a_{j_p\,-m_p}^\dagger$ is its time reversal. The symmetry group of H is the $SO(5)$ group [19]. By performing the transformation of the particle-creation and annihilation operators to the quasiparticle representation, for protons and neutrons separately, the resulting Hamiltonian is written as follows:

$$\begin{aligned} H = {} & E_p N_p + E_n N_n + \lambda_1 A^\dagger A + \lambda_2 (A^\dagger A^\dagger + AA) \\ & - \lambda_3 (A^\dagger B + B^\dagger A) - \lambda_4 (A^\dagger B^\dagger + BA) + \lambda_5 B^\dagger B \\ & + \lambda_6 (B^\dagger B^\dagger + BB), \end{aligned} \tag{3}$$

where E_p and E_n are the quasiparticle energies and the operators and matrix elements of Eq. (3) are defined by the following:

$$\begin{aligned} N_q &= \sum_{m_q} \alpha_{j_q m_q}^\dagger \alpha_{j_q m_q}, \quad q = n, p, \\ A^\dagger &= \left[\alpha_p^\dagger \otimes \alpha_n^\dagger\right]_{M=0}^{J=0}, \\ B^\dagger &= \left[\alpha_p^\dagger \otimes \alpha_{\bar{n}}\right]_{M=0}^{J=0}, \\ \lambda_1 &= 4\Omega\left[\chi\left(u_p^2 v_n^2 + v_p^2 u_n^2\right) - \kappa\left(u_p^2 u_n^2 + v_p^2 v_n^2\right)\right], \\ \lambda_2 &= 4\Omega(\chi + \kappa) u_p v_p u_n v_n, \\ \lambda_3 &= 4\Omega(\chi + \kappa) u_n v_n \left(u_p^2 - v_p^2\right), \\ \lambda_4 &= 4\Omega(\chi + \kappa) u_p v_p \left(u_n^2 - v_n^2\right), \\ \lambda_5 &= 4\Omega\left[\chi\left(u_p^2 u_n^2 + v_p^2 v_n^2\right) - \kappa\left(u_p^2 v_n^2 + v_p^2 u_n^2\right)\right], \\ \lambda_6 &= -\lambda_2. \end{aligned} \tag{4}$$

In the single-shell space, the quasiparticle energies are given by the following expressions:

$$E_p = \frac{\Omega G}{2}, \quad E_n = \frac{\Omega G}{2}, \tag{5}$$

and the corresponding occupation factors are as follows:

$$v_p = \sqrt{\frac{N_p}{2\Omega}}, \quad u_p = \sqrt{1 - v_p^2}, \tag{6}$$

$$v_n = \sqrt{\frac{N_n}{2\Omega}}, \quad u_n = \sqrt{1 - v_n^2}. \tag{7}$$

The Hamiltonian expressed in Eq. (3) contains all the terms generated by the transformation to the quasiparticle basis. In the context of the BCS approximation, the expectation value of the commutator

$$[A, A^\dagger] = 1 - \frac{1}{2\Omega}(N_p + N_n), \tag{8}$$

is

$$\langle [A, A^\dagger] \rangle = 1, \tag{9}$$

because the quasiparticle vacuum is annihilated by the operators N_q. In other words, up to this point the mapping of the initial Hamiltonian expressed in Eq. (1) to the quasiparticle basis expressed in Eq. (3) is exact.

A. pn-QRPA

The linearization procedure (pn-QRPA) allows us to write the pair contribution of the quasiparticle Hamiltonian in Eq. (3) as follows:

$$H_{\text{pn-pairs}} = (E_p + E_n) A^\dagger A + \lambda_1 A^\dagger A + \lambda_2 (A^\dagger A^\dagger + AA), \tag{10}$$

in the harmonic form

$$H_{\text{pn-QRPA}} = \omega \Gamma^\dagger \Gamma, \tag{11}$$

where

$$\Gamma^\dagger = X A^\dagger - Y A \tag{12}$$

is the one phonon creation operator. The new vacuum, $|0_{\text{pn-QRPA}}\rangle$, is annihilated by the operator Γ

$$\Gamma\, |0_{\text{pn-QRPA}}\rangle = 0, \tag{13}$$

and

$$[\Gamma, \Gamma^\dagger] = 1. \tag{14}$$

The equation of motion

$$[H_{\text{pn-QRPA}}, \Gamma^\dagger] = \omega \Gamma^\dagger, \tag{15}$$

fixes the eigenvalue ω, and the amplitudes X and Y are normalized as follows:

$$X^2 - Y^2 = 1, \tag{16}$$

as a consequence of Eq. (14). The equation of motion [Eq. (15)] can be written in matrix form by commuting with Γ to the left, leading to the following:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{17}$$

The matrix elements $\mathcal{A}, \mathcal{B}$ are defined by the following:

$$\begin{aligned} \mathcal{A} &= \langle [A, [H_{\text{pn-pairs}}, A^\dagger]] \rangle, \\ &= E_p + E_n + \lambda_1, \\ \mathcal{B} &= -\langle [A, [H_{\text{pn-pairs}}, A]] \rangle, \\ &= 2\lambda_2. \end{aligned} \tag{18}$$

Equation (17) is valid under two conditions:

(a) the interaction does not include exchange terms and
(b) the expectation value of the quasiparticle number operator, on the pn-QRPA vacuum, vanishes.

The following eigenvalue:

$$\omega_{\text{pn-QRPA}} = \sqrt{(E_p + E_n + \lambda_1)^2 - 4\lambda_2^2}, \tag{19}$$

vanishes for

$$\kappa_{\omega=0}^{\text{pn-QRPA}} = \frac{(G/4) + \chi(u_p v_n - v_p u_n)^2}{(u_p u_n + v_p v_n)^2}. \tag{20}$$

The strong dependence of ω on κ, shown by the *pn-QRPA* solution, is similar to the dependence exhibited by the exact solution [19]. Also, the overall agreement among the exact solution, the quasiparticle solution, and the pn-QRPA is noticeable [19]. From the agreement found in Ref. [19], it is evident that the pn-QRPA method gives the correct value of ω at leading order and that the agreement with the exact solution improves with the inclusion of the remaining terms of the Hamiltonian. This is the case of the solution labeled the quasiparticle solution in Ref. [19].

B. pn-RQRPA

Because the commutator [Eq. (8)] contains the proton and neutron quasiparticle number operators, it seems natural to accommodate them in the matrix equation [Eq. (17)] by replacing the definition of the quasiparticle pair operators such that they do commute to unity. The replacement of the expectation value of the commutator, as done in the standard QRPA, by the commutator itself is done by performing a renormalization of the pair operators [9–11] as follows:

$$\begin{aligned}\tilde{A}^\dagger &= D^{-1/2}A^\dagger,\\ \tilde{A} &= D^{-1/2}A,\\ \langle[\tilde{A},\tilde{A}^\dagger]\rangle &= 1,\\ D = \langle[A,A^\dagger]\rangle &= 1-\left\langle\frac{N_p+N_n}{2\Omega}\right\rangle.\end{aligned} \tag{21}$$

The expectation value that appears in the definition of D is taken by a new vacuum. To construct the set of equations of the renormalized proton-neutron QRPA, one introduces the following phonon creation operator:

$$\Gamma_R^\dagger = X_R\tilde{A}^\dagger - Y_R\tilde{A}, \tag{22}$$

where

$$X_R^2 - Y_R^2 = 1. \tag{23}$$

Thus, a new matrix equation is obtained as follows:

$$\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}_R \begin{pmatrix}X\\ Y\end{pmatrix}_R = \omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}_R, \tag{24}$$

with

$$\begin{aligned}\mathcal{A}_\mathcal{R} &= \langle[A,[H_{\text{pn-pairs}},A^\dagger]]\rangle,\\ &= E_p+E_n+\lambda_1 D,\\ \mathcal{B}_\mathcal{R} &= -\langle[A,[H_{\text{pn-pairs}},A]]\rangle,\\ &= 2\lambda_2 D.\end{aligned} \tag{25}$$

The factor D is determined by the following condition:

$$\begin{aligned}D &= 1-\langle 0_{\text{pn-RQRPA}}|\frac{N_p+N_n}{2\Omega}|0_{\text{pn-RQRPA}}\rangle\\ &= \frac{1}{1+\frac{Y_R^2}{\Omega}}.\end{aligned} \tag{26}$$

and it is a function of the amplitude Y.

Equation (26) is a direct consequence of Eq. (21), and it can be viewed as the vacuum condition of the pn-RQRPA. In fact, because the Hamiltonian is the same for the pn-QRPA and pn-RQRPA, the renormalization implied by Eq. (25) should come only from the definition of the vacuum. Otherwise, the renormalization factor D should be equal to unity. It means that there is not a gradual transition from the pn-RQRPA to the pn-QRPA or vice versa. They are different approximations and have been obtained under different assumptions. The pn-QRPA is a valid approximation within a domain of small amplitude vibrations and the pn-RQRPA is a procedure that aims at crossing a phase transition without changing the Hamiltonian. The drawbacks of the pn-RQRPA procedure have been discussed extensively in Refs. [12,20]. We refer the reader to these references for further details concerning the comparison of the pn-QRPA and the pn-RQRPA.

The comparison between the results of the pn-QRPA, for the matrix elements of $\mathcal{A}_\mathcal{R}$ and $\mathcal{B}_\mathcal{R}$, with the results of the previous equations, $\mathcal{A}$ and $\mathcal{B}$, shows that the renormalization, represented by D, is *a plain renormalization* of the couplings λ_1 and λ_2. This renormalization should be consistent with the requirement that the corrections introduced by considering nonvanishing vacuum expectation values of N_p and N_n should be of the order $1/\Omega$ with respect to leading-order terms included in the Hamiltonian. Otherwise one has to consider the full Hamiltonian and not only the pair part of it. The renormalization of the couplings shifts the point where the eigenvalue vanishes, but this does not necessarily preserve the structure of the eigenfunction. In fact, the value of κ for which the eigenvalue vanishes is as follows:

$$\kappa_{\omega=0}^{\text{pn-RQRPA}} = \frac{(G/4D)+\chi(u_pv_n-v_pu_n)^2}{(u_pu_n+v_pv_n)^2}. \tag{27}$$

Near $\omega=0$ the value of D is approximately $D\approx 0.80$, and it leads to the following estimate:

$$\kappa_{\omega=0}^{\text{pn-RQRPA}} > \kappa_{\omega=0}^{\text{pn-QRPA}}. \tag{28}$$

The comparison with the exact results [12,14] shows that the pn-RQRPA and exact wave functions differ significantly. This indicates that the renormalization procedure violates the consistency of the QRPA approach severely. The asymmetric treatment of the QRPA matrix and of the QRPA norm reflects on the wave function. The obvious consequence of it is the violation of the sum rule

$$[\beta^+,\beta^-] = N-Z, \tag{29}$$

which in the pn-RQRPA takes the following form:

$$[\beta^+,\beta^-] = D(N-Z). \tag{30}$$

This means that, because $D\neq 1$, the oscillator sum rule (Ikeda Sum Rule) is inevitably violated. This has been shown by a comparison between exact and pn-RQRPA wave functions in Ref. [14] and by the comparison between the matrix elements of the operators β^- and β^+, calculated in the pn-QRPA and the pn-RQRPA in Ref. [12]. We return to the discussion of these features later.

C. pn-FRQRPA

The fully renormalized pn-QRPA of Refs. [17,18] goes beyond the renormalization scheme of the pn-RQRPA. It is an ad hoc procedure that postulates the use of all the terms resulting from the transformation of $a_p^\dagger a_n$ to the quasiparticle basis in the definition of the phonon. Then:

$$\begin{aligned}\tilde{A}^\dagger &= D^{-1/2}(A^\dagger+\alpha B^\dagger+\beta B),\\ \tilde{A} &= (\tilde{A}^\dagger)^\dagger,\end{aligned} \tag{31}$$

with

$$\alpha = -\frac{u_n v_n}{v_p^2 - v_n^2}, \qquad \beta = \frac{u_p v_p}{v_p^2 - v_n^2}. \tag{32}$$

The corresponding phonon creation operator is written as follows:

$$\Gamma^\dagger_{\rm FR} = X_{\rm FR}\tilde{A}^\dagger - Y_{\rm FR}\tilde{A}. \tag{33}$$

The commutator between the operators $\tilde{A}^\dagger$ and $\tilde{A}$ is taken as unity,

$$\langle[\tilde{A}, \tilde{A}^\dagger]\rangle = 1, \tag{34}$$

a condition that is enforced by the choice of the renormalization factor D such that

$$\begin{aligned} D_{\rm FR} &= \langle[A + \alpha B + \beta B^\dagger, A^\dagger + \alpha B^\dagger + \beta B]\rangle \\ &\approx 1 - \left\langle\frac{N_p + N_n}{2\Omega}\right\rangle + (\alpha^2 - \beta^2)\left\langle\frac{N_n - N_p}{2\Omega}\right\rangle. \end{aligned} \tag{35}$$

The above expression is valid if the quasineutron-quasiproton correlations are created by the action of pair of $A^\dagger$ operators. Attempts to build the vacuum associated with the operator $\Gamma^\dagger_{\rm FR}$ fail badly, even at the level of the boson approximation (see the next section). The normalization of the phonon operator is therefore given by the following:

$$X^2_{\rm FR} - Y^2_{\rm FR} = 1, \tag{36}$$

as in the previous cases. The eigenvalue problem is written in matrix form as follows:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}_{\rm FR} \begin{pmatrix} X \\ Y \end{pmatrix}_{\rm FR} = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}_{\rm FR}, \tag{37}$$

with

$$\begin{aligned} (D\mathcal{A})_{\rm FR} &= \langle[\tilde{A}, [H, \tilde{A}^\dagger]]\rangle, \\ &= (E_p + E_n) f_1 + (E_p - E_n)(\alpha^2 + \beta^2) f_2 \\ &\quad + \lambda_1 f_1^2 - \lambda_3 2\alpha f_1 f_2 + \lambda_4 2\beta f_1 f_2 \\ &\quad + \lambda_5(\alpha^2 + \beta^2) f_2^2 + \lambda_6 4\alpha\beta f_2^2, \\ (D\mathcal{B})_{\rm FR} &= -\langle[\tilde{A}, [H, \tilde{A}]]\rangle, \\ &= (E_p - E_n)\alpha\beta f_2 + \lambda_2 2 f_1^2 + \lambda_3 2\beta f_1 f_2 \\ &\quad - \lambda_4 2\alpha f_1 f_2 - \lambda_5 \alpha\beta f_2^2 + \lambda_6(\alpha^2 + \beta^2) f_2^2. \end{aligned} \tag{38}$$

The factors f_i are given by the following expressions:

$$\begin{aligned} f_1 &= \langle 0_{\text{pn-FRQRPA}}|\left(1 - \frac{N_p + N_n}{2\Omega}\right)|0_{\text{pn-FRQRPA}}\rangle \\ &\approx 1 - \frac{D^{-2}_{\rm FR} Y^2_{\rm FR}}{2\Omega\left(1 + Y^2_{\rm FR}\right)}\left(2 - \frac{2 - (\alpha^2 + \beta^2)}{2\Omega}\right), \\ f_2 &= \langle 0_{\text{pn-FRQRPA}}|\frac{N_n - N_p}{2\Omega}|0_{\text{pn-FRQRPA}}\rangle \\ &\approx \frac{D^{-2}_{\rm FR} Y^2_{\rm FR}}{2\Omega\left(1 + Y^2_{\rm FR}\right)}\frac{\alpha^2 - \beta^2}{2\Omega}. \end{aligned} \tag{39}$$

The leading-order expression for D yields the following:

$$\begin{aligned} D_{\rm FR} &= \langle 0_{\text{pn-FRQRPA}}|\left(1 - \frac{N_p + N_n}{2\Omega}\right. \\ &\quad \left. + (\alpha^2 - \beta^2)\frac{N_n - N_p}{2\Omega}\right)|0_{\text{pn-FRQRPA}}\rangle \\ &\approx f_1 + (\alpha^2 - \beta^2) f_2. \end{aligned} \tag{40}$$

These factors are functions of the BCS parameters u and v, the couplings χ and κ, and the quantities α and β of Eq. (32).

The above two equations, Eqs. (39) and (40), are the equivalent of Eq. (26) (see Sec. II B). Notice that to reobtain the results of Sec. II B, one should use the values $\alpha = \beta = 0$, for which $f_2 = 0$. By keeping the lowest order approximation in $1/\Omega$, in the expression of f_1, one gets $f_1 \to D$. However, there is a conceptual difference between the pn-RQRPA and the pn-FRQRPA. In the pn-RQRPA the quasiparticle pair operators are the standard pair operators $A^\dagger$ and A of Eq. (4), and the vacuum is changed while keeping the Hamiltonian unchanged. In the pn-FRQRPA, to the contrary, both the vacuum and the Hamiltonian are changed because the quasiparticle pairs of Eq. (31), differ from the quasiparticle pairs of both the pn-QRPA and pn-RQRPA. The difference is the inclusion of the one quasiparticle operators B and $B^\dagger$ in Eq. (31).

The value of κ for which $\omega = 0$ is then as follows:

$$\kappa^{\rm FR}_{\omega=0} \approx \frac{(G/4f_1) + \chi(u_p v_n - v_p u_n)^2}{(u_p u_n + v_p v_n)^2}. \tag{41}$$

Because f_1 of Eq. (40) goes to 1, as a function of $1/\Omega$, faster than $D_{\rm FR}$ [see also Eq. (40)], the value of $\kappa^{\rm FR}_{\omega=0}$ of Eq. (41) is closer to $\kappa^{\text{pn-QRPA}}_{\omega=0}$ than to $\kappa^{\text{pn-RQRPA}}_{\omega=0}$. We return to this comparison under Sec. III.

To illustrate the scope and differences between the pn-QRPA, pn-RQRPA, and pn-FRQRPA we test them, under Sec. II D, by means of the calculation of the strength of charge-dependent operators.

D. Transitions

We now proceed with the study of Fermi transitions for each of the considered approximations. The Fermi operator is given by the following:

$$\beta^- = \sum_{m_p = m_n} a^\dagger_{m_p} a_{m_n}, \tag{42}$$

and after it is transformed to the quasiparticle basis, it reads as follows:

$$\beta^- = \sqrt{2\Omega}(u_n u_p B^\dagger - v_n v_p B - u_p v_n A^\dagger - u_n v_p A). \tag{43}$$

This expression includes quasiparticle-pair terms

$$\beta^-_{\rm pair} = \sqrt{2\Omega}(-u_p v_n A^\dagger - u_n v_p A) \tag{44}$$

and scattering terms

$$\beta^-_{\rm scattering} = \sqrt{2\Omega}(u_n u_p B^\dagger - v_n v_p B). \tag{45}$$

The transformation of the pair terms to the phonon basis yields

$$\beta^-_{\rm pair} = -(u_p v_n X + u_n v_p Y)\Gamma^\dagger - (u_p v_n Y + u_n v_p X)\Gamma. \tag{46}$$

The commutator

$$\left[\beta^{+}_{\text{pair}}, \beta^{-}_{\text{pair}}\right], \tag{47}$$

is independent of the transformation to the phonon basis. Because the combination of products of the amplitudes X and Y disappears from Eq. (47) because of the pseudo-orthogonality of the pn-QRPA basis, and the combination of factors u and v reduces to quadratic factors, the vacuum expectation value of Eq. (47), is the same in the quasiparticle and in the phonon basis

$$\left\langle\left[\beta^{+}_{\text{pair}}, \beta^{-}_{\text{pair}}\right]\right\rangle = N - Z. \tag{48}$$

This result is the Ikeda Sum Rule. The standard pn-QRPA fulfills it exactly, as we have shown before.

In the pn-RQRPA this sum rule is not obeyed, because in this approximation

$$\left\langle\left[\beta^{+}_{\text{pair}}, \beta^{-}_{\text{pair}}\right]_R\right\rangle = D(N - Z), \tag{49}$$

a result that is trivially obtained from Eq. (47) by replacing $\Gamma^\dagger$ with $\Gamma^\dagger_R$.

In the case of the pn-FRQRPA the sum rule has the following value:

$$\begin{aligned}\langle[\beta^+, \beta^-]_{\text{FR}}\rangle = {} & 2\Omega v_n^2 + \left(u_n^2 - v_n^2\right)\mathcal{N}_n \\ & - 2\Omega v_p^2 + \left(u_p^2 - v_p^2\right)\mathcal{N}_p,\end{aligned} \tag{50}$$

and this result coincides with the Ikeda Sum Rule $(N - Z)$ if the following conditions are satisfied:

$$\begin{aligned}N &= 2\Omega v_n^2 + \left(u_n^2 - v_n^2\right)\mathcal{N}_n, \\ Z &= 2\Omega v_p^2 + \left(u_p^2 - v_p^2\right)\mathcal{N}_p,\end{aligned} \tag{51}$$

with

$$\begin{aligned}\mathcal{N}_n &= \langle 0_{\text{pn-FRQRPA}}|N_n|0_{\text{pn-FRQRPA}}\rangle, \\ \mathcal{N}_p &= \langle 0_{\text{pn-FRQRPA}}|N_p|0_{\text{pn-FRQRPA}}\rangle.\end{aligned} \tag{52}$$

Notice that Eq. (51) implies the finding of new values u^2 and v^2, which in turn will change the structure of the $|0_{\text{pn-FQRPA}}\rangle$ vacuum by the change of the backward-going amplitude Y. Thus, the pn-FRQRPA would differ from the pn-QRPA and pn-RQRPA if the nontrivial solutions ($\mathcal{N}_n \neq 0$, $\mathcal{N}_p \neq 0$) of Eq. (52) are found. Before performing the test of consistency of the pn-FRQRPA numerically (see Sec. III), we show, by using boson mapping, that Eqs. (51) and (52) do not lead to nontrivial solutions.

E. Boson mapping

The quasiparticle creation and annihilation operators $\alpha^\dagger_p$, α_p, $\alpha^\dagger_n$, α_n can be arranged in the following set of pairs:

$$\begin{aligned}&\alpha^\dagger_p\alpha^\dagger_p, \quad \alpha_p\alpha_p, \\ &\alpha^\dagger_n\alpha^\dagger_n, \quad \alpha_n\alpha_n, \\ &\alpha^\dagger_p\alpha_p, \quad \alpha^\dagger_n\alpha_n, \\ &\alpha^\dagger_p\alpha^\dagger_n, \quad \alpha_n\alpha_p, \\ &\alpha^\dagger_p\alpha_n, \quad \alpha^\dagger_n\alpha_p.\end{aligned} \tag{53}$$

These 10 operators and their commutators form the $SO(5)$ algebra. The Hamiltonian of Eq. (3) is indeed a bilinear form (sum of products of pairs) of them. The boson image of Eq. (3) can be obtained by applying Holstein-Primakoff boson mapping [21,22]. The images of the 10 pairs of quasiparticle operators are as follows:

$$\begin{aligned}A^\dagger_{pp} &= \frac{1}{\sqrt{\Omega}} b^\dagger_p (\Omega - n_p - n_f)^{1/2}, \\ A_{pp} &= \left(A^\dagger_p\right)^\dagger, \\ N_p &= 2n_p + n_f, \\ A^\dagger_{nn} &= \frac{1}{\sqrt{\Omega}} b^\dagger_n (\Omega - n_n - n_f)^{1/2}, \\ A_{nn} &= \left(A^\dagger_n\right)^\dagger, \\ N_n &= 2n_n + n_f, \\ A^\dagger_{pn} &= \frac{1}{\sqrt{2\Omega}} b^\dagger_f (\Omega - n_n - n_f)^{1/2} (\Omega - n_p - n_f)^{1/2} \Phi(n_f) \\ &\quad - \frac{1}{\sqrt{2\Omega}} \Phi(n_f) b^\dagger_p b^\dagger_n b_f, \\ A_{pn} &= \left(A^\dagger_{pn}\right)^\dagger, \\ B^\dagger_{pn} &= \frac{1}{\sqrt{2\Omega}} b^\dagger_f \Phi(n_f)(\Omega - n_p - n_f)^{1/2} b_n \\ &\quad + \frac{1}{\sqrt{2\Omega}} b^\dagger_p (\Omega - n_n - n_f)^{1/2} \Phi(n_f) b_f, \\ B_{pn} &= \left(B^\dagger_{pn}\right)^\dagger,\end{aligned} \tag{54}$$

where $b^\dagger_p$, $b^\dagger_n$, and $b^\dagger_f$ are bosons and n_p, n_n and n_f are number operators defined as follows:

$$\begin{aligned}n_p &= b^\dagger_p b_p, \\ n_n &= b^\dagger_n b_n, \\ n_f &= b^\dagger_f b_f.\end{aligned} \tag{55}$$

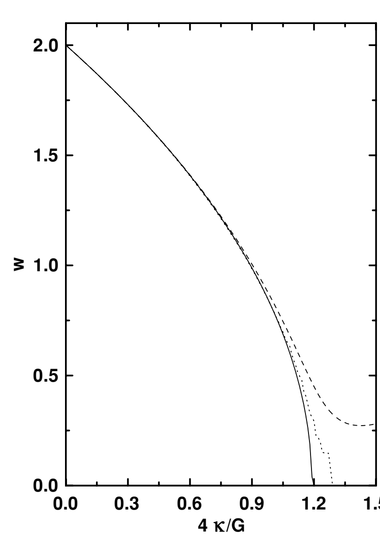

FIG. 1. Excitation energy as a function of κ. The results displayed correspond to $N_n = 14$, $N_p = 6$, $\Omega = 10$ and $\chi = 0.0$ MeV. Solid lines correspond to the pn-QRPA approximation, dashed lines correspond to the pn-RQRPA approximation, whereas the pn-FRQRPA results are shown in dotted lines.

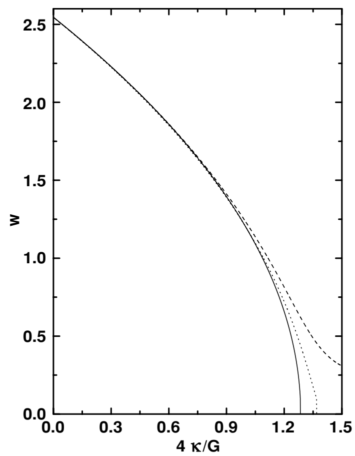

FIG. 2. The same as Fig. 1, for $\chi = 0.025$ MeV.

The bosons are commuting objects

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}, \tag{56}$$

and $\phi(n_f)$ is the operator

$$\Phi(n_f) = \left[\frac{(2\Omega + 2 - n_f)}{(\Omega + 1 - n_f)(\Omega - n_f)}\right]^{1/2}. \tag{57}$$

To leading order in Ω one obtains the following:

$$\begin{aligned} A_{pn}^\dagger &= b_f^\dagger, \\ A_{pn} &= \left(A_{pn}^\dagger\right)^\dagger, \\ B_{pn}^\dagger &= \frac{1}{\sqrt{2\Omega}}\left(b_f^\dagger b_n + b_p^\dagger b_f\right), \\ B_{pn} &= \left(B_{pn}^\dagger\right)^\dagger. \end{aligned} \tag{58}$$

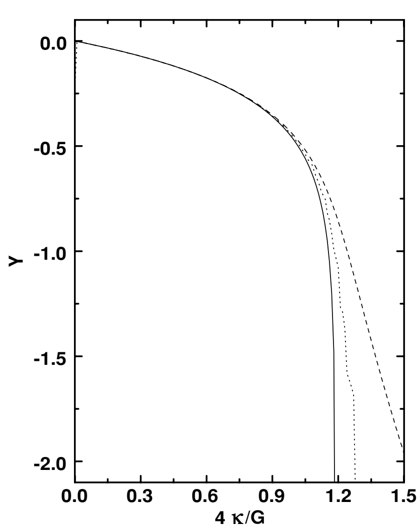

FIG. 3. Backward amplitude as a function of κ. The results displayed correspond to $N_n = 14$, $N_p = 6$, $\Omega = 10$, and $\chi = 0.0$ MeV. Solid lines correspond to the pn-QRPA approximation, and dashed lines correspond to the pn-RQRPA approximation, whereas the pn-FRQRPA results are shown in dotted lines.

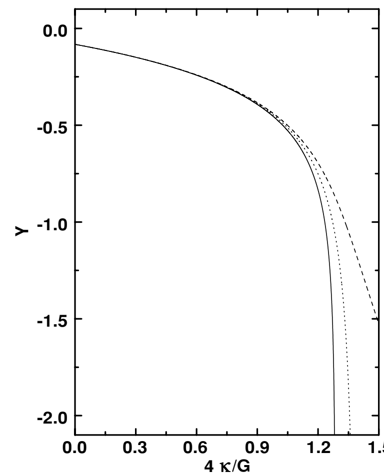

FIG. 4. The same as Fig. 3, for $\chi = 0.025$ MeV.

The equivalent of the pn-FRQRPA approximation, in this boson mapping, is given by the following replacement:

$$\begin{aligned} \tilde{A}^\dagger = D^{-1/2}\Bigg(& b_f^\dagger + \alpha\frac{1}{\sqrt{2\Omega}}\left(b_f^\dagger b_n + b_p^\dagger b_f\right) \\ &+ \beta\frac{1}{\sqrt{2\Omega}}\left(b_n^\dagger b_f + b_f^\dagger b_p\right)\Bigg). \end{aligned} \tag{59}$$

If we now define the phonon creation operator as done in (33), transform the operator $\tilde{A}^\dagger$ and $\tilde{A}$ to the boson basis, and request the condition

$$\Gamma_{\mathrm{FR}}\, |0_{\text{pn-FRQRPA}}\rangle = 0, \tag{60}$$

we obtain

$$\begin{aligned} \Gamma_{\mathrm{FR}}\, |0_{\text{pn-FRQRPA}}\rangle \approx{}& D^{-1/2}(D^{-1} - 1) y b_f^\dagger\, |0_{\text{pn-FRQRPA}}\rangle \\ &+ D^{-3/2}\frac{y}{4x\Omega}(x(\alpha^2 + \beta^2) - 2y\alpha\beta) b_f^\dagger\, |0_{\text{pn-FRQRPA}}\rangle \\ &+ D^{-3/2}\frac{y^2}{2x} b_f^{\dagger\,3}\, |0_{\text{pn-FRQRPA}}\rangle \end{aligned}$$

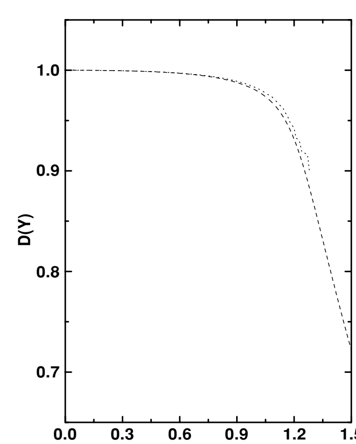

FIG. 5. Parameter D as a function of κ. The results displayed correspond to $N_n = 14$, $N_p = 6$, $\Omega = 10$, and $\chi = 0.0$ MeV. Dashed lines correspond to the pn-RQRPA approximation, whereas the pn-FRQRPA results are shown in dotted lines.

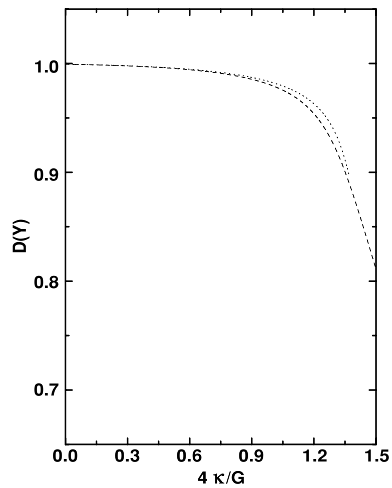

FIG. 6. The same as Fig. 5, for $\chi = 0.025$ MeV.

$$+ D^{-3/2}\frac{y}{2x}\frac{1}{\sqrt{2\Omega}}(2x\alpha - 3y\beta)b_n^\dagger b_f^\dagger\ |0_{\text{pn-FRQRPA}}\rangle$$
$$+ D^{-3/2}\frac{y}{2x}\frac{1}{\sqrt{2\Omega}}(2x\beta - 3y\alpha)b_p^\dagger b_f^\dagger\ |0_{\text{pn-FRQRPA}}\rangle$$
$$= 0. \tag{61}$$

To fulfill this condition, (i) D should be equal to unity, regardless of the values of α and β, or (ii) $y = 0$.

Clearly, the boson picture of the pn-FRQRPA fails the consistency test of the method, because the solution (i), for which $D = 1$, means that one is working with the standard QRPA, and the solution (ii) implies zero-ground-state correlations.

In the next section we explore, numerically, the consistency of the pn-FRQRPA.

III. RESULTS AND DISCUSSION

In this section we present and discuss the results of the calculations we have performed by applying the formalism of the previous sections. The model space consists of a single shell with $\Omega = 10$, both for protons and neutrons, with $N_n = 14$ and $N_p = 6$, neutrons and protons, respectively. For the coupling constant χ we have taken the values $\chi = 0.0$ and $\chi = 0.025$ MeV. The coupling constant κ was varied between 0.0 and 1.5 $G/4$. Figures 1 and 2 show the dependence of the eigenvalue ω on $4\kappa/G$, for $\chi = 0.0$ and $\chi = 0.025$ MeV, respectively, for the pn-QRPA, the pn-RRPA and the pn-FRQRPA. As shown in these figures, the value of $\kappa_{\omega=0}^{\text{pn-FRQRPA}}$ differs slightly with respect to the value $\kappa_{\omega=0}^{\text{pn-QRPA}}$, for both values of χ. Because the solution of the pn-FRQRPA was searched for consistently, the results of Figs. 1 and 2 indicate the absence of nontrivial values of Eqs. (51) and (52). Figures 3 and 4 show the values of the backward-going amplitude Y, also as a function of $4\kappa/G$, and for the same two values of χ of Figs. 1 and 2. Again for this quantity the difference between the results of the pn-QRPA and the pn-FRQRPA is very small. Finally, Figs. 5 and 6 show the value of D for the pn-RQRPA and for pn-FRQRPA. These figures show that the pn-FRQRPA equations cannot be computed passed the value where ω vanishes. Again, this is a consequence of Eqs. (51) and (52), which do not yield nontrivial values of f_1 and f_2 [see Eq. (39)].

IV. CONCLUSIONS

In this work, we have discussed the validity of the pn-FRQRPA of Refs. [17,18] for the case of a schematic Hamiltonian in a simple but nontrivial limit. The comparison of the results obtained by using the standard pn-QRPA and the pn-FRQRPA shows that:

(a) In spite of the complications that are inherent to the pn-FRQRPA formalism, its results are almost indistinguishable from the pn-QRPA ones in the region of collapse.
(b) The pn-FRQRPA cannot surpass the collapse point, as the pn-RQRPA does, if one imposes consistency between the approximations and the structure of the vacuum.

This has been shown both numerically and analytically. The results obtained by applying a boson expansion method support the present claim about the failure of the pn-FRQRPA approximation.

ACKNOWLEDGMENTS

This work has been supported by the CONICET (Argentina) and by CONACYT (Mexico). M. R. acknowledges financial support of the Fundacion Antorchas.

[1] M. Baranger, Phys. Rev. **120**, 957 (1960) .
[2] P. Ring and P. Shuck, *The Nuclear Many-Body Problem* (Springer-Verlag, New York, 1980).
[3] D. J. Rowe, Phys. Rev. **175**, 1283 (1968).
[4] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin Reading, MA, 1975), vol. 2.
[5] J. A. Halbleib and R. A. Sorensen, Nucl. Phys. **A98**, 542 (1967).
[6] D. Cha, Phys. Rev. C **27**, 2269 (1983).
[7] P. Vogel and M. R. Zirnbauer, Phys. Rev. Lett. **57**, 3148 (1986).
[8] O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. **B194**, 11 (1987).
[9] K. Hara, Progr. Theor. Phys. **32**, 88 (1964).
[10] F. Catara, N. Dinh Dang, and M. Sambataro, Nucl. Phys. **A579**, 1 (1994).
[11] J. Toivanen and J. Suhonen, Phys. Rev. Lett. **75**, 410 (1995); J. Toivanen and J. Suhonen, Phys. Rev. C **55**, 2314 (1997).
[12] D. R. Bes, O. Civitarese, N. N. Scoccola, Phys. Lett. **B446**, 93 (1999).
[13] O. Civitarese and M. Reboiro, Phys. Rev. C **57**, 3062 (1998).
[14] J. G. Hirsch, O. Civitarese, and M. Reboiro, Phys. Rev. C **60**, 024309 (1999).

[15] A. Bobyk, W. A. Kaminski, F. Simkovic, Phys. Rev. C **63**, 051301 (2001); F. Simkovic, J. Schwieger, G. Pantis, Amand Faessler, Found. Phys. **27**, 1275 (1997); F. Simkovic, A. A. Raduta, M. Veselsky, and A. Faessler, Phys. Rev. C **61**, 044319 (2000).
[16] J. Suhonen and O. Civitarese, Phys. Rep. **300**, 123 (1998).
[17] L. Pacearescu, V. Rodin, F. Simkovic, and A. Faessler, Phys. Rev. C **68**, 064310 (2003).
[18] V. Rodin and A. Faessler, Phys. Rev. C **66**, 051303 (2002).
[19] J. G. Hirsch, P. Hess, and O. Civitarese, Phys. Rev. C **56**, 199 (1997).
[20] D. R. Bes and O. Civitarese, Nucl. Phys. **A705**, 297 (2002).
[21] J. A. Evans and N. Krauss, Phys. Lett. **B37**, 455 (1971).
[22] O. Civitarese, F. Montani, and M. Reboiro, Phys. Rev. C **60**, 024305 (1999).